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New Versions of Caristi's Fixed Point Theorems in Soft-Cone Metric Spaces

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Abstract:

In this work, we introduce a few versions of Caristi's fixed point theorems in soft-cone metric spaces which extend Caristi's fixed point theorems in metric spaces. Analogs of such fixed-point theorems are demonstrated here. Our work broadens a lot of results around here of exploration.

Keywords: *Caristi's fixed point, Soft-cone metric space, unique fixed point, and self-mappings.*

1. Introduction

Mohammad *et, al* [16] studied sufficient conditions for the existence of a common fixed point of multivalued-mappings satisfying contractive type conditions in cone metric spaces. Molodtsov [17] started the thought of delicate sets as another numerical instrument for taking care of vulnerabilities and has exhibited various uses of this hypothesis in finding numerous functional issues in different bearings like as financial matters, designing, and so on. Maji *et al.* [14, 15] demonstrated soft set theory in detail and presented an application of soft sets in decision making problems. Chen *et al.* [3] worked on a new definition of reduction and addition of parameters of soft sets. Das and Samanta demonstrated the notion of soft real set and number [5], soft complex set and number [6], soft metric space [7, 8], soft normed linear space [9, 10]. Chiney and Samanta [4] demonstrated the notion of vector soft topology. In this paper, concept of soft cone metric space which is based on Caristi's soft elements is discussed.

In the last decades, Caristi's fixed point theorem has been generalized and extended in several directions and the related references therein. The following are basic definitions and theorems

2. Preliminaries and Definitions

Definition 2.1. [16] Let V be an initial universe and A be a set of parameters. Let $P(V)$ denote the power set of V. A pair (F, A) is called a soft set over V, where F is a mapping given by $F: A \to P(V)$.

Definition 2.2. [14] Let (F, A) and (G, A) be two soft sets over a common initial universe V.

a) (F, A) is said to be null soft set (denoted by ϕ), if $\forall\, \lambda \in A, F(\lambda) = \phi$. And$(F, A)$ is said to be an absolute soft set (denoted by $\tilde{V}$), if $\forall\, \lambda \in A, F(\lambda) = V$.

b) (F, A) is said to be a soft subset of(G, A) if $\forall\, \lambda \in A, F(\lambda) \subseteq G(\lambda)$ and it is denoted by $(F, A) \tilde{\subseteq} (G, A)$. (F, A) is said to be a soft upper set of (G, A) if (G, A) is a soft subset of(F, A). We denote it by $(F, A) \tilde{\supseteq} (G, A)$. (F, A) and (G, A) is said to be equal if$(F, A)$is asoft subset of $(G, A)$and$(G, A)$is a soft subset of (F, A).

c) The union of $(F, A)$and (G, A) over $V$is(H, A) defined as $H(\lambda) = F(\lambda) \cup G(\lambda), \forall\, \lambda \in$ A.We write$(F, A) \tilde{\cup} (G, A) = (H, A)$.

d) The intersection of (F, A) and (G, A)over $V$is (H, A) defined as$H(\lambda) = F(\lambda) \cap G(\lambda), \forall\, \lambda \in$ A. We write $(F, A) \tilde{\cap} (G, A) = (H, A)$.

e) The Cartesian product (H, A) of $(F, A)$and (G, A) over V denoted by $(H, A) = (F, A) \tilde{\times} (G, A)$,is defined as $H(\lambda) = F(\lambda) \times G(\lambda), \forall\, \lambda \in$ A.

f) The complement of (F, A) is defined as $(F, A)^{C} = (F^{C}, A)$ where $F^{C}: A \to P(V)$is a mapping given by $F^{C}(\lambda) = D \backslash F(\lambda), \forall\, \lambda \in$ A for all $\lambda \in$A. Clearly, we have $\widetilde{V^{C}} = \varphi$ and $\widetilde{F^{C}} = \tilde{V}$.

g) The difference$(H, A)$of$(F, A)$and (G, A)denoted by$(F, A)\tilde{\backslash}(G, A) = (H, A)$isdefined as $H(\lambda) = F(\lambda) \backslash G(\lambda), \forall\, \lambda \in$ A.

