Fuzzy p- Laplace Transform: Property and Applications

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ABSTRACT:

In this paper we propose a noval fuzzy integral transform namely Fuzzy $p-$ Laplace Transform (F$p-$LT), we apply it to solve initial Value Problem and Volterra integral equation in fuzzy system as an analytic solution method. We prove the related elementary properties and theorems in detail. Also, the (F$p-$LT) method is illustrated by solving two examples in Differential Calculus.

Keywords: *Fuzzy number, Fuzzy Laplace Transform, Fuzzy differential equation, Hukuhara difference, Fuzzy $p-$ Laplace Transform.*

1. Introduction

Different types of integral transforms are available in the fields of engineering and mathematics. The Laplace integral transform is a more widely used transform that can examine new research strategies in pure and practical mathematics. In [26], Mohd Saif proposed a new definition of Laplace Transform called Modified Laplace Transform. In fuzzy environment, [6] Chang and Zadeh was introduced the concept of differentiability for function. In [12], Dubois and Prade use the Zadeh's extension principle concept in their study, following Chang and Zadeh's ideas. In [2], Allahviranloo and Ahmadi was solve Fuzzy Differential Equations and associated FIVP by Fuzzy Laplace Transform. In [22], Salahshour discussed how to solve fuzzy Volterra integral equations using the Fuzzy Laplace Transform. The following is how the paper is structured:

Important definitions, theorems, and other information are supplied in section 2 and will be used later in the study. In part 3, we look at the basic features and theorems of the Fuzzy $p-$ Laplace Transform. We present two applications of the Fuzzy $p-$ Laplace Transform to solve mathematical problems using the derivative and convolution theorems in part 4, and we reach a conclusion in section 5.

2. Basic Concepts

We'll look at some basic concepts and theorems inside this section:

Definition 1 [30] A fuzzy membership function of bounded support $\xi: \Re \to [0,1]$ is called a fuzzy number and it is a fuzzy subset of the $\Re$ with

- upper semicontinuous,
- convex,
- normal.

The − level arrangement of fuzzy numbers is characterised as in[30, 20].

Definition 2 The − level arrangement of a fuzzy number $\xi \in \Re_\intercal$, $\in [0,1]$,

$$[\xi]_\alpha = \begin{cases} \{\wp_1 \in \Re \,/\, \xi(\wp_1) \geq \alpha\}, & \text{if } \alpha \in (0,1], \\ \text{cl}(\text{supp}\, \xi), & \text{if } \alpha = 0. \end{cases}$$

where, $\Re_\intercal$ represents the set of all fuzzy numbers on $\Re$.

Evidently, the − level arrangement is a closed and bounded interval $[\underline{\xi}_\alpha, \overline{\xi}_\alpha]$ of a fuzzy number , where $\underline{\xi}_\alpha, \overline{\xi}_\alpha$ signifies the left - right hand terminal points of $[\xi]_\alpha$ respectively. For all $\wp_2 \in \Re$, consider it as a fuzzy number $\widetilde{\wp}_2$ represented by

$$\widetilde{\wp}_2(s) = \begin{cases} 1 & \text{if } s = z \\ 0 & \text{if } s \neq z \end{cases}$$

Remark 1 [3, 20]Let $\mathcal{P} = \mathcal{P}_1 \times \ldots \times \mathcal{P}_m$ be Cartesian product of universe and $\varkappa_1, \ldots, \varkappa_m$ be m fuzzy numbers in $\mathcal{P}_1, \ldots, \mathcal{P}_m$ respectively. A fuzzy function $\intercal: \mathcal{P} \to \mathcal{Q}$, $q = \intercal(\wp_1, \ldots, \wp_m)$. Then, at that point the extension principle permits us to characterize a fuzzy set B in $\mathcal{Q}$ be

$$B = \{(q, u_B(q)) \,|\, q = \intercal(\wp_1, \ldots, \wp_n), (\wp_1, \ldots, \wp_n) \in \mathcal{P}\},$$

Where,

$$u_B(q) = \begin{cases} \sup\limits_{(\wp_1, \ldots, \wp_m) \in \intercal^{-1}(q)} \min\{u_{\varkappa_1}(\wp_1), \ldots, u_{\varkappa_m}(\wp_m)\} & \text{if } \intercal^{-1} \neq 0 \\ 0 & \text{if } \intercal^{-1} = 0 \end{cases}$$

Where $\intercal^{-1}$ means the inverse of $\intercal$.

The extension principle is decreased to (for specific $m = 1$)

$$B = \{(q, u_B(q)) \,|\, q = \intercal(\wp), \wp \in \mathcal{P}\},$$

Where,

$$u_B(q) = \begin{cases} \sup\limits_{\wp \in \intercal^{-1}(q)} u_\varkappa(\wp) & \text{if } \intercal^{-1} \neq 0 \\ 0 & \text{if } \intercal^{-1} = 0. \end{cases}$$

Expansion of addition operation on $\Re_\intercal$ as per Zadeh's extension principle[2] is characterized by

$$(\Omega \oplus \theta)(\wp_1) = \sup_{\wp_2 \in \Re} \min\{\Omega(\wp_2), \theta(\wp_1 - \wp_2)\}, \wp_1 \in \Re$$

and scalar multiplication[1] is given by

$$(\varrho \odot \Omega)(\wp_1) = \begin{cases} \Omega(\frac{\wp_1}{\varrho}), & \varrho > 0 \\ \tilde{0}, & \varrho = 0, \end{cases}$$

where $\tilde{0} \in \Re_T$. The accompanying properties are valid, $\forall \ \alpha$ levels.

$$[\Omega \oplus \theta]_\alpha = [\Omega]_\alpha + [\theta]_\alpha$$

and

$$[\varrho \odot \Omega]_\alpha = \varrho[\Omega]_\alpha$$

The terminal points of the intervals $[\Omega]_\alpha$ determine a fuzzy number, as we can see from the properties of fuzzy numbers.

Definition 3 [15, 24] A fuzzy number Ω in parametric structure is a couple $(\underline{\Omega}, \overline{\Omega})$ of capacities $\underline{\Omega}_\alpha, \overline{\Omega}_\alpha$, $0 \leq \alpha \leq 1$ which satify the accompanying conditions as

1. $\underline{\Omega}_\alpha$ and $\overline{\Omega}_\alpha$ are a left continuous, right continuous at 0, and bounded,
2. $\overline{\Omega}_\alpha$ and $\underline{\Omega}_\alpha$ are non-increasing and non-decreasing function in $(0,1]$ respectively,
3. $\underline{\Omega}_\alpha \leq \overline{\Omega}_\alpha$, $0 \leq \alpha \leq 1$

Definition 4 [18, 20] A Ω is known as triangular fuzzy number if a membership function of a fuzzy number Ω has the accompanying structure:

$$\Omega(\wp_1) = \begin{cases} 0, & \text{if } \wp_1 < p_1, \\ \dfrac{\wp_1 - p_1}{q_1 - p_1}, & \text{if } p_1 \leq \wp_1 < q_1, \\ \dfrac{r_1 - \wp_1}{r_1 - q_1}, & \text{if } q_1 \leq \wp_1 \leq r_1, \\ 0, & \text{if } \wp_1 > r_1, \end{cases}$$

furthermore, its -levels are just $[u]_\alpha = [p_1 + \alpha(q_1 - p_1), r_1 - \alpha(r_1 - q_1)]$, $\alpha \in [0,1]$

Definition 5 [15, 20, 27] For self-assertive $[\Omega]_\alpha = [\underline{\Omega}_\alpha, \overline{\Omega}_\alpha]$ and $[\theta]_\alpha = [\underline{\theta}_\alpha, \overline{\theta}_\alpha]$ and $\varrho > 0$, we characterize addtion, substraction and scalar multiplication by ϱ for $[\Omega]_\alpha$ and $[\theta]_\alpha$ as

