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Geodesics, Metrics, Distance functions on M and T(M) forms a Topological Manifold

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Abstract

The concept of geodesics on Topological Manifold in order to metric space or vector space. We are discussed the geodesics on Manifold M and Tangent bundle T(M). These M and T(M) are Topological Manifold as well as Riemannian Manifolds. The convergence property of sequence of Tangents Vector at a point with each and every point of M. The distance functions are studied in Tangent vector field (Tangent bundle) with the help of Lipchitz condition and distance functions. We redefined geodesics on M and Norms (geodesics) on T(M). Maximal geodesic on T(M) are studied with help of co-ordinate transformation. Also we studied geodesic space with help of expontial function which is Tangent bundles.

Keywords: *Geodesics, Norm, Tangent bundle, Continuous linear maps, Semi-Riemannian Manifold, Lipchitz functions Lipchitz conditions.*

1. INTRODUCTION

In recent decades, the study of variation in metric, i.e. geodesic on M and T(M) has a part of development in Topological Manifold. According to William. M. Boothby [15] T(M) is a Topological Manifold of dimension 2n as of M is of dimension n. Here T(M) is Tangent bundles on M.

In Differential Geometry Singurdur Helgacon [13] defined a total geodesic sub-manifolds. Jeffrey M.lee [9] studied Uniqueness properties of geodesics joining between any two points in space, also defined the completeness properties of geodesics on M . We extended these concepts on Topological Manifold T(M) . By using Cauchy sequence in metric space we defined on T(M).

We study the geodesics on M and T (M) both are Topological Manifolds. We extended these concepts on Riemannian Manifold, Lorentz Manifolds as case study.

In section 2 we recall some basic definitions and results and redefined required.

In section 3 we discussed geodesic as continues cover with piece-wise smoothness called broken geodesics segment also we showed that by using geodesic concept M and T(M)complete . In section 4, we study semi Riemannian Manifold i.e. Riemannian and Lorentz manifold .these concept is discussed by [Jeffrey lee] [9] on Topological Manifold M, but we extended these concept on Tangent bundles. We redefined some concept with your need .In theorem 4.9 we proved the relation between totally geodesic on M and parallel translation in T(M) with the help Tangents to M.

We used some Notation for this paper which we used. We listed as –

Notations

TM = T(M) → Tangent bundle .

Tp(M) →Tangent space at P , P ∈ M

V = V_p →vector at P

$D_h\gamma^{\cdot}$ – Component of $\gamma^{\cdot\cdot}(t)$ in $T_p(M)$

$dist(x_{p_i},\ x_{p_j})$ – distance between x_{p_i} and x_{p_j} vectors.

$\widetilde{D}_p$ – The set of all vectors $V \in\ T_p(M)$

Exp_M – Exponential mapping for M

$Exp_{T(M)}$ – Exponential mapping for T(M)

$L(\gamma)$ – Length of Curve γ
$C(p)$ – (Time Cone at p, $p \in M) \in T_p(M)$
$\mathfrak{X}(M)$ – Set of all Vector bundles of M
$\|x_p\|$ – Length of vector $x_p \in T_p(M)$

2. BASIC DEFINITIONS

Definition 2.1 [9] Metric Space (Euclidean Spaces)

The Cartesian product $\mathbb{R}^n = \mathbb{R} \times \dots \times \mathbb{R}$ of n copies of the real line is known as n- dimension Euclidean space. It is the set of ordered n- tuples of real numbers.

$\mathbb{R}^n$ is an n-dimensional vector space with the usual operation of scalar multiplication and vector addition. The geometric properties of $\mathbb{R}^n$ are derived from the Euclidean dot product$x.y = x_1y_1 + \cdots + x_ny_n$. Where a point in $\mathbb{R}^n$ is denoted by $(x_1, \dots, x_n)$

The norm or length of a vector $x \in \mathbb{R}^n$ is given by

$|x| = (x.x)^{\frac{1}{2}} = ((x_1)^2 + \cdots + (x_n)^2)^{\frac{1}{2}}$

Given two points $x, y \in \mathbb{R}^n$, the line segment between them is the set $\{tx + (1-t)y : 0 \le t \le 1\}$.