Definition 2.3. [5, 7] Let A be a non-empty parameter set and Z be a non-empty set. Then afunction $h: A \to Z$ is said to be a soft element of Z. A soft element h of Z is said to belongs to a soft set$(F, A)$of Z which is denoted by $h \tilde{\in} (F, A)$if $h(\lambda) \in F(\lambda), \forall \lambda \in$ A. Thus for a soft set $(F, A)$of Z with respect to the index set A, we have $F(\lambda) = \{\mathrm{h}(\lambda): \mathrm{h} \tilde{\in} (F, A), \lambda \in$ A. In that case, h is alsosaid to be a soft element of the soft set (F, A). Thus every singleton soft set (a soft set (F, A) of E for which F(λ) is a singleton set, $\forall\, \lambda \in$ A) can be identified with a soft element by simply identifyingthe singleton set with the element that it contains$\forall\, \lambda \in$ A.

Definition 2.4. [5,7] Let R be the set of real numbers andAbe a set of parameters and $B(R)$ be the collection of non-empty bounded subsets of R. Then a mapping $F: A \to B(R)$ is called a softreal set, denoted by (F, A). If specifically (F, A) is a singleton soft set, then after identifying $(F, A)$with the corresponding soft element, it will be called a soft real number.The set of all soft real numbers is denoted by$R(A)$ and the set of non-negative soft real numbersby$R(A)^*$.

Let $\tilde{r}$ and $\tilde{s}$ be two soft real numbers. Then the following statements hold:

- $\tilde{r} \tilde{\leq} \tilde{s}$, if $\tilde{r}(\lambda) \leq \tilde{s}(\lambda), \forall \lambda \in A$,
- $\tilde{r} \tilde{<} \tilde{s}$, if $\tilde{r}(\lambda) < \tilde{s}(\lambda), \forall \lambda \in A$,
- $\tilde{r} \tilde{\geq} \tilde{s}$, if $\tilde{r}(\lambda) \geq \tilde{s}(\lambda), \forall \lambda \in A$,
- $\tilde{r} \tilde{>} \tilde{s}$, if $\tilde{r}(\lambda) > \tilde{s}(\lambda), \forall \lambda \in A$.

Proposition 2.5. [6]

(a) For any soft sets$(F, A), (G, A) \in S(\tilde{V})$, we have$(F, A) \tilde{\subset} (G, A)$ if and only if every soft element of $(F, A)$is also a soft elements of (G, A).

(b) Any collection of soft elements of a soft set can generate a soft subset of that soft set. The soft set constructed from a collection B of soft elements is denoted by $SS(B)$.

(c) For any soft set $(F, A) \in S(\tilde{E}), SS(SE(F, A)) = (F, A)$; whereas for a collection B of soft elements, $SE(SS(B)) \supset B$,but, in general,$SE(SS(B)) \neq B$.

Definition 2.6. [10, 11]

(a) A sequence $\{\widetilde{z_n}\}$ of soft elements in a soft normed linear space $(\tilde{Z}, \|.\|, A)$ is said to be convergent and converges to a soft element $\tilde{z}$ if$\| \widetilde{z_n} - \tilde{z} \| \to \bar{0}$ as $n \to \infty$. This meansfor every $\tilde{\epsilon} \tilde{>} \bar{0}$, chosen arbitrarily, $\exists$ a natural number $N = N(\tilde{\epsilon})$ such that $\bar{0} \tilde{\leq} \| \widetilde{z_n} - \tilde{z} \| \tilde{<} \tilde{\epsilon}$whenever $n > N$i.e. $n > N \Rightarrow \widetilde{z_n} \in B(\tilde{z}, \tilde{\epsilon})$, (where $B(\tilde{z}, \tilde{\epsilon})$ is an open ball with centre $\tilde{z}$ and radius$\tilde{\epsilon}$).

(b) A sequence $\{\widetilde{z_n}\}$ of soft elements in a soft normed linear space$(\tilde{Z}, \|.\|, A)$ is said to be a Cauchy sequence in $\tilde{Z}$ if corresponding to every $\tilde{\epsilon} \tilde{>} \bar{0}$$\exists$ a natural number$N = N(\tilde{\epsilon})$such that$\| \widetilde{z_n} - \widetilde{z_n} \| \tilde{\leq} \tilde{\epsilon}, \forall$ m, n $> N$i.e.$\| \widetilde{z_n} - \widetilde{z_n} \| \to \bar{0}$as n, m $\to \infty$.

(c) Let$(\tilde{Z}, \|.\|, A)$be a soft normed linear space. Then $\tilde{Z}$ is said to be complete if every Cauchy sequence of soft elements in $\tilde{Z}$ converges to a soft element of $\tilde{Z}$. Every complete soft normed linear space is called a soft Banach space.