1. **Addition** , $[\Omega]_\alpha \oplus [\theta]_\alpha = [\underline{\Omega}_\alpha + \underline{\theta}_\alpha, \overline{\Omega}_\alpha + \overline{\theta}_\alpha]$
2. **substraction**, $[\Omega]_\alpha \ominus [\theta]_\alpha = [\underline{\Omega}_\alpha - \overline{\theta}_\alpha, \overline{\Omega}_\alpha - \underline{\theta}_\alpha]$
3. **scalar multiplication**, $$\varrho \odot [\Omega]_\alpha = \begin{cases} [\varrho\underline{\Omega}_\alpha, \varrho\overline{\Omega}_\alpha] & \varrho \geq 0, \\ [\varrho\overline{\Omega}_\alpha, \varrho\underline{\Omega}_\alpha] & \varrho < 0, \end{cases}$$

If $\varrho < 0$ for arbitrary $\varrho = -1$ then $\varrho \odot [\Omega]_\alpha = -[\Omega]_\alpha$

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Definition 6 [8, 20]Let Ω and θ be two fuzzy integrals and $\mathfrak{D}_f(\Omega,\theta)$ denotes the distance between them is characterized as follow

$$\mathfrak{D}_f(\Omega,\theta) = \sup_{\alpha\in[0,1]} d_H(\Omega_\alpha,\theta_\alpha),$$

where,

$$d_H(\Omega_\alpha,\theta_\alpha) = \max\{|\underline{\Omega}_\alpha - \underline{\theta}_\alpha|,|\overline{\Omega}_\alpha - \overline{\theta}_\alpha|\}$$

is the Hausdorff distance among $[\Omega]_\alpha$ and $[\theta]_\alpha$

From Definition 6, $\mathfrak{D}_f$ is a metric space. The elementary properties of metric space $\mathfrak{D}_f$ listed as follow:

1. $\mathfrak{D}_f(\Omega \oplus \Phi, \theta \oplus \Phi) = \mathfrak{D}_f(\Omega,\theta), \forall \Omega,\theta,\Phi \in \mathfrak{R}_\intercal,$
2. $\mathfrak{D}_f(\varrho \odot \Omega, \varrho \odot \theta) = |\varrho|\mathfrak{D}_f(\Omega,\theta), \forall \varrho \in \mathfrak{R}, \Omega,\theta \in \mathfrak{R}_\intercal,$
3. $\mathfrak{D}_f(\Omega \oplus \theta, \Phi \oplus \tau) \leq \mathfrak{D}_f(\Omega,\Phi) + \mathfrak{D}_f(\theta,\tau), \forall \Omega,\theta,\Phi,\tau \in \mathfrak{R}_\intercal$
4. $(\mathfrak{D}_f, \mathfrak{R}_\intercal)$ is a complete metric space.

Definition 7 [11, 20]Let $\intercal$ is mapping from $\mathfrak{R}$ to $\mathfrak{R}_\intercal$. A function $\intercal$ is called continuous at $\wp_0 \in \mathfrak{R}$ if $> 0, \exists > 0$, such that $\mathfrak{D}_f(\intercal(\wp), \intercal(\wp_0)) <$ whenever $|\wp - \wp_0| <$.

Theorem 1 [10, 20, 25, 28]For any fixed $\in [0,1]$, let $\intercal(\flat)$ be a function on $[0,\infty)$ in fuzzy environment with parametric $[\intercal(\flat)]_\alpha = [\underline{\intercal}_\alpha(\flat), \overline{\intercal}_\alpha(\flat)]$. Assume that, for every $h \geq k$, $\underline{\intercal}_\alpha(\flat)$, $\overline{\intercal}_\alpha(\flat)$ are fuzzy Riemann- integrable on $[k,h]$ and let we take two positive constants $\underline{\mathcal{H}}_\alpha$, $\overline{\mathcal{H}}_\alpha$ such that for every $h \geq k$, $\int_k^h |\underline{\intercal}_\alpha(\flat)| d\flat \leq \underline{\mathcal{H}}_\alpha$ and $\int_k^h |\overline{\intercal}_\alpha(\flat)| d\flat \leq \overline{\mathcal{H}}_\alpha$ Then $\intercal(\flat)$ is improper fuzzy Riemann- integrable on $[0,\infty)$. Furthermore, We write:

$$\int_k^\infty \intercal(\flat) d\flat = \left(\int_k^\infty \underline{\intercal}_\alpha(\flat) d\flat, \int_k^\infty \overline{\intercal}_\alpha(\flat) d\flat\right)$$

Proposition 1 [20, 29]Let $\intercal(\flat)$ and $g(\flat)$ be fuzzy valued Riemann integrable functions on $[k,\infty)$, then $\intercal(\flat) \oplus G(\flat)$ is also fuzzy valued Riemann- integrable function on $[k,\infty)$. furthermore, we have

$$\int_I (\intercal(\flat) \oplus G(\flat)) d\flat = \int_I \intercal(\flat) d\flat + \int_I G(\flat) d\flat$$

Definition 8 [7, 23]Let $\mu, \theta \in \mathfrak{R}_\intercal$. If $\mu = \theta + \Phi$ with the end goal that $\exists\ \Phi \in \mathfrak{R}_\intercal$, Φ is the Hukuhara difference(H-Difference) of μ and θ, and is denoted by $\Phi = \mu \ominus \theta$. Note that $\mu + (-1)\theta \neq \mu \ominus \theta$.

Definition 9 [1] Let be $\intercal: (a_0, b_0) \to \mathfrak{R}_\intercal$ and $\flat_0 \in (a_0, b_0)$. Then we are saying that $\intercal$ is strongly generalized differentiablity at $\flat_0$ if there exist $\intercal'(\flat_0) \in \mathfrak{R}_\intercal$ such

1. for all $h > 0$ extremely small, then $\exists\ \intercal(\flat_0 + h) \ominus \intercal(\flat_0), \intercal(\flat_0) \ominus \intercal(\flat_0 - h)$ and therefore the limits hold within the metric $\mathfrak{D}_f$.

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$$\lim_{h \searrow 0} \frac{\daleth(b_0 + h) \ominus \daleth(b_0)}{h} = \lim_{h \searrow 0} \frac{\daleth(b_0) \ominus \daleth(b_0 - h)}{h} = \daleth'(b_0)$$

2. for all $h > 0$ extremely small, $\exists$ $\daleth(b_0) \ominus \daleth(b_0 + h)$, $\daleth(b_0 - h) \ominus \daleth(b_0)$ and therefore the limits hold within the metric $\mathfrak{D}_f$.

$$\lim_{h \searrow 0} \frac{\daleth(b_0) \ominus \daleth(b_0 + h)}{-h} = \lim_{h \searrow 0} \frac{\daleth(b_0 - h) \ominus \daleth(b_0)}{-h} = \daleth'(b_0)$$

3. for all $h > 0$ extremely small, then $\exists$ $\daleth(b_0 + h) \ominus \daleth(b_0)$, $\daleth(b_0 - h) \ominus \daleth(b_0)$ and therefore the limits hold within the metric $\mathfrak{D}_f$.

$$\lim_{h \searrow 0} \frac{\daleth(b_0 + h) \ominus \daleth(b_0)}{h} = \lim_{h \searrow 0} \frac{\daleth(b_0 - h) \ominus \daleth(b_0)}{-h} = \daleth'(b_0)$$

4. for all $h > 0$ extremely small, $\exists$ $\daleth(b_0) \ominus \daleth(b_0 + h)$, $\daleth(b_0) \ominus \daleth(b_0 - h)$ and therefore the limits hold within the metric $\mathfrak{D}_f$.

$$\lim_{h \searrow 0} \frac{\daleth(b_0) \ominus \daleth(b_0 + h)}{-h} = \lim_{h \searrow 0} \frac{\daleth(b_0) \ominus \daleth(b_0 - h)}{h} = \daleth'(b_0)$$

Theorem 2 [7, 17]Let $\daleth: [a_0, b_0] \to \Re_\daleth$ be fuzzy membership function and $\forall \in [0,1]$ it represented as $[\daleth(b)]_\alpha = [\underline{\daleth}_\alpha(b), \overline{\daleth}_\alpha(b)]$.