Continuity and convergence in Euclidean spaces are defined in the usual ways. A map f : U → V between subsets of Euclidean spaces is continuous, if for any $x \in U$ and any $\varepsilon > 0, \exists\, \delta > 0$ such that

$|x - y| < \delta$

$\Rightarrow |f(x) - f(y)| < \varepsilon$

A sequence $\{x_i\}$ of points in $\mathbb{R}^n$ converges to $x \in \mathbb{R}^n$ if, for any $\varepsilon > 0,\ \exists\, N$ such that $i \ge N$ implies $|x_i - x| < \varepsilon.$

If M is a set, then a metric on M is a function : $M \times M \to \mathbb{R}$, also called a distance function, which satisfies following three properties.

i. Symmetry: for all $x, y \in M, d(x,y) = d(y,x)$
ii. Positivity: for all $x, y \in M, d(x,y) \ge 0$ and $d(x,y) = 0$ $if\ and\ only\ if$ $x = y$
iii. Triangle inequality: for all $x, y, z \in M, d(x,z) \le d(x,y) + d(y,z)$

The pair (M, d) is called a metric space.

Definition 2.2 [9][6] Continuous

If (M_1, d_1) and (M_2, d_2) are two metric spaces, then a map f: $M_1 \to M_2$ is said to be continuous if for every x, y ϵ M_1 and every > 0 , there exists δ > 0 , such that d_1 (x , y) < δ implies $d_2(f(x) - fy<\varepsilon$. Similarly, if { x_i } is a sequence of points in a metric space (M, d) it is said to converge to $x \in M$. Written $x_i \to$ x or $\lim_{i\to\infty} x_i = x$ if for any $\varepsilon > 0$, there exists N such that i ≥ N implies $d(x_i\,, x) < \varepsilon.$

Definition 2.3 [9] [Cauchy Sequence]

A sequence { x_i } in a metric space is said to be Cauchy if, for every δ > 0 there exists N such that i, j ≥ N implies $d(x_i\,, x) < \varepsilon$.

Definition 2.4[Completeness]

A metric space in which every Cauchy sequence converges is said to be complete.

Definition 2.5 [8] [14] [15]

Suppose γ: I → M is a smooth curve that is self parallel i.e $\nabla_{\partial t}\gamma^{\cdot} = 0$ along γ. We call γ a geodesic.

If γ: [a ,b] → M is a curve which is the restriction of a geodesic defined on an open interval containing [a , b] then we call γ a (parameterized) closed geodesic segment or just a geodesic, in short .

If γ: $[a, \infty)$ → M (resp. γ : (- ∞ ,a] → M) is the respestriction of a geodesic, then γ is positive (resp. negative) geodesic ray.

Definition 2.6 [8]Complete Geodesic

If the domain of geodesic is ℝ, then we call γ a complete geodesic.

If M is an n- Manifold and the image of a geodesic γ is contained in the domain of some chart with co-ordinate functions $x^1, \dots, x^n$, then the condition for γ to be a geodesic is

$$\frac{d^2x^i o\gamma(t)}{dt^2} + \sum \Gamma jk^i(\gamma(t))\frac{dx^j o\gamma(t)}{dt}, \qquad \frac{dx^k o\gamma(t)}{dt} = 0 \qquad \text{for all t } \epsilon \text{ I and } 1 \le i \le n \qquad \dots (1)$$

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is called local equations of geodesic or local geodesic equation.

We can convert these local geodesic equations into a system of 2n first order equation by the usual reduction of order trick.

i.e. let v - denote vector,

$$\frac{dx^i}{dt} = v^i, \qquad 1 \le i \le n$$

The equation (1) implies with observation

$\frac{dv^i}{dt} + \sum_{i,j} \Gamma jk^i \, v^j \, v^k = 0, \; i \le i \le n$ ……….(2)

Here v^i, as coordinates on TM

This first order system of equation (2) is a local expression of the equation for the integral curves of a vector field on TM.

Lemma 2.7 [8][5]

For each $v \in T_p(M)$, there is an open interval I containing 0 and a unique geodesic γ: I→ M , such that $\gamma^{\cdot}(0) = v, \; \gamma(0) = p$

Definition 2.8 [8][5]

A geodesic γ : I → M is called Maximal, if there is no other geodesic with open interval domain I strictly containing I that agree with γ on I

Theorem 2.9 [8]

For any γ ϵ TM there is a unique maximal geodesic γ_v with $\gamma^{\cdot}{}_v(0) = v$

Definition 2.10 [8][7]

A continuous curve γ: [a, b] → M is called a broken geodesic segment if it is a piece wise smooth curve whose smooth segments are geodesic segments.