Definition 2.7. [7] Let Z be a non-empty set and A be non-empty a parameter set. A mapping $d: SV(\tilde{Z}) \times SV(\tilde{Z}) \rightarrow R(A)^*$is said to be a soft metric on the soft set $\tilde{Z}$ if d satisfies the following conditions:

- $d(\widetilde{z_1}, \widetilde{z_2}) \geqq \bar{0}, \forall\, \widetilde{z_1}, \widetilde{z_2} \in \tilde{Z}$.
- $d(\widetilde{z_1}, \widetilde{z_2}) = \bar{0}$, if and only if $\widetilde{z_1} = \widetilde{z_2}$.
- $d(\widetilde{z_1}, \widetilde{z_2}) = d(\widetilde{z_2}, \widetilde{z_1}), \forall\, \widetilde{z_1}, \widetilde{z_2} \in \tilde{Z}$.
- $d(\widetilde{z_1}, \widetilde{z_2}) \leqq d(\widetilde{z_1}, \widetilde{z_3}) + d(\widetilde{z_3}, \widetilde{z_2}), \forall\, \widetilde{z_1}, \widetilde{z_2}, \widetilde{z_3} \in \tilde{Z}$.

The soft $\tilde{Z}$with a soft metric $d$on $\tilde{Z}$ is said to be a soft metric space and denoted by $(\tilde{Z}, d, A)$ or $(\tilde{Z}, d)$.

Proposition 2.8. [11] Let $(\tilde{Z}, \|.\|, A)$be soft normed linear space. Let us define$d: \tilde{Z} \times \tilde{Z} \rightarrow R(A)^*byd(\widetilde{z_1}, \widetilde{z_2}) = k\widetilde{z_1} - \widetilde{z_2}k, \forall\, \widetilde{z_1}, \widetilde{z_2} \tilde{\in} \tilde{Z}$. Then d is a soft metric on $\tilde{Z}$.

3. Soft Cone Metric Spaces:

Definition 3.1. Let$(\tilde{V}, \|.\|, A)$be a soft real Banach space and$(P, A) \in S(\tilde{V})$ be a soft subset of $\tilde{V}$.Then (P, A) is called a soft cone if and only if

(1) (P, A) is closed, $(P, A) \neq \varphi$and $(P, A) \neq SS\{(\theta)$,

(2) $\tilde{a}, \tilde{b} \in R(A)^*, \widetilde{z_1}, \widetilde{z_2} \in (P, A) \Longrightarrow \tilde{a}\widetilde{z_1} + \tilde{b}\widetilde{z_2} \in (P, A)$,

(3) $\widetilde{z_1} \tilde{\in} (P, A)$ and $-\widetilde{z_1} \tilde{\in} (P, A)$implies $\widetilde{z_1} = \theta$.

Given a soft cone $(P, A) \in S(\tilde{V})$, we define a soft partial ordering $\tilde{\leqslant}$with respect to $(P, A)$by $\widetilde{z_1} \tilde{\leqslant} \widetilde{z_2}$if and only if $\widetilde{z_2} - \widetilde{z_1} \tilde{\in} (P, A)$. We write$\widetilde{z_1} \tilde{<} \widetilde{z_2}$ whenever, $\widetilde{z_1} \tilde{\leqslant} \widetilde{z_2}$ and $\widetilde{z_1} \neq \widetilde{z_2}$, while$\widetilde{z_1} \tilde{\ll} \widetilde{z_2}$will stand for$\widetilde{z_2} - \widetilde{z_1} \tilde{\in} Int(P, A)$ where $Int(P, A)$denotes the interior of(P, A).The cone (P, A) is called normal if there is a number $k > 0$, such that $\forall\, \widetilde{z_1}, \widetilde{z_2} \in \tilde{V}$, we have$\theta \tilde{\leqslant} \widetilde{z_1} \tilde{\leqslant} \widetilde{z_2} \Longrightarrow \| \widetilde{z_1} \| \leqslant k \| \widetilde{z_2} \|$**2**

The least positive number satisfying this inequality is called the soft normal constant of (P, A). The soft cone (P, A) is called regular if every increasing sequence which is bounded from above is convergent. Equivalently the cone (P, A) is called regular if every decreasing sequence which is bounded from below is convergent. Regular soft cones are soft normal and there exist soft normal cones which are not regular. Throughout the Banach space $\tilde{V}$ and the cone (P, A) will be omitted.