1. If the function $\daleth$ is (1)- differentiable then $\underline{\daleth}_\alpha$ and $\overline{\daleth}_\alpha$ are lower function and upper function are differentible respectively and $[\daleth'(b)]_\alpha = \left[\underline{\daleth}_\alpha{}'(b), \overline{\daleth}_a{}'(b)\right]$,

2. If the function $\daleth$ is (2)- differentiable then $\underline{\daleth}_\alpha$ and $\overline{\daleth}_\alpha$ are lower function and upper function are differentible respectively and, $[\daleth'(b)]_\alpha = \left[\overline{\daleth}_\alpha{}'(b), \underline{\daleth}_\alpha{}'(b)\right]$

3 Main Results : Fuzzy $p -$ Laplace Transform

Definition 10 [1]Suppose that, $\eth(\wp) \odot p^{-\varrho\wp}$ be a improper fuzzy valued Riemann integrable function on $[0, \infty)$ then

$$\int_0^\infty \eth(\wp) \odot p^{-\varrho\wp} d\wp$$

where $\eth$ be continuous fuzzy function, is known as Fuzzy $p -$ Laplace transform(Fp –LT) and is defined for function of exponential order of functions in the set $\mathcal{P}$ defined by

$$\mathcal{P} = \{\eth(\wp) | \exists A, \rho_1, \rho_2 > 0, \left|\eth(\wp) < Ae^{\frac{|\wp|}{\rho_j}}\right| \text{if } \wp \in (-1)^j \times [0, \infty)\} \quad (1)$$

as

$$\mathcal{L}_p\{\eth(\wp)\} = f_p(\varrho) = \int_0^\infty \eth(\wp) \odot p^{-\varrho\wp} d\wp \quad (2)$$

where, $\mathrm{Re}(\varrho) > 0, p \in (0,\infty)\backslash\{1\}$ and $\mathcal{L}_p$ represents the Fuzzy $\mathrm{p}-$ Laplace operator.

Remark 2 When $\mathrm{p} = \mathrm{e}$, the Fuzzy $\mathrm{p}-$ Laplace Transform coverges to famous Fuzzy Laplace transform in [1]

we can write (2) as

$$\int_0^\infty ð(\wp) \odot \mathrm{p}^{-\varrho\wp} d\wp = \left(\int_0^\infty \underline{ð}_\alpha(\wp)\ \mathrm{p}^{-\varrho\wp} d\wp, \int_0^\infty \overline{ð}_\alpha(\wp)\ \mathrm{p}^{-\varrho\wp} d\wp\right) \tag{3}$$

where,

$$\begin{aligned} \mathcal{L}_p\{\underline{ð}_\alpha(\wp)\} &= \int_0^\infty \underline{ð}_\alpha(\wp)\ \mathrm{p}^{-\varrho\wp} d\wp \\ \mathcal{L}_p\{\overline{ð}_\alpha(\wp)\} &= \int_0^\infty \overline{ð}_\alpha(\wp)\ \mathrm{p}^{-\varrho\wp} d\wp \end{aligned} \tag{4}$$

In above (4) denotes the $\mathrm{p}-$ Laplace Transform in [26]. So we write (4) in this manner

$$\begin{aligned} \mathcal{L}_p\{ð(\wp)\} &= \left(l_p\{\underline{ð}_\alpha(\wp)\}, l_p\{\overline{ð}_\alpha(\wp)\}\right) \\ &= \left(\underline{f}_p(\varrho), \overline{f}_p(\varrho)\right) \end{aligned} \tag{5}$$

Now we defined the inversion formula for the Fuzzy $\mathrm{p}-$ Laplace Transform as follow

$$ð(\wp) = \mathcal{L}_p^{-1}\{f_p(\varrho)\} = \frac{1}{2\pi i} \odot \int_{c-i\infty}^{c+i\infty} \mathrm{p}^{\varrho\wp} \odot f_p(\varrho) d\varrho \tag{6}$$

where, $\mathrm{Re}(\varrho) > 0, p \in (0,\infty)\backslash\{1\}$, $c > 0$ and $\mathcal{L}_p^{-1}$ denotes the inverse Fuzzy $\mathrm{p}-$ Laplace operator.

Theorem 3 [1] If $ð(\wp)$ be the bounded piecewise continuous fuzzy valued function on $[0,\infty)$ and of exponential order $\mho$ then Fuzzy $\mathrm{p}-$ Laplace Transform $f_p(\varrho) = \mathcal{L}_p\{ð(\wp)\}$ exists for $\mathrm{Re}(\varrho) > \mho, |p| - 1 > 0$ and converges absolutely.

Proof. We have any positive real number $\wp_0$ i.e. $0 \le \wp \le \wp_0$[19]

$$\begin{aligned} f_p(\varrho) = \mathcal{L}_p\{ð(\wp)\} &= \int_0^\infty ð(\wp) \odot \mathrm{p}^{-\varrho\wp} d\wp \\ &= \int_0^{\wp_0} ð(\wp) \odot \mathrm{p}^{-\varrho\wp} d\wp + \int_{\wp_0}^\infty ð(\wp) \odot \mathrm{p}^{-\varrho\wp} d\wp \end{aligned} \tag{7}$$

we know that, $ð(\wp)$ be continuous [4]Fuzzy valued function and $\mathrm{p}^{-\varrho\wp} \odot ð(\wp)$ is fuzzy Riemann integrable on $[0, \wp_0]$.

That is, in (7) First integral is exists i. e. $\int_0^{\wp_0} \mathrm{p}^{-\varrho\wp} \odot ð(\wp) d\wp$ and second integral $\mathrm{p}^{-\varrho\wp} \odot ð(\wp)$ is improper fuzzy Riemann integrable on $[\wp_0, \infty)$, then we want to show that $\int_{\wp_0}^\infty ð(\wp) \odot \mathrm{p}^{-\varrho\wp} d\wp$ also exists.

Since, $ð(\wp)$ is of exponential order $\mho$ for $\wp > \wp_0$

Consider

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$$
\begin{aligned}
\left|\int_{\wp_0}^{\infty} ð(\wp) \odot p^{-\varrho\wp} d\wp\right| &\leq \int_{\wp_0}^{\infty} |ð(\wp) \odot p^{-\varrho\wp}| d\wp \\
&= \int_{\wp_0}^{\infty} p^{-\varrho\wp} \odot |ð(\wp)| d\wp \\
&< \int_{\wp_0}^{\infty} p^{-\varrho\wp} \odot A e^{Ʊ\wp} d\wp \\
&= A \int_{\wp_0}^{\infty} e^{-(\varrho \ln p - Ʊ)\wp} d\wp \\
&= A \left[\frac{e^{-(\varrho \ln p - Ʊ)\wp}}{-(\varrho \ln p - Ʊ)}\right]_{\wp_0}^{\infty}, \quad \text{if} \quad \varrho \ln p > Ʊ \\
&= \frac{A e^{-(\varrho \ln p - Ʊ)\wp_0}}{(\varrho \ln p - Ʊ)}, \quad \text{if} \quad \varrho \ln p > Ʊ
\end{aligned}
$$

i.e. $\left|\int_{\wp_0}^{\infty} ð(\wp) \odot p^{-\varrho\wp} d\wp\right| \leq \frac{A e^{-(\varrho \ln p - Ʊ)\wp_0}}{(\varrho \ln p - Ʊ)}, \quad \text{if} \quad \varrho \ln p > Ʊ$

if we choose $\wp_0$ extremely large then $\frac{e^{-(\varrho \ln p - Ʊ)\wp_0}}{(\varrho \ln p - Ʊ)}$ converges to finite quantity.

Therefore, overall (7) becomes exists and $\int_0^{\infty} ð(\wp) \odot p^{-\varrho\wp} d\wp$ converges absolutely, complete the proof.

3.1 Elementary properties of (Fp −LT)

In this section, we discuss some the elementary properties (Fp −LT).