Lemma: 2.11 [15]

Let γ(t) , a < t < b be a non trivial geodesic and Let t' be a new parameter with respect to t' the curve will be a geodesic if and only if t = ct' +d, c ≠ 0 and d constant . In particular the arc length is always such a parameter.

Definition 2.12 [5]

A curve γ: (a, b) → M on a surface M in $\mathbb{R}^3$ is said to be geodesic if its acceleration vector is perpendicular to the Tangent plane to M i.e

$$\frac{dv^{\cdot}(t)}{dt} = 0 \;\; or \; (D_h v^{\cdot})(t) = 0$$

Where $D_h v^{\cdot}$ denote the components of $v^{\cdot\cdot}(t)$ in T_p (M).

3. GEODESIC ON M AND T(M)

In this section, we discussed the Basic concept need for your further work.

A curve in M is of course a curve in TM, but a curve in TM contained in M is not necessarily a curve in a M, because it may not even be continuous.

The Lemma 3.4 gives idea clearly.

Definition 3.1

A Cauchy sequence in metric space T(M) is sequence $x_{p_1}, x_{p_2}, \dots, x_{p_n}$ such that $\forall\, \varepsilon > 0, \exists\, N > 0$ such that $n, m > N \;\; \Rightarrow dist\left(x_{p_n}, x_{p_m}\right) < \varepsilon$

Definition 3.2 Complete Space

A metric space TM is called complete if every Cauchy sequence in TM converges in x

Let us assume that TM is path connected space with path γ: [0 ,1] → T(M)

$$\gamma(0) = x_{p_i} , \; \gamma(1) = x_{p_j} \;\; ; \;\; \forall i \ne j$$

We redefined the metric space in T(M). A space TM is said to be metric space if

$$\widehat{dist}\left(x_{p_i}, \; x_{p_j}\right) = \begin{cases} 1, & if \; dist\left(x_{p_i}, \; x_j\right) > 1 \\ dist\left(x_{p_i}, \; x_p\right), & if \; dist\left(x_{p_i}, \; x_p\right) \le 1 \end{cases}$$

Then dist and $\widehat{dist}$ give rise to the same topological on T(M) .

Definition 3.3

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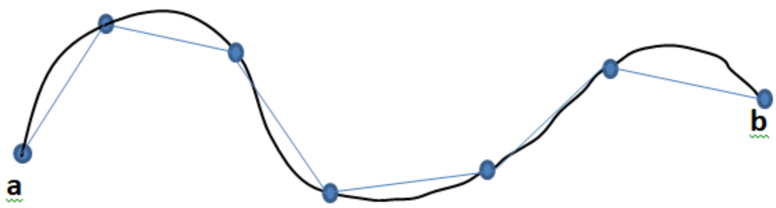
A continuous curve γ: [a,b] → M is called a broken geodesic segment if it is a piece-wise smooth curve whose smooth segments are geodesic segments . If t_* is a point of [a, b] where γ is not smooth. We call $\gamma(t_*)$ a break point (A smooth geodesic segment is considered a special case).



Piece wise smooth, but not smooth

The existence of geodesic passing through a point p ∈ M at parameter value zero with any specified velocity allows us to define a very important map. Let $\widetilde{D}_p$ denote the set of all $v \in T_pM$ such that the geodesic γ_v is defined at least on the interval [0, 1]. The exponential map $exp_p: \widetilde{D}_p \to M$ is defined by

$$exp_p v = \gamma_v(1)$$

Lemma 3.4

Let φ be a differentiable mapping of Manifold V into the Manifold T(M) such that $\varphi(V)$ is contained in the sub manifold M. If the mapping $\varphi: V \to M$ is continuous, it is also differentiable.

Proof:

Let $x_p \in V$ be a vector in T(M). [13]

M be a sub manifold of a manifold N and $p \in M$ then there exists a co-ordinates system $\{x_1, \dots, x_n\}$ valid on an open neighborhood, V of P in N such that

$x_1(p) = \cdots = x_n(p) = 0$ and such that the set .

$U = \{q \in V: x_j(q) = 0, for\ m+1 \le j \le n\}$

Together with the restrictions of $(x_1, \dots, x_n)$ to U form a local chart on M , containing p.