Definition 3.2. Let Z be a non-empty set and $\tilde{Z}$ be absolute soft set. A mapping $d: SV(\tilde{Z}) \times SV(\tilde{Z}) \rightarrow SV(\tilde{Z})$ is said to be a soft cone metric on $\tilde{Z}$if $d$satisfies the following axioms:

(d1) $d(\widetilde{z_1}, \widetilde{z_2}) \tilde{\in} \tilde{Z}, i.e. \theta \tilde{<} d(\widetilde{z_1}, \widetilde{z_2}), \forall\, \widetilde{z_1}, \widetilde{z_2} \tilde{\in} \tilde{Z}$ and $d(\widetilde{z_1}, \widetilde{z_2}) = \theta$ if and only if $\widetilde{z_1} = \widetilde{z_2}$.

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(d2) $d(\widetilde{z_1},\widetilde{z_2}) = d(\widetilde{z_2},\widetilde{z_1}), \forall\, \widetilde{z_1},\widetilde{z_2} \,\tilde{\in}\, \tilde{Z}$.

(d3) $d(\widetilde{z_1},\widetilde{z_2}) \,\tilde{\lessapprox}\, d(\widetilde{z_1},\widetilde{z_3}) + d(\widetilde{z_3},\widetilde{z_2}), \forall\, \widetilde{z_1},\widetilde{z_2},\widetilde{z_3} \,\tilde{\in}\, \tilde{Z}$.

Then, the soft set $\tilde{Z}$with a soft cone metric d on $\tilde{Z}$ is called a soft cone metric space and is denoted by $(\tilde{Z}, d, A)$.

Hence, it is obvious that soft cone metric spaces generalize soft metric spaces.

Definition 3.3: Let $(\tilde{Z}, d, A)$ be a soft cone metric space and $S: \tilde{Z} \to \tilde{Z}$ be a mapping if there is a positive soft real number $\tilde{t}$ with $0 \leq \tilde{t} < 1$ such that

$$d(S\tilde{u}, S\tilde{v}) \leq \tilde{t} d(\tilde{u}, \tilde{v}), \forall\, \tilde{u}, \tilde{v} \in \tilde{Z}$$

4. Main Results:

Theorem 4.1: Let $(\tilde{Z}, d, A)$ be a complete soft cone metric space. $S: \tilde{Z} \to \tilde{Z}$ be a self-mapping satisfying

$$d(S\tilde{u}, S\tilde{v}) \geq \alpha d(\tilde{u}, \tilde{v}) + \beta \frac{d(S\tilde{u}, \tilde{u})[1 + d(S\tilde{v}, \tilde{v})]}{1 + d(\tilde{u}, \tilde{v})} + \gamma \frac{[d(S\tilde{u}, \tilde{u}) + d(S\tilde{v}, \tilde{v})]}{d(S\tilde{u}, \tilde{v})} + \delta \frac{[d(\tilde{u}, S\tilde{u}) + d(\tilde{v}, S\tilde{v})]d(\tilde{u}, \tilde{v})}{d(\tilde{u}, S\tilde{v})} + \zeta d(S\tilde{u}, \tilde{v}) \qquad (4.1)$$

For all $\tilde{u}, \tilde{v} \in \tilde{Z}$ and $\alpha, \beta, \gamma, \delta, \zeta \geq 0$ with $\alpha > 1$ and $\beta \leq 1$ also $d(S\tilde{u}, \tilde{v}) = 0$. then S has a unique fixed point.

Proof. Since $\alpha >$ and $\beta, \gamma, \delta, \zeta \geq 0$ then obviously $\alpha + \beta + \gamma + \delta + \zeta > 1$. for $\widetilde{z_0} \in \tilde{Z}$, we define a sequence $\{\widetilde{z_n}\}$ in $\tilde{Z}$ by the following way

$$\widetilde{z_n} = S_{\widetilde{z_{n+1}}} \text{ for } n = 0,1,2,3, \dots \dots \dots$$

Consider

$$d(\widetilde{z_{n-1}}, \widetilde{z_n}) = d\left(S_{\widetilde{z_n}}, S_{\widetilde{z_{n+1}}}\right).$$