Property 1 If $\beta_1, \beta_2 \in \mathbb{C}$, $\mathcal{L}_p\{ð(\wp)\} = f_p(\varrho)$ and $\mathcal{L}_p\{G(\wp)\} = g_p(\varrho)$ then

$$\mathcal{L}_p\{\beta_1 \odot ð(\wp) \oplus \beta_2 \odot G(\wp)\} = \beta_1 \odot f_p(\varrho) \oplus \beta_2 \odot g_p(\varrho) \tag{8}$$

Proof. Consider

$$
\begin{aligned}
\mathcal{L}_p\{\beta_1 \odot ð(\wp) \oplus \beta_2 \odot G(\wp)\} &= \int_0^{\infty} \{\beta_1 \odot ð(\wp) \oplus \beta_2 \odot G(\wp)\} \odot p^{-\varrho\wp} d\wp \\
&= \beta_1 \odot \int_0^{\infty} ð(\wp) \odot p^{-\varrho\wp} d\wp \oplus \beta_2 \odot \int_0^{\infty} G(\wp) \odot p^{-\varrho\wp} d\wp \\
&= \beta_1 \odot f_p(\varrho) \oplus \beta_2 \odot g_p(\varrho)
\end{aligned}
$$

Corollary 1 If $\mathcal{L}_p\{ð_i(\wp)\} = f_{i_p}(\varrho)$ and $\beta_i \in \mathbb{C}$ then $\mathcal{L}_p\{\sum_{i=0}^{n} \beta_i \odot ð_i(\wp)\} = \sum_{i=0}^{n} \beta_i \odot \mathcal{L}_p\{ð_i(\wp)\}$

Property 2 If $\mathcal{L}_p\{ð(\wp)\} = f_p(\varrho)$ and for any non zero $a \in \mathbb{C}$ then

$$\mathcal{L}_p\{e^{a\wp} \odot ð(\wp)\} = f_p(\varrho \ln p - a) \tag{9}$$

Proof. By using equation (2)

$$
\begin{aligned}
\mathcal{L}_p\{e^{a\wp} \odot ð(\wp)\} &= \int_0^{\infty} e^{a\wp} \odot p^{-\varrho\wp} \odot ð(\wp) d\wp \\
&= \int_0^{\infty} ð(\wp) \odot e^{-(\varrho \ln p - a)\wp} d\wp \\
&= f_p(\varrho \ln p - a), \qquad k > 0, |p| - 1 > 0
\end{aligned}
$$

Remark 3 If $p = e$ then equation (9) converges to the first shifting property of the Fuzzy Laplace transform.

Property 3 For any $c > 0$, $\mathcal{L}_p\{ð(\wp)\} = f_p(\varrho)$ and $u(\wp) = \begin{cases} 1, & \wp \geq 0, \\ 0, & \wp < 0 \end{cases}$ then

$$\mathcal{L}_p\{\eth(\wp - c) \odot u(\wp - c)\} = p^{-\varrho c}\mathcal{L}_p\{\eth(\wp)\} \tag{10}$$

Proof. By using equation (2)

$$\mathcal{L}_p\{\eth(\wp - c) \odot u(\wp - c)\} = \int_0^\infty \{\eth(\wp - c) \odot u(\wp - c)\} \odot p^{-\varrho\wp}d\wp$$

put $\wp - c = \mu d\wp = d\mu$ we get,

$$= \int_{-\mu}^{\infty} \eth(\mu) \odot u(\mu) \odot p^{-\varrho(\mu+c)}d\mu$$
$$= p^{-\varrho c} \odot \int_{-\mu}^{\infty} \eth(\mu) \odot u(\mu) \odot p^{-\varrho\mu}d\mu$$
$$= p^{-\varrho c} \odot \left\{\int_{-\mu}^{0} \eth(\mu) \odot (0) \odot p^{-\varrho\mu}d\mu \oplus \int_0^\infty \eth(\mu) \odot (1) \odot p^{-\varrho\mu}d\mu\right\}$$
$$= p^{-\varrho c} \odot \int_0^\infty \eth(\mu) \odot p^{-\varrho\mu}d\mu$$

by changing variable $\mu \to \wp$ we get,

$$= p^{-\varrho c} \odot \mathcal{L}_p\{\eth(\wp)\}, \quad \frac{\ln p}{\varrho} > 0$$

Remark 4 If $p = e$ then equation (10) converges to second shifting property of Fuzzy Laplace transform

Property 4 If $\beta \in \mathbb{C}$ and $\mathcal{L}_p\{\eth(\wp)\} = f_p(\varrho)$ then $\mathcal{L}_p\{\eth(\beta\wp)\} = \frac{1}{\beta} \odot f_p\left(\frac{\varrho}{\beta}\right)$ (11)

Proof. By using equation (2)

$$\mathcal{L}_p\{\eth(\beta\wp)\} = \int_0^\infty \{\eth(\beta\wp)\} \odot p^{-\varrho\wp}d\wp$$

put $\beta\wp = \mu d\wp = \frac{d\mu}{\beta}$ we get,

$$= \frac{1}{\beta} \odot \int_0^\infty \{\eth(\mu)\} \odot p^{-\varrho\frac{\mu}{\beta}}d\mu$$

by changing variable $\mu \to \wp$ we get,

$$= \frac{1}{\beta} \odot f_p\left(\frac{\varrho}{\beta}\right)$$

Remark 5 If $p = e$ then equation (11) converges to Change of scale property of Fuzzy Laplace transform

Theorem 4 [19]Let $\eth^{\jmath}(\wp)$ be an integrable fuzzy valued function and $\eth(\wp)$ is primitive of $\eth^{\jmath}(\wp)$ on $[0, \infty)$ and then,

$$\mathcal{L}_p\{\eth^{\jmath}(\wp)\} = (\varrho \ln p)^{\jmath} \odot \mathcal{L}_p\{\eth(\wp)\} \ominus \sum_{l=0}^{\jmath-1} (\varrho \ln p)^{\jmath-l-1} \odot \eth^{l}(0) \tag{12}$$

Proof. For $\alpha \in [0,1]$ be arbitrary, then

$$(\varrho \ln p)^{\jmath} \odot \mathcal{L}_p\{\eth(\wp)\} \ominus \sum_{l=0}^{\jmath-1} (\varrho \ln p)^{\jmath-l-1} \odot F^{l}(0) = \Big((\varrho \ln p)^{\jmath}\ l_p\{\underline{\eth}_\alpha(\wp)\} - \sum_{l=0}^{\jmath-1} (\varrho \ln p)^{\jmath-l-1}\ \underline{\eth}^{l}(0),$$
$$(\varrho \ln p)^{\jmath}\ l_p\{\overline{\eth}_\alpha(\wp)\} - \sum_{l=0}^{\jmath-1} (\varrho \ln p)^{\jmath-l-1}\ \overline{\eth}^{l}(0)\Big)$$
$$= \left(l_p\{\underline{\eth}^{\jmath}(\wp)\}, l_p\left\{\overline{\eth}^{\jmath}(\wp)\right\}\right)$$

where,

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$$l_p\{\underline{ð}^{ȷ}(\wp)\} = \int_0^{\infty} \underline{ð}^{ȷ}(\wp) \odot p^{-ϱ\wp} d\wp$$

$$l_p\left\{\overline{ð}^{ȷ}(\wp)\right\} = \int_0^{\infty} \overline{ð}^{ȷ}(\wp) \odot p^{-ϱ\wp} d\wp$$

We use the Mathematical induction to prove (12)

for $ȷ = 1$, equation (12) hold , we get

$$\mathcal{L}_p\{ð'(\wp)\} = (ϱ \ln p) \odot \mathcal{L}_p\{ð(\wp)\} \ominus ð(0) \qquad (13)$$

Now assume that equation (12) hold for $ȷ = a$, we get

$$\mathcal{L}_p\{ð^a(\wp)\} = (ϱ \ln p)^a \odot \mathcal{L}_p\{ð(\wp)\} \ominus \sum_{l=0}^{a-1} (ϱ \ln p)^{a-l-1} \odot F^l(0) \qquad (14)$$

we want to show that equation (12) is hold for $ȷ = a + 1$

by using (13) and (14), we get[Fine Rosenberger, 2007]

$$\begin{aligned}\mathcal{L}_p\{(ð^a(\wp))'\} &= (ϱ \ln p) \odot \mathcal{L}_p\{ð^{(a)}(\wp)\} \ominus ð^{(a)}(0) \\ &= (ϱ \ln p) \odot \{(ϱ \ln p)^a \odot \mathcal{L}_p\{ð(\wp)\} \ominus \\ &\quad \sum_{l=0}^{a-1} (ϱ \ln p)^{a-l-1} \odot ð^l(0)\} \ominus ð^{(a)}(0) \\ &= (ϱ \ln p)^{a+1} \odot \mathcal{L}_p\{ð(\wp)\} \ominus \sum_{l=0}^{a} (ϱ \ln p)^{a-l} \odot ð^l(0)\end{aligned}$$

Thus equation (12) is hold for $ȷ = a + 1$ complete the proof.