In view of above statement (proposition 3.2 [13]) there exists a co-ordinate system

$x_{p_1}, x_{p_2}, \dots, x_{p_{2m}}$ valid on an open neighborhood N of $\varphi(p)$ in T(M) such that the set

$N_m = \left\{r \in N: x_{p_j}(r) = 0\ fo\ m < j \le 2m\right\}$ together with the restriction of $(x_{p_1}, x_{p_2}, \dots, x_{p_m})$ to N_m form a local chart on M , containing $\varphi(p)$. By the continuity of φ there exists a local chart (w,) around p such that φ (w) ⊂ N_m . The co- ordinates x_{p_j} $(\varphi(q)), (1 \le j \le 2m)$ depends differentially on the co-ordinates of q ∈ w . In particular this hold for the co-ordinates x_{p_j} $(\varphi(q)),\ 1 \le j \le 2m.$

So the mapping φ:V→ M is differentiable . This completes the proof of Lemma.

As an immediate consequence of this Lemma we have the following statement.

Suppose that V and M are sub Manifolds of T(M) and V ⊂ M .

If M has the relative Topology of T(M) then V is the sub Manifold of T(M)

For your convenient, assume that T(M) is admits Riemannian structure also T(M) is Riemannian Manifold (detail in section 4) .

As M a connected sub Manifold. The Riemannian structure of T(M) include a Riemannian structure of M Let $d_{T(M)}$ and d_M - denote the distance functions in T(M) and M respectively.

It is obvious that

$d_{T(M)} = ||x_p - x_q|| \le d_M(p,q)$; where || || is normal in T(M) is distance function in vector field T(M) for p,q ∈ M.

In order to distinguish between geodesics in T(M) and M. We shall call them T(M) geodesics (norm) and M-geodesics respectively.

Note: 3.5

In T(M) distance between two vectors x_p and x_q is denotes a norm in T(M).

Lemma: 3.6

Let γ be a curve in M and suppose γ is an T(M) – geodesic .then γ is an M geodesic .

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Proof:
Let o and p be any points on γ. Say $o = \gamma(x_o)$ and $p = \gamma(x_p)$
Let N_o be a spherical normal neighborhood of x_o in T(M) . If x_p is sufficiently close to x_o the geodesic segment.
$\gamma_{op}: t \to \gamma(t),\ ||t - x_o|| \leq ||x - x_p||$ is contained in N_o. The length of γ_{op} satisfies length of $\gamma_{op} = ||xo \quad xp|| < d_M(o, p) \leq L(\gamma op)$ in T(M) .
i.e. L ($\gamma_{xo\ xp}$) = dT(M) $^{(Xo\ Xp)}$ ≤ d_m(o,p) ≤ L (γ_{op})
Consequently L(γ_{op})= d_M (o ,p) thus γ_{op} is curve of shortest length in M joining O and P. Hence the curve γ is a geodesic in M
∴ γ is M geodesic .

Lemma: 3.7
Suppose M is a sub Manifold of T(M) and geodesic at point p ∈ M . If γ is an M – geodesic through p then γ is also an T(M) is complete then M is complete.

Proof :
Let Γ be the maximal T(M) – geodesic Tangent to γ at P . Then Γ ⊂ M. So by Lemma: 3.6 Γ is an M – geodesic . Hence γ ⊂ Γ.
Now suppose T(M) is complete and Let Exp $_{T(M)}$ and Exp $_M$ denote the exponential mapping at x_p for T(M), p for M respectively. By assumption Exp $_{T(M)}$ is defined on the entire T(M_p). Since M is a geodesic at p. Exp $_M$ is the restriction of Exp $_{T(M)}$ to M_P in particular, M is complete at p in the sense of the following .
In a complete Riemannian Manifold (see section 4) M with metric d, each pair p, q ∈ M can be joined by a geodesic of length d (p, q).
This shows M is complete. This is also by definition [section 4, Geodesically complete].
As M and T(M) are subset of $\mathbb{R}^n$ and $\mathbb{R}^{2n}$ respectively. Each subset of M and T(M) are real .
This shows that each geodesic emanating from a point p ∈ T_p(M) is all of ℝ than we say that M is geodesically complete at p.

4. Geodesic And Norms On [M and T(M)] Riemannian And Lorentz Geometry

In this section we assumed that M and T(M) are Topological Manifold which is also Riemannian Manifold . Also we define the semi – Riemannian geometry which includes Riemannian geometry and Lorentz geometry.