Using (4.1) and the defined construction of the sequence we have

$$\geq \alpha d(\widetilde{z_n}, \widetilde{z_{n+1}}) + \beta \frac{d\left(S_{\widetilde{z_n}}, \widetilde{z_n}\right)[1 + d(S_{\widetilde{z_{n+1}}}, \widetilde{z_{n+1}})]}{1 + d(\widetilde{z_n}, \widetilde{z_{n+1}})} + \gamma \frac{\left[d\left(S_{\widetilde{z_n}}, \widetilde{z_n}\right) + d\left(S_{\widetilde{z_{n+1}}}, \widetilde{z_{n+1}}\right)\right] d(\widetilde{z_n}, \widetilde{z_{n+1}})}{d(S_{\widetilde{z_n}}, \widetilde{z_{n+1}})} + \delta \frac{\left[d\left(\widetilde{z_n}, S_{\widetilde{z_n}}\right) + d\left(\widetilde{z_{n+1}}, S_{\widetilde{z_{n+1}}}\right)\right] d(\widetilde{z_{n+1}}, \widetilde{z_n})}{d\left(\widetilde{z_n}, S_{\widetilde{z_{n+1}}}\right)} + \zeta d(S\widetilde{z_n}, \widetilde{z_{n+1}})$$

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$$= \alpha d(\widetilde{z_n}, \widetilde{z_{n+1}}) + \beta \frac{d(\widetilde{z_{n-1}}, \widetilde{z_n})[1 + d(\widetilde{z_n}, \widetilde{z_{n+1}})]}{1 + d(\widetilde{z_n}, \widetilde{z_{n+1}})}$$

$$+ \gamma \frac{[d(\widetilde{z_{n-1}}, \widetilde{z_n}) + d(\widetilde{z_n}, \widetilde{z_{n+1}})]d(\widetilde{z_n}, \widetilde{z_{n+1}})}{d(\widetilde{z_{n-1}}, \widetilde{z_{n+1}})}$$

$$+ \delta \frac{[d(\widetilde{z_n}, \widetilde{z_{n-1}}) + d(\widetilde{z_n}, \widetilde{z_{n+1}})]d(\widetilde{z_{n+1}}, \widetilde{z_n})}{d(\widetilde{z_n}, \widetilde{z_{n-1}})} + \zeta d(S\widetilde{z_n}, \widetilde{z_{n+1}})$$

By simplification and using the fact that

$$d(\widetilde{z_n}, \widetilde{z_{n+1}}) \leq \frac{[d((\widetilde{z_{n-1}}, \widetilde{z_n}) + d(\widetilde{z_n}, \widetilde{z_{n+1}})]}{d(\widetilde{z_{n-1}}, \widetilde{z_{n+1}})} \leq \frac{[d((\widetilde{z_{n-1}}, \widetilde{z_n}) + d(\widetilde{z_n}, \widetilde{z_{n+1}})]d(\widetilde{z_{n+1}}, \widetilde{z_n})}{(\widetilde{z_{n+1}}, \widetilde{z_{n-1}})}$$

$$\leq \frac{[d(\widetilde{z_n}, \widetilde{z_{n-1}}) + d(\widetilde{z_n}, \widetilde{z_{n+1}})]d(\widetilde{z_n}, \widetilde{z_{n-1}})}{d(\widetilde{z_n}, \widetilde{z_{n-1}})}$$

We have

$$(\alpha + \delta)d(\widetilde{z_n}, \widetilde{z_{n+1}}) \leq (1 - \beta)d(\widetilde{z_{n-1}}, \widetilde{z_n}).$$

$$d(\widetilde{z_n}, \widetilde{z_{n+1}}) \leq \left(\frac{1 - \beta}{\alpha + \delta}\right) d(\widetilde{z_{n-1}}, \widetilde{z_n}).$$

Let

$$\left(\frac{1 - \beta}{\alpha + \delta}\right) = h < 1.$$

So the above inequality takes the form

$$d(\widetilde{z_n}, \widetilde{z_{n+1}}) \leq hd(\widetilde{z_{n-1}}, \widetilde{z_n}).$$

Also

$$d(\widetilde{z_{n-1}}, \widetilde{z_n}) \leq hd(\widetilde{z_{n-2}}, \widetilde{z_{n-1}}).$$

Hence

$$d(\widetilde{z_n}, \widetilde{z_{n+1}}) \leq h^2 d(\widetilde{z_{n-2}}, \widetilde{z_{n-1}}).$$

Continuing the same procedure we have

$$d(\widetilde{z_n}, \widetilde{z_{n+1}}) \leq h^2 d(x_0, x_1).$$

Since $h < 1$ and taking $n \to \infty$, $h^n \to 0$. Hence

$$\lim_{n\to\infty} d\,(\widetilde{z_n}, \widetilde{z_{n+1}}) = 0$$

Which proves that $\{\widetilde{z_n}\}$ is a Cauchy sequence in complete soft cone metric space $\tilde{Z}$.

$$S\lim_{n\to\infty}\widetilde{z_n} = S\tilde{u} \;\Rightarrow\; \lim_{n\to\infty}S\widetilde{z_n} = S\tilde{u} \Rightarrow\; \lim_{n\to\infty}\widetilde{z_{n-1}} = S\tilde{u} \Rightarrow S\tilde{u} = \tilde{u}.$$

Hence $\tilde{u}$ is fixed point of S.