Property 5 Fuzzy $p -$ Laplace Transform of Integral

If $\mathcal{L}_p\{ð(\wp)\} = f_p(ϱ)$ then $\mathcal{L}_p\left\{\int_0^{\wp} ð(u)du\right\} = \frac{1}{ϱ \ln p} f_p(ϱ)$

Property 6 Multiplication by powers of $\wp$

If $\mathcal{L}_p\{ð(\wp)\} = f_p(ϱ)$ then

$$\mathcal{L}_p\{\wp^n \odot ð(\wp)\} = (-1)^n \odot \frac{d^n}{dϱ^n} f_p(ϱ), \quad \text{for } n = 1,2,3,\dots \qquad (15)$$

Property 7 Division by $\wp$

If $\mathcal{L}_p\{ð(\wp)\} = f_p(ϱ)$ then

$$\mathcal{L}_p\left\{\frac{1}{\wp^n} \odot ð(\wp)\right\} = \underbrace{\int_ϱ^{\infty} \int_ϱ^{\infty} \dots \int_ϱ^{\infty}}_{n-\text{times}} f_p(ϱ) dϱ^n \qquad (16)$$

Property 8 [5, 26]**Convolution**

Let $(\aleph_1 * \aleph_2)$ denotes the convolution of $\aleph_1(\wp)$ and $\aleph_2(\wp)$ and is defined as

$$(\aleph_1 * \aleph_2)(\wp) = \int_0^{\wp} \aleph_1(\wp - u)\aleph_2(u)du \quad (17)$$

where, $\aleph_1(\wp)$ and $\aleph_2(\wp)$ are continuous fuzzy valued and of exponential[Ahmed Farooq, 2014] ordered functions.

Note that, $(\aleph_1 * \aleph_2)$ is commutative. That is we can write

$$\int_0^{\wp} \aleph_1(\wp - u)\aleph_2(u)du = \int_0^{\wp} \aleph_1(\wp)\aleph_2(\wp - u)du \quad (18)$$

Theorem 5 If $\mathcal{L}_p\{\eth(\wp)\} = f_p(\varrho)$ and $\mathcal{L}_p\{G(\wp)\} = g_p(\varrho)$ then

$$\mathcal{L}_p\{(\eth * G)(\wp)\} = f_p(\varrho)\ g_p(\varrho) \quad (19)$$

Proof. Using the (2) and the use of (17), (18) we can write,

$$\mathcal{L}_p\{(\eth * G)(\wp)\} = \int_0^{\infty} p^{-\varrho\wp} \odot \left\{\int_0^{\wp} \eth(u) \odot G(\wp - u)du\right\} d\wp$$

rearranging the order and limit of integration we have,

$$\mathcal{L}_p\left\{\int_0^{\wp} \eth(u) \odot G(\wp - u)du\right\} = \int_0^{\infty} \left(\eth(u) \odot \int_u^{\wp} p^{-\varrho\wp} \odot G(\wp - u)d\wp\right) du \quad (20)$$

substitute $\wp - u = z \quad d\wp = dz$

$$\int_u^{\wp} p^{-\varrho\wp} \odot G(\wp - u)d\wp = \int_0^{\infty} p^{-\varrho(u+z)} \odot G(z)dz$$

$$= p^{-\varrho u} \odot \int_0^{\infty} p^{-\varrho z} \odot G(z)dz$$

$$= p^{-\varrho u} \odot g_p(\varrho) \quad (21)$$

Now using (21) in (20) we get ,

$$\mathcal{L}_p\{(\eth * G)(\wp)\} = f_p(\varrho)\ g_p(\varrho)$$

4 Applications

4.1 Method to Solve the initial Value Problem using (Fp −LT)

Consider the initial Value Problem in fuzzy environment

$$\begin{aligned} \curlywedge'(\wp) &= \eth(\wp, \curlywedge(\wp)) \\ \curlywedge(0) &= (\underline{\curlywedge}_\alpha(0), \overline{\curlywedge}_\alpha(0)), \quad 0 \le \alpha \le 1 \end{aligned} \quad (22)$$

where, $F: [\wp_0, T] \times \Re_\intercal \rightarrow \Re_\intercal$ is a continuous fuzzy valued function. According to Kaleva[16] Theorem 2 provides the systematic process to solve the equation (22).

Now, we Apply Fuzzy p − Laplace transform on equation (22), we get

$$\mathcal{L}_p\{\curlywedge'(\wp)\} = \mathcal{L}_p\{\eth(\wp, \curlywedge(\wp))\} \quad (23)$$

1. If $\curlywedge(\wp)$ is (1) − differentiable then by above Theorem 2 we have

$[\curlywedge'(\wp)]_\alpha = \left[\underline{\curlywedge}_\alpha'(\wp), \overline{\curlywedge}_\alpha'(\wp)\right]$ and

$$\mathcal{L}_p\{\curlywedge'(\wp)\} = (\varrho \ \ln \ p) \odot \mathcal{L}_p\{\curlywedge(\wp)\} \ominus \curlywedge(0) \tag{24}$$

using (24), equation (23) becomes

$$\begin{aligned} l_p\{\underline{\eth}_\alpha(\wp,\curlywedge(\wp))\} &= (\varrho \ \ln \ p) l_p\{\underline{\curlywedge}_\alpha(\wp)\} - \underline{\curlywedge}_\alpha(0) \\ l_p\{\overline{\eth}_\alpha(\wp,\curlywedge(\wp))\} &= (\varrho \ \ln \ p) l_p\{\overline{\curlywedge}_\alpha(\wp)\} - \overline{\curlywedge}_\alpha(0) \end{aligned} \tag{25}$$

where

$$\underline{\eth}_\alpha(\wp,\curlywedge(\wp)) = \mathrm{Min}\{\eth(\wp,\mu)/\mu \in (\underline{\curlywedge}_\alpha(\wp), \overline{\curlywedge}_\alpha(\wp))\}$$

$$\overline{\eth}_\alpha(\wp,\curlywedge(\wp)) = \mathrm{Max}\{\eth(\wp,\mu)/\mu \in (\overline{\curlywedge}_\alpha(\wp), \overline{\curlywedge}_\alpha(\wp))\}$$

To solve the system (25) which is equivalent to (23), first we assume that

$$\begin{aligned} l_p\{\underline{\curlywedge}_\alpha(\wp)\} &= A^1(\varrho) \\ l_p\{\overline{\curlywedge}_\alpha(\wp)\} &= B^1(\varrho) \end{aligned} \tag{26}$$

where, $A^1(\varrho)$ and $B^1(\varrho)$ are the solutions of the system (25). By taking the inverse (Fp −LT), we get

$$\begin{aligned} \underline{\curlywedge}_\alpha(\wp) &= l_p^{-1}\{A_\alpha{}^1(\varrho)\} \\ \overline{\curlywedge}_\alpha(\wp) &= l_p^{-1}\{B_\alpha{}^1(\varrho)\} \end{aligned} \tag{27}$$

1. If $\curlywedge(\wp)$ is $(2)-$ differentiable then by above Theorem 2 we have

$[\curlywedge'(\wp)]_\alpha = \left[\overline{\curlywedge}_\alpha'(\wp), \underline{\curlywedge}_\alpha'(\wp)\right]$ and

$$\mathcal{L}_p\{\curlywedge'(\wp)\} = -(-\curlywedge(0)) \ominus \left(-(\varrho \ \ln \ p) \odot \mathcal{L}_p\{\curlywedge(\wp)\}\right) \tag{28}$$

using (28), equation (23) becomes

$$\begin{aligned} l_p\{\underline{\eth}_\alpha(\wp,\curlywedge(\wp))\} &= (\varrho \ \ln \ p) l_p\{\underline{\curlywedge}_\alpha(\wp)\} - \underline{\curlywedge}_\alpha(0) \\ l_p\{\overline{\eth}_\alpha(\wp,\curlywedge(\wp))\} &= (\varrho \ \ln \ p) l_p\{\overline{\curlywedge}_\alpha(\wp)\} - \overline{\curlywedge}_\alpha(0) \end{aligned} \tag{29}$$

where

$$\underline{\eth}_\alpha(\wp,\curlywedge(\wp)) = \mathrm{Min}\{\eth(\wp,\mu)/\mu \in (\underline{\curlywedge}_\alpha(\wp), \overline{\curlywedge}_\alpha(\wp))\}$$

$$\overline{\eth}_\alpha(\wp,\curlywedge(\wp)) = \mathrm{Max}\{\eth(\wp,\mu)/\mu \in (\overline{\curlywedge}_\alpha(\wp), \overline{\curlywedge}_\alpha(\wp))\}$$