Definition 4.1 (Distance in Riemannian Manifold)
Let p, q ∈ M, consider the set path (p, q) consisting of all piece wise smooth curve that begin at p and end at q. We define the Riemannian distance from p to q as $(p, q) = inf\{L(c) : c \in path\ (p, q)\}$.

Definition 4.2 (open geodesic ball)
If p ∈ M is a point in a Riemannian Manifold and R > 0 then the set B_R (P) (also denoted by B (P , R)) defined by B_R(P) ={q ∈ M : dist (p ,q) < R} is called an open geodesic ball centered at p with Radius R .

Definition 4.3
Let p, q ∈ M are points in Riemannian Manifold the distance function is defined which is metric Topological defined as –

i. Distance (p, q) = distance (q, p) ……..... symmetric
ii. Distance (p, q) ≤ distance (p, x) + distance (x, q) …………. Triangle inequality .
iii. Distance (p, q) ≥ 0
iv. Distance (p, q) = 0 if and only if p = q .

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By definition, a curve segment in a Riemannian Manifold say $\gamma : [a, b] \rightarrow M$ is a shortest curve if $(\gamma) = \text{dist}(\gamma(a), \gamma(b))$.
We say that such a curve is length minimizing such curves must be geodesic

Definition : 4.4

Let M be a surface in $\mathbb{R}^3$. Then M has an induced Riemannian metric from $\mathbb{R}^3$. We shall consider an appropriate condition under which a given curve $\gamma : (a, b) \rightarrow M$ can be throughout of as a generalization of a line segment on the plane. The velocity $\gamma^{\cdot}(t)$ of the curve (as motion) is contained in the Tangent plane at $\gamma(t)$.

i.e $\gamma^{\cdot}(t) \in T_{\gamma(t)}M$, $t \in (a, b)$

But the acceleration vector $\gamma^{\cdot\cdot}(t) = \frac{d}{dt}\gamma^{\cdot}$ goes out of the tangent plane at $\gamma(t)$ in general. Denoting by N_p the orthogonal complement of the tangent space $T_p(M)$ in $T_p\mathbb{R}^3$, we have $T_p\mathbb{R}^3 = T_p(M) \oplus N_p$. Accordingly we get $\gamma^{\cdot\cdot}(t) = D_h\gamma^{\cdot}(t) + (D_n\gamma^{\cdot})(t)$. Where $D_h\gamma^{\cdot}$ and $D_n\gamma$ denoate the components of $\gamma^{\cdot\cdot}(t)$ in T_p and in N_p respectively. In fact

We have $D_h\gamma^{\cdot} = \gamma^{\cdot\cdot} - < \gamma^{\cdot\cdot}, n > n$

Where n is a unit normal field at $\gamma(t)$.

A curve $\gamma : (a, b) \rightarrow M$ on a surface M in $\mathbb{R}^3$ is said to be a geodesic if its acceleration vector is perpendicular to the tangent plane to M that is $D_h\gamma^{\cdot}(t) = 0$.

Next, we consider Manifold means Topological Manifold M or T(M) . Which is Riemannian Manifold and Lorentz Manifold which is called Semi – Riemannian- Manifold .

Definition 4.5

Let (M, g) be a Semi- Riemannian Manifold. Suppose that $\gamma: I \rightarrow M$ is smooth curve that is self parallel in the sense that $\nabla\partial_t\gamma^{\cdot} = 0$ along γ.

Definition 4.6 (Geodesically Complete)

If the domain of every Maximal geodesic emanating from a point $p \in T_p(M)$ is all of $\mathbb{R}$, then we say that M is geodesically complete, if and only if, it is geodesically complete at each of its points.

We defined

Definition 4.7 (Totally geodesic)

Let T(M) be a Riemannian Manifold and M is a connected sub Manifold of T(M). Let $p \in M$. the sub Manifold M is said to be geodesic at p, if each T(M) geodesic which is Tangent to M at p is a curve in M . The sub Manifold M is called Totally geodesic if it is geodesic at each of its points.

Definition 4.8 (Geodesically complete)

Let M and $dT_p(M)$ – are Semi – Riemannian Manifold are said to be geodesically complete iff it is geodesically complete at of its each points .

Proposition 4.8:

Suppose M is a totally Geodesic sub Manifold of T(M) and Let I denotes the identity mapping of M into T(M) for each $p \in M$. For each $p \in M$ there exists an open neighborhood U_p of P in M on which I is distance preserving that is $d_M(q_1, q_2) = d_{T(M)}(x_{q_1}, x_{q_2})$ for $q_1, q_2 \in U_p$.