Uniqueness: Consider

$$d(\tilde{u},\tilde{v}) = d(S\tilde{u},S\tilde{v})$$
$$\geq \alpha d(\tilde{u},\tilde{v}) + \beta\frac{d(S\tilde{u},\tilde{u})[1+d(S\tilde{v},\tilde{v})]}{1+d(\tilde{u},\tilde{v})} + \gamma\frac{[d(S\tilde{u},\tilde{u})+d(S\tilde{v},\tilde{v})]d(\tilde{u},\tilde{v})}{d(S\tilde{u},\tilde{v})}$$
$$+\delta\frac{[d(\tilde{u},S\tilde{v})+d(\tilde{u},S\tilde{v})]}{1-[d(\tilde{u},S\tilde{v})d(\tilde{u},S\tilde{v})]}.$$

Using the above proved facts we have

$$d(\tilde{u},\tilde{v}) \geq \alpha d(\tilde{u},\tilde{v})$$

Which is again contradiction thus $d(\tilde{u},\tilde{v}) = 0$ similarly we can show that $d(\tilde{v},\tilde{u}) = 0$ implies $\tilde{u} = \tilde{v}$. Thus fixed point of S is unique.

Corollary 2.1: Let $(\tilde{Z}, d, A)$ be a complete soft cone metric space. $S: \tilde{Z} \to \tilde{Z}$ be a self-mapping satisfying

$$d(S\tilde{u},S\tilde{v}) \geq \alpha d(\tilde{u},\tilde{v})$$

For all

$\tilde{u},\tilde{v} \in \tilde{Z}$ with $\alpha > 1$. Then S has a unique fixed point.

Theorem 2.2: Let $(\tilde{Z}, d, A)$ be a complete soft cone metric space. $S: \tilde{Z} \to \tilde{Z}$ be a self-mapping satisfying

$$d(S\tilde{u},S\tilde{v}) \leq \alpha\psi d(\tilde{u},\tilde{v}) + \beta\psi \max\{d(\tilde{u},S\tilde{u}), d(\tilde{u},\tilde{v})\} + \gamma\psi\frac{d(\tilde{u},S(\tilde{u}))d(\tilde{v},S(\tilde{v}))}{d(\tilde{u},\tilde{v})}$$
$$+\delta\psi\left(\frac{d(\tilde{u},\tilde{v})\left[1+\sqrt{d(\tilde{u},\tilde{v})d(\tilde{u},S\tilde{u})}\right]^2}{(1+d(\tilde{u},\tilde{v}))^2}\right)$$
$$+\zeta\Psi\left(\frac{d\left(\widetilde{Su},(\tilde{u})\right)+d(\tilde{v},S(\tilde{v}))}{d(\tilde{u},\tilde{v})}\right) \qquad (2.2)$$

For all $\tilde{u},\tilde{v} \in \tilde{Z}, \alpha,\beta,\gamma,\delta,\zeta \geq 0$ with $\alpha+\beta+\gamma+\delta+\zeta < 1$ and ψ is a comparison function as defined in . Then S has a unique fixed point.

Proof. Let $\tilde{z}_0$ be arbitrary point in $\tilde{Z}$. Define a sequence $\{\tilde{z}_n\}$ in $\tilde{Z}$ by the rule

$$\widetilde{z_{n+1}} = S\tilde{z}_n \quad n = 0,1,2,3 \ldots\ldots$$

Consider

$$d(\tilde{z}_n,\tilde{z}_{n+1}) = d(S\tilde{z}_{n-1},S\tilde{z}_n).$$

Now using (2.2) we have

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$$\leq \alpha\psi \mathrm{d}(\widetilde{z_{n-1}},\widetilde{z_n}) + \beta\psi \max\{\mathrm{d}(\widetilde{z_{n-1}},\widetilde{Sz_{n-1}})\mathrm{d}(\widetilde{z_{n-1}},\widetilde{z_n})\} + \gamma\psi \frac{\mathrm{d}(\widetilde{z_{n-1}},\widetilde{Sz_{n-1}})\mathrm{d}\left(\widetilde{z_n,S\widetilde{z_n}}\right)}{\mathrm{d}(\widetilde{z_{n-1}},\widetilde{z_n})}$$