To solve the system (29) which is equivalent to (23), first we assume that

$$\begin{aligned} l_p\{\underline{\curlywedge}_\alpha(\wp)\} &= A_\alpha{}^2(\varrho) \\ l_p\{\overline{\curlywedge}_\alpha(\wp)\} &= B_\alpha{}^2(\varrho) \end{aligned} \tag{30}$$

where, $A^2(\varrho)$ and $B^2(\varrho)$ are the solutions of the system (29). By taking the inverse (Fp −LT), we get

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$$\begin{aligned} \underline{\lambda}_\alpha(\wp) &= l_p^{-1}\{A_\alpha{}^2(\varrho)\} \\ \overline{\lambda}_\alpha(\wp) &= l_p^{-1}\{B_\alpha{}^2(\varrho)\} \end{aligned} \tag{31}$$

Simillarly, we can follow the same process for 2^{nd}, 3^{rd},..., m^{th} order Fuzzy Differential equations.

4.2 NUMERICAL EXAMPLE

Example 1 Consider the IVP in fuzzy environment

$$\begin{aligned} \lambda'(\wp) &= -\lambda(\wp) - b, \quad 0 \le \wp \le T, \\ \lambda(0) &= (\alpha - 2, 2 - \alpha), \\ \text{where,} \quad & b = (\alpha - 2, 2 - \alpha) = (\underline{b}_\alpha, \overline{b}_\alpha), 0 \le \alpha \le 1 \end{aligned} \tag{32}$$

Consider Case II of strongly generalised H-differentiability, as defined in Theorem 2. Then, using the fuzzy $p-$ Laplace Transform, we obtain

$$\mathcal{L}_p\{\lambda'(\wp)\} = \mathcal{L}_p\{-\lambda(\wp) - b\} \tag{33}$$

we know that,

$$\mathcal{L}_p\{\lambda'(\wp)\} = -(\lambda(0)) \ominus \left(-(\varrho \ln p) \odot \mathcal{L}_p\{\lambda(\wp)\}\right) \tag{34}$$

we can rewrite equation (33) as follow and for any specific $\in [0,1]$, we obtain the following $-$level representation in system form

$$\begin{aligned} -\left(\underline{\lambda}_\alpha(0)\right) - \left(-(\varrho \ln p) l_p\{\underline{\lambda}_\alpha(\wp)\}\right) &= -l_p\{\underline{\lambda}_\alpha(\wp)\} - \underline{b}_\alpha l_p\{1\} \\ -\left(\overline{\lambda}_\alpha(0)\right) - \left(-(\varrho \ln p) l_p\{\overline{\lambda}_\alpha(\wp)\}\right) &= -l_p\{\overline{\lambda}_\alpha(\wp)\} - \overline{b}_\alpha l_p\{1\} \end{aligned} \tag{35}$$

$$\begin{aligned} l_p\{\underline{\lambda}_\alpha(\wp)\} &= \frac{1}{(\varrho \ln p+1)} \underline{\lambda}_\alpha(0) - \underline{b}_\alpha \frac{1}{(\varrho \ln p)(\varrho \ln p+1)} \\ l_p\{\overline{\lambda}_\alpha(\wp)\} &= \frac{1}{(\varrho \ln p+1)} \overline{\lambda}_\alpha(0) - \overline{b}_\alpha \frac{1}{(\varrho \ln p)(\varrho \ln p+1)} \end{aligned} \tag{36}$$

After simplification of (36)

$$\begin{aligned} \underline{\lambda}_\alpha(\wp) &= l_p^{-1}\left\{\frac{1}{(\varrho \ln p+1)}\right\} \underline{\lambda}_\alpha(0) - \underline{b}_\alpha l_p^{-1}\left\{\frac{1}{(\varrho \ln p)(\varrho \ln p+1)}\right\} \\ \overline{\lambda}_\alpha(\wp) &= l_p^{-1}\left\{\frac{1}{(\varrho \ln p+1)}\right\} \overline{\lambda}_\alpha(0) - \overline{b}_\alpha l_p^{-1}\left\{\frac{1}{(\varrho \ln p)(\varrho \ln p+1)}\right\} \end{aligned} \tag{37}$$

Finally, we obtain the required solution of (32) as follow

$$\begin{aligned} \underline{\lambda}_\alpha(\wp) &= (\alpha - 2)[2e^{-\wp} - 1] \\ \overline{\lambda}_\alpha(\wp) &= (2 - \alpha)[2e^{-\wp} - 1] \end{aligned} \tag{38}$$

The Solutions obtained using Fuzzy $p-$ Laplace Transform(F$p-$LT) for the case II proposed in this section are shown in Figure 1-2. We can see that for solution converges as $\wp$ increases. Figure 1 shows the $-$level representation of $\underline{\lambda}_\alpha(\wp)$ and $\overline{\lambda}_\alpha(\wp)$ where, $0 \le \alpha \le 1$

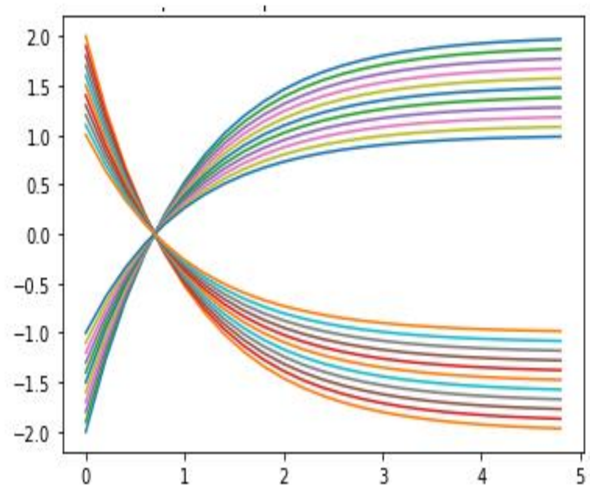

Figure 1: α - cut representation of solution of Example 1. for different

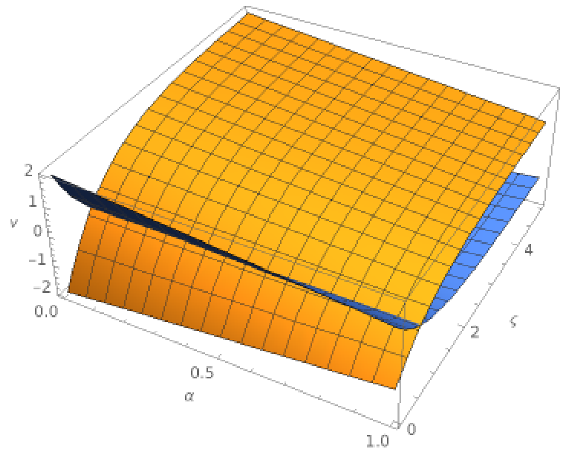

Figure 2: 3D Graphical representation of Equation (38).

4.3 Method to find the solution Volterra integral equation

Consider the Volterra integral equation of the second kind[19, 22] in fuzzy environment

$$u(\wp) = \eth(\wp) + \int_0^{\wp} (z - \wp) \odot u(z)dz, \wp \in [0, T], T < \infty \tag{39}$$

where, $(z - \wp)$ is convolution kernel function in fuzzy environment and $\eth$ is a continuous fuzzy valued function. i.e. $\eth(\wp) \in \Re_T$.