$$+ \delta\psi\left(\frac{\mathrm{d}(\widetilde{z_{n-1}},\widetilde{z_n})\left[1+\sqrt{\mathrm{d}(\widetilde{z_{n-1}},\widetilde{z_n})\mathrm{d}(\widetilde{z_{n-1}},\widetilde{Sz_{n-1}})}\right]^2}{(1+\mathrm{d}(\widetilde{z_{n-1}},\widetilde{z_n}))^2}\right)$$

$$+ \zeta\Psi\left(\frac{\mathrm{d}\left(\widetilde{Sz_{n-1}},\widetilde{z_{n-1}}\right)+\mathrm{d}\left(\widetilde{z_n,S\widetilde{z_n}}\right)}{\mathrm{d}(\widetilde{z_{n-1}},\widetilde{z_n})}\right)$$

$$=\leq \alpha\psi \mathrm{d}(\widetilde{z_{n-1}},\widetilde{z_n}) + \beta\psi \max\{\mathrm{d}(\widetilde{z_{n-1}},\widetilde{z_{n-1}})\mathrm{d}(\widetilde{z_{n-1}},\widetilde{z_n})\} + \gamma\psi \frac{\mathrm{d}(\widetilde{z_{n-1}},\widetilde{z_{n-1}})\mathrm{d}\left(\widetilde{z_n,\widetilde{z_n}}\right)}{\mathrm{d}(\widetilde{z_{n-1}},\widetilde{z_n})}$$

$$+ \delta\psi\left(\frac{\mathrm{d}(\widetilde{z_{n-1}},\widetilde{z_n})\left[1+\sqrt{\mathrm{d}(\widetilde{z_{n-1}},\widetilde{z_n})\mathrm{d}(\widetilde{z_{n-1}},\widetilde{z_{n-1}})}\right]^2}{(1+\mathrm{d}(\widetilde{z_{n-1}},\widetilde{z_n}))^2}\right)$$

$$+ \zeta\psi\left(\frac{\mathrm{d}(\widetilde{z_{n-1}},\widetilde{z_{n-1}})\mathrm{d}\left(\widetilde{z_n,\widetilde{z_n}}\right)}{\mathrm{d}(\widetilde{z_{n-1}},\widetilde{z_n})}\right).$$

$$= (\alpha+\beta+\gamma+\delta+\zeta)\Psi d(\widetilde{z_n},\widetilde{z_{n+1}}).$$

Since $\psi(\mathrm{t}) \leq \mathrm{t}\forall\mathrm{t} \geq 0$ so

$$d(\widetilde{z_n},\widetilde{z_{n+1}}) \leq (\alpha+\beta+\gamma+\delta+\zeta)d(\widetilde{z_{n-1}},\widetilde{z_n}).$$

Hence

$$d(\widetilde{z_n},\widetilde{z_{n+1}}) \leq kd(\widetilde{z_{n-1}},\widetilde{z_n})$$

where $k = (\alpha+\beta+\gamma+\delta+\zeta)$

Continuing in the same we get

$$d(\widetilde{z_n},\widetilde{z_{n+1}}) \leq k^n d(\widetilde{z_0},\widetilde{z_1})$$

Taking limit $n \to \infty\ so\ k^n \to 0$. therefore

$$\lim_{n\to\infty} d(\widetilde{z_n},\widetilde{z_{n+1}}) = 0$$

Which prove that $\{\tilde{z}_n\}$ is a Cauchy sequence in complete soft cone metric space $\tilde{Z}$. so there must exists $\tilde{h} \in \tilde{Z}$ such that

$$\lim_{n\to\infty} \widetilde{z_n} = \tilde{h}$$

Also since S is continuous function so we have

$$S\tilde{h} = S\lim_{n\to\infty} \widetilde{z_n} = \lim_{n\to\infty} S\widetilde{z_n} + \lim_{n\to\infty} \widetilde{z_{n+1=}}\tilde{h}.$$

Therefore $\tilde{h}$ is the fixed point of S.

Uniqueness. Let $u \neq v$ are two distinct fixed points of S then consider

$$d(\tilde{u},\tilde{v}) = d(S\tilde{u},S\tilde{v}).$$

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$$\leq \alpha\psi d(\tilde{u},\tilde{v}) + \beta\psi \max\{d(\tilde{u},S\tilde{u}), d(\tilde{u},\tilde{v})\} + \gamma\psi \frac{d(\tilde{u},(S\tilde{u}))d\left(\tilde{v},(\widetilde{Sv})\right)}{d(\tilde{u},\tilde{v})}$$

$$+ \delta\psi\left(\frac{\mathrm{d}(\tilde{u},\tilde{v})\left[1+\sqrt{\mathrm{d}(\tilde{u},\tilde{v})\mathrm{d}(\tilde{u},S\tilde{u})}\right]^2}{(1+\mathrm{d}(\tilde{u},\tilde{v}))^2}\right)$$

$$+ \zeta\psi\left(\frac{d\left(\widetilde{S(u)},(\tilde{u})\right)d\left(\widetilde{S(v)},(\tilde{v})\right)}{d(\tilde{u},\tilde{v})}\right)$$

$$= \alpha\psi d(\tilde{u},\tilde{v}) + \beta\psi \max\{d(\tilde{u},\tilde{u}), d(\tilde{u},\tilde{v})\} + \gamma\psi \frac{d(\tilde{u},(\tilde{u}))d(\tilde{v},(\tilde{v}))}{d(\tilde{u},\tilde{v})}$$

$$+ \delta\psi\left(\frac{\mathrm{d}(\tilde{u},\tilde{v})\left[1+\sqrt{\mathrm{d}(\tilde{u},\tilde{v})\mathrm{d}(\tilde{u},\tilde{u})}\right]^2}{(1+\mathrm{d}(\tilde{u},\tilde{v}))^2}\right)$$

$$+ \zeta\psi\left(\frac{d\left((\widetilde{u}),(\tilde{u})\right)d\left((\widetilde{v}),(\tilde{v})\right)}{d(\tilde{u},\tilde{v})}\right) \qquad (2.3)$$

$$d(\tilde{u},\tilde{v}) \leq \alpha\psi d(\tilde{u},\tilde{v}) + \beta\psi\mathrm{d}(\tilde{u},\tilde{v}) + \gamma\psi\frac{\mathrm{d}(\tilde{u},\tilde{u}) + \mathrm{d}(v,\tilde{v})}{\mathrm{d}(\tilde{u},\tilde{v})} + \delta\psi\frac{\mathrm{d}(\tilde{u},\tilde{v})}{\left(1+\mathrm{d}(\tilde{u},\tilde{v})\right)^2}$$

$$+ \zeta\Psi\left(\frac{\mathrm{d}(\tilde{u},\tilde{u}) + \mathrm{d}(v,\tilde{v})}{\mathrm{d}(\tilde{u},\tilde{v})}\right) \qquad (2.4)$$

Thus (2.4) becomes

$$\mathrm{d}(\tilde{u},\tilde{v}) \leq (\alpha+\beta+\gamma+\delta+\zeta)\Psi\mathrm{d}(\tilde{u},\tilde{v}).$$

Since $\alpha+\beta+\gamma+\delta+\zeta < 1$ so the above inequality is possible only if $\mathrm{d}(\tilde{u},\tilde{v}) = 0$. similarly we can show that $\mathrm{d}(\tilde{u},\tilde{v}) = 0$ which implies that $\tilde{u} = \tilde{v}$. *hence fixed point* S is unique.

Example 2.1: Let $\tilde{Z} = \mathbb{R}$ and the complete soft cone metric on $\tilde{Z}$ is defined by

$$\mathrm{d}(\tilde{u},\tilde{v}) = |\tilde{u}| \text{for all } \tilde{u},\tilde{v} \in \tilde{Z}$$

and $S\tilde{u} = \frac{\tilde{u}}{12}$ for all $\tilde{u} \in \tilde{Z}$ *then*

$$d(S\tilde{u},S\tilde{v}) = \left|\frac{\tilde{u}}{24}\right| \leq \left|\frac{\tilde{u}}{12}\right| = \frac{1}{12}|\tilde{u}| = \frac{1}{6}\frac{1}{6}|\tilde{u}| = \alpha\psi\big(d(\tilde{u},\tilde{v})\big)$$

Thus for $\alpha = \frac{1}{6}$ *and* $\psi(\mathrm{t}) = \frac{t}{6}$ for all $t \geq 0$ satisfy all the conditions having $\tilde{u} = 0$ is the unique fixed point of S.

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