Apply Fuzzy $p -$ Laplace Transform on (1), we obtain

$$\mathcal{L}_p\{u(\wp)\} = \mathcal{L}_p\{\eth(\wp)\} + \mathcal{L}_p\left\{\int_0^{\wp} (z - \wp) \odot u(z)dz\right\} \tag{40}$$

where, $\wp \in [0, T], T < \infty$

Now using (19) and (2), we get for any fixed $\alpha \in [0,1]$,

$$\begin{aligned} l_p\{\underline{u}_\alpha(\wp)\} &= l_p\{\underline{\eth}_\alpha(\wp)\} + \underline{l_p\{\ _\alpha(\wp)\}l_p\{u_\alpha(\wp)\}},\ 0 \le \alpha \le 1 \\ l_p\{\overline{u}_\alpha(\wp)\} &= l_p\{\overline{\eth}_\alpha(\wp)\} + \overline{l_p\{\ _\alpha(\wp)\}l_p\{u_\alpha(\wp)\}},\ 0 \le \alpha \le 1 \end{aligned} \tag{41}$$

from (41), we have the following cases

1. if $\eta_\alpha(\wp)$ and $u_\alpha(\wp)$ are positive, then we get

$$\underline{l_p\{\eta_\alpha(\wp)\}l_p\{u_\alpha(\wp)\}} = l_p\left\{\underline{\eta_\alpha(\wp)}\right\}l_p\left\{\underline{u_\alpha(\wp)}\right\}$$

$$\overline{l_p\{\eta_\alpha(\wp)\}l_p\{u_\alpha(\wp)\}} = l_p\{\overline{\eta_\alpha(\wp)}\}l_p\{\overline{u_\alpha(\wp)}\}$$

or 2. if $\eta_\alpha(\wp)$ is positive and $u_\alpha(\wp)$ is negative, then we get

$$\underline{l_p\{\eta_\alpha(\wp)\}l_p\{u_\alpha(\wp)\}} = l_p\{\overline{\eta_\alpha(\wp)}\}l_p\left\{\underline{u_\alpha(\wp)}\right\}$$

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$$\overline{l_p\{\eta_\alpha(\wp)\}l_p\{u_\alpha(\wp)\}} = l_p\left\{\underline{\eta_\alpha(\wp)}\right\}l_p\left\{\overline{u_\alpha(\wp)}\right\}$$

or 3. if $\eta_\alpha(\wp)$ is negative and $u_\alpha(\wp)$ is positive, then we get

$$\underline{l_p\{\eta_\alpha(\wp)\}l_p\{u_\alpha(\wp)\}} = l_p\left\{\underline{\eta_\alpha(\wp)}\right\}l_p\left\{\overline{u_\alpha(\wp)}\right\}$$

$$\overline{l_p\{\eta_\alpha(\wp)\}l_p\{u_\alpha(\wp)\}} = l_p\left\{\overline{\eta_\alpha(\wp)}\right\}l_p\left\{\underline{u_\alpha(\wp)}\right\}$$

or 4. if $\eta_\alpha(\wp)$ and $u_\alpha(\wp)$ are negative, then we get

$$\underline{l_p\{\eta_\alpha(\wp)\}l_p\{u_\alpha(\wp)\}} = l_p\left\{\overline{\eta_\alpha(\wp)}\right\}l_p\left\{\overline{u_\alpha(\wp)}\right\}$$

$$\overline{l_p\{\eta_\alpha(\wp)\}l_p\{u_\alpha(\wp)\}} = l_p\left\{\underline{\eta_\alpha(\wp)}\right\}l_p\left\{\underline{u_\alpha(\wp)}\right\}$$

where,

$$\underline{\eta_\alpha(\wp)} = \underline{\eta}_\alpha(\wp) \quad \text{and} \quad \overline{\eta_\alpha(\wp)} = \overline{\eta}_\alpha(\wp)$$

$$\underline{u_\alpha(\wp)} = \underline{u}_\alpha(\wp) \quad \text{and} \quad \overline{u_\alpha(\wp)} = \overline{u}_\alpha(\wp)$$

Note that, zero does not exist in support.

Rewriting system (3), using any one of above cases, we get a simple form

$$\begin{aligned} l_p\{\underline{u}_\alpha(\wp)\} &= \frac{l_p\{\underline{\delta}_\alpha(\wp)\}}{1 - l_p\{\underline{\eta}_\alpha(\wp)\}} \\ l_p\{\overline{u}_\alpha(\wp)\} &= \frac{l_p\{\overline{\delta}_\alpha(\wp)\}}{1 - l_p\{\overline{\eta}_\alpha(\wp)\}} \end{aligned} \tag{42}$$

Finally, we get the solution by applying inverse of Fuzzy $p -$ Laplace Transform

$$\begin{aligned} \underline{u}_\alpha(\wp) &= l_p^{-1}\left\{\frac{l_p\{\underline{\delta}_\alpha(\wp)\}}{1 - l_p\{\underline{\eta}_\alpha(\wp)\}}\right\} \\ \overline{u}_\alpha(\wp) &= l_p^{-1}\left\{\frac{l_p\{\overline{\delta}_\alpha(\wp)\}}{1 - l_p\{\overline{\eta}_\alpha(\wp)\}}\right\} \end{aligned} \tag{43}$$

4.4 NUMERICAL EXAMPLE

Consider the second kind Volterra integral equation in fuzzy environment

$$u(\wp) = (\alpha + 2, 2 - \alpha) \odot e^{-3\wp} + \int_0^{\wp} \sin(\wp - z) \odot u(z)dz, \wp \in [a_0, b_0] \tag{44}$$

Applying Fuzzy $p -$ Laplace Transform on (6) we get

$$\mathcal{L}_p\{u(\wp)\} = \mathcal{L}_p\{(\alpha + 2, 2 - \alpha) \odot e^{-3\wp}\} + \mathcal{L}_p\left\{\int_0^{\wp} \sin(\wp - z) \odot u(z)dz\right\} \tag{45}$$

Using Convolution Theorem 5, equation (7) reduces to

$$\mathcal{L}_p\{u(\wp)\} = \mathcal{L}_p\{(\alpha + 2, 2 - \alpha) \odot e^{-3\wp}\} + \mathcal{L}_p\{\sin(\wp)\} \odot \mathcal{L}_p\{u(\wp)\} \tag{46}$$

for any fixed $\in [0,1]$, which is equivalently

$$\begin{aligned} l_p\{\underline{u}(\wp)\} &= l_p\{(\alpha+2)\cdot e^{-3\wp}\} + l_p\{\sin(\wp)\}\cdot l_p\{\underline{u}(\wp)\} \\ l_p\{\overline{u}(\wp)\} &= l_p\{(2-\alpha)\cdot e^{-3\wp}\} + l_p\{\sin(\wp)\}\cdot l_p\{\overline{u}(\wp)\} \end{aligned} \tag{47}$$

After some simplifications we get

$$\begin{aligned} l_p\{\underline{u}(\wp)\} &= (\alpha+2)[-\tfrac{1}{9}(\varrho\ \ln\ p)^{-1} + \tfrac{1}{3}(\varrho\ \ln\ p)^{-2} + \tfrac{10}{9}(\varrho\ \ln\ p+3)^{-1}] \\ l_p\{\overline{u}(\wp)\} &= (2-\alpha)[-\tfrac{1}{9}(\varrho\ \ln\ p)^{-1} + \tfrac{1}{3}(\varrho\ \ln\ p)^{-2} + \tfrac{10}{9}(\varrho\ \ln\ p+3)^{-1}] \end{aligned} \tag{48}$$

By taking inverse (Fp −LT) we get, the required solution.

$$\begin{aligned} \underline{u}(\wp) &= (\alpha+2)\left(-\tfrac{1}{9}+\tfrac{1}{3}\wp+\tfrac{10}{9}e^{-3\wp}\right) \\ \overline{u}(\wp) &= (2-\alpha)\left(-\tfrac{1}{9}+\tfrac{1}{3}\wp+\tfrac{10}{9}e^{-3\wp}\right) \end{aligned} \tag{49}$$

The Solutions obtained using method Fuzzy $p-$ Laplace Transform (Fp −LT) by the proposed process in this paper are shown in Figure 3-4. Figure 3 shows the −level representation of $\underline{u}_\alpha(\wp)$ and $\overline{u}_\alpha(\wp)$ where $0 \le \alpha \le 1$.

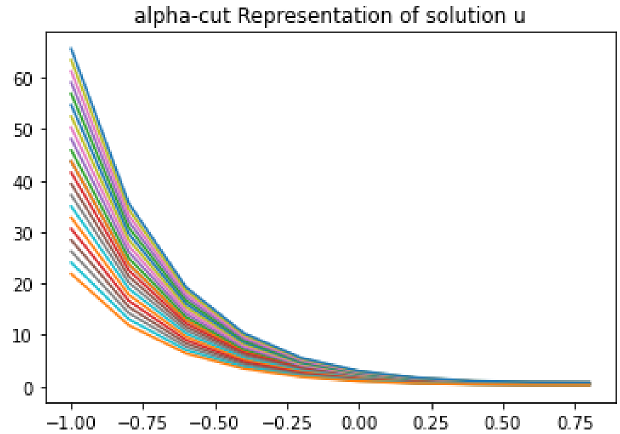

Figure 1: α - cut representation of solution of Example 2. for different

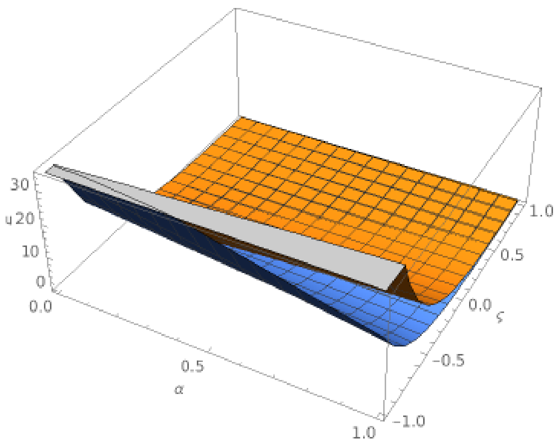

Figure 2: 3D Graphical representation of Equation (49).

5 CONCLUSION

The Fuzzy $p-$ Laplace Transform is proposed and applied to FIVP and Fuzzy convolution volterra integral equation of second kind. The solutions obtained by this transform will be more accurate and precised, and verifying by graphically. In this paper, our main goal is to demostrate a (Fp −LT) is more effective and working nicely on $p \in (0,\infty)\backslash\{1\}$.

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Conflict of interest
The authors declare no potential conflict of interest.

REFERENCE

[1] Allahviranloo, T., Ahmadi, M. B. Fuzzy Laplace transforms. *Soft Computing, 14(3)*, 2010, 235–243. https://doi.org/10.1007/s00500-008-0397-6

[2] T. Allahviranloo, S. Abbasbandy, M. R. Balooch Shahryari, S. Salahshour, D. Baleanu, On Solutions of Linear Fractional Differential Equations with Uncertainty, *Abstract and Applied Analysis, vol. 2013*, Article ID 178378, 2013, 13 pages. https://doi.org/10.1155/2013/178378

[3] Ahmad, L., Farooq, M., Ullah, S., & Abdullah, S. Solving fuzzy two-point boundary value problem using fuzzy Laplace transform. 2014, arXiv preprint arXiv:1403.0571.

[4] Bushnaq, S., Ullah, Z., Ullah, A., & Shah, K. Solution of fuzzy singular integral equation with Abel's type kernel using a novel hybrid method. *Advances in Difference Equations, 2020(1)*, 2020, 1-13.

[5] Castillo, O., Jana, D. K., Giri, D., & Ahmed, A. (Eds.). *Recent Advances in Intelligent Information Systems and Applied Mathematics*. 2020

[6] Chang, S. S., & Zadeh, L. A. On fuzzy mapping and control. *In Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh*, 1996, (pp. 180-184).

[7] Citil, H. G. An approach to a fuzzy problem using the fuzzy Laplace transform under the generalized differentiability. *Malaya Journal of Matematik (MJM), 8(1)*, 2020, 82-88.

[8] De Baets, M. Z. A. B.A predator-prey model with fuzzy initial populations,IFSA-EUSFLAT conference, 2009,1310-1314

[9] Dubey, B., & George, R. K. A note on the evolution of solutions of a system of ordinary differential equations with fuzzy initial conditions and fuzzy-inputs. *Journal of Uncertain Systems, 7(4)*,2013, 294-302.

[10] Dutta, H., Hammouch, Z., Bulut, H., & Baskonus, H. M. 4th International Conference on Computational Mathematics and Engineering Sciences (CMES-2019). In Conference proceedings CMES (p. 31).

[11] Dubois, D. J. *Fuzzy sets and systems: theory and applications* (Vol. 144), Academic press, Waltham, MA, USA, 1980.

[12] Dubois, D., & Prade, H., Towards fuzzy differential calculus. *Fuzzy sets and systems, (8) :1-7*, 1982, 105-116, 225-233.

[13] Ezzati, R., & Ziari, S. *Numerical solution and error estimation of fuzzy Fredholm integral equation using fuzzy Bernstein polynomials*. *Australian Journal of Basic and Applied Sciences, 5(9)*,2011, 2072-2082.

[14] Fine, B., & Rosenberger, G. *Number theory*. Birkhũser Boston, 2007

[15] Friedman, M., Ma, M., & Kandel, A. *Numerical solutions of fuzzy differential and integral equations*. *Fuzzy sets and Systems, 106(1)*, 1999, 35-48.

[16] Kaleva, O. A note on fuzzy differential equations. *Nonlinear Analysis: Theory, Methods and Applications, 64(5)*, 2006, 895-900.

[17] Khastan, A., Bahrami, F., Ivaz, K. New results on multiple solutions for th-order fuzzy differential equations under generalized differentiability. *Boundary value problems,* 2009,1-13.

[18] Ma, M., Friedman, M., Kandel, A.Numerical solutions of fuzzy differential equations, *Fuzzy sets and systems*, *105(1)*, 1999, 133-138.

[19] Maitama, S., & Zhao, W. Homotopy analysis Shehu transform method for solving fuzzy differential equations of fractional and integer order derivatives. *Computational and Applied Mathematics*, *40(3)*, 2021, 1-30.

[20] Rahman, Norazrizal A.A., and Muhammad Z. Ahmad, Applications of the Fuzzy Sumudu Transform for the Solution of First Order Fuzzy Differential Equations *Entropy 17, no. 7,* 2015, 4582-4601. https://doi.org/10.3390/e17074582

[21] Rabiei, F., Abd Hamid, F., Rashidi, M. M., & Ismail, F. Numerical simulation of fuzzy differential equations using general linear method and B-series. *Advances in Mechanical Engineering*, *9(9)*, 2017.

[22] Salahshour, S., Khezerloo, M., Hajighasemi, S., Khorasany, M. Solving fuzzy integral equations of the second kind by fuzzy Laplace transform method,*Int.J. Industrial Mathematics, 4 (1)*,2012, 21-29.

[23] Salahshour, S., Allahviranloo, T. Applications of fuzzy Laplace transforms. *Soft computing*, *17(1)*,2013, 145-158.

[24] Salahshour, S., Allahviranloo, T., & Abbasbandy, S. Solving fuzzy fractional differential equations by fuzzy Laplace transforms. *Communications in Nonlinear Science and Numerical Simulation*, *17(3)*,2012, 1372-1381.

[25] Salahshour, S., Ahmadian, A., Senu, N., Baleanu, D., & Agarwal, P. On analytical solutions of the fractional differential equation with uncertainty: application to the Basset problem. *Entropy*, *17(2)*,2015, 885-902.

[26] Saif, M., Khan, F., Nisar, K. S., Araci, S.Modified Laplace transform and its properties,*J. Math. Computer Sci*,*21*,2020, 127-135

[27] Valenčáková, V. A note on the conditional probability of IF-events, *Mathematica Slovaca*, *59(2)*,2009, 251-260.

[28] Wu, H. C. The improper fuzzy Riemann integral and its numerical integration. *Information Sciences*, *111(1-4)*,1998, 109-137.

[29] Wu, H. C. The fuzzy Riemann integral and its numerical integration. *Fuzzy Sets and Systems*, *110(1)*,2000, 1-25.

[30] Xu, J., Liao, Z., and Hu, Z. A class of linear differential dynamical systems with fuzzy initial condition. *Fuzzy sets and systems*, *158(21)*,2007, 2339-2358.