Proof

Let $B_p(x_p)$ be a minimizing convex normal ball around x_p in T(M) . Since I is continuous the intersection $B_p(x_p) \cap M$ is an open subset of M .

Let U_p be a minimizing convex normal ball around p in M such that $U_p \subset B_p(x_p)$. Let q, r be arbitrary points in U_p and γ_{qr} the M- geodesic inside U_p joining q and r, then $d_M(q, r) = L(\gamma_{qr})$. Consider the maximal T(M) – geodesic Γ such that Γ and γ_{qr} have the same tangent vector at q. Since M is totally geodesic. Γ ⊂ M from Lemma 3.6 follows that $\gamma_{qr} \subset \Gamma$. So γ_{qr} is an T(M) – geodesic . Since $\gamma_{qr} \subset B_p (x_p)$ it follows that $d_{T(M)} (x_q , x_r) = L(\gamma_{qr})$.

Theorem 4.9

Let T(M) be a Riemannian Manifold and M a connected complete Manifold of T(M) . Then M is totally geodesic if and only if T(M) is parallel translation along curve in M always transports Tangents to M into Tangents to T(M) .

Let m = dim M

dim TM = 2m and 0 be an arbitrary point in M.

We have a Lemma [13]. As M be a submanifold of a manifold N and Let $p \in M$, then there exists a coordinate system $\{x_1, \dots, x_n\}$ valid on an open neighborhood V of p in N such that $x_1(p) = \cdots = x_n(p) = 0$ and such that the set $U = \{q \in V: x_j(q) = 0\ for\ m+1 \le j \le n\}$ to U form a local chart on M containing p.

In this above statement there exists an open neighborhood M of x_0 in T(M) on which a coordinate system $\{x_{p_1}, \dots, x_{p_{2m}}\}$ is valid such that the set $U = \left\{q \in M: x_{p_j}(q) = 0\right\} for\ m+1 \le j \le m$ is a normal neighborhood of 0 in M and such that the restrictions of $\{x_1, \dots, x_m\}$ to U from a coordinate system on U.

Let $p \in U$ and Let $\gamma: t \to \gamma(t)$ be a curve in U such that $p.\gamma(0)$

Let $\gamma(t)$ be family of tangent vectors to T(M) which is T(M) parallel along the curve γ and such that $\gamma(0) \in M_p$ writing

$\gamma(t) = \sum_{a=1}^{m} \gamma^a(t) \frac{\partial}{\partial x_a}$ the coefficients $\gamma^a(t)$ satisfy the equation

$\gamma^{\cdot a}(t) + \sum_{bc}^{2m} \Gamma bc^a x_{(b)}^{\cdot}(t) \gamma^{(c)}(t) = 0$; where $a = i, b = j, c = k, 1 \le a, b, c \le m$

...(2)

$\gamma^a(0) = 0,\ x_a(t) = 0,\ m+1 \le a \le 2m$

Let $\pi: t \to \pi(t)$ be an T(M) – geodesic tangent to M at P, t being the arc parameter measured from P. If t is sufficiently small and we write $x_a(t)$ for $x_a(\pi(t))$

$\ddot{x_a}(t) + \sum_{b,c=1}^{2m} \Gamma bc^a x_b(t) \dot{x_c}(t) = 0$; where $1 < a, b, c \le 2m$; and $\dot{x_a}(0) = 0, m < a \le 2m$

...(3)

In the computation below we adopt the following range of indices

$$1 \le i, j, k \le m, m+1 \le \alpha, \beta, \gamma \le 2m$$

Suppose now M is totally geodesic, then the T(M) – geodesic π above is a curve in M. Hence an M-geodesic.

For small t, $\pi(t)$, lies in U, so $x_\alpha(t) = 0$ since every M-geodesic is now an M-geodesic (3), implies $\Gamma jk^\alpha(p) = 0,\ p \in U$ …………(4)

For the curve γ above, we have $\dot{x_\alpha}(t) = 0$. In view of (4) we obtain

$Y^{\cdot\alpha}(t) + \sum_{j,\beta} \Gamma j\beta^\alpha x_j^{\cdot}(t) Y\, Y^\beta(t) = 0$ on γ ..(5)

Now $Y(0) \in M_p$ so $Y^\beta(0) = 0$. Owing to the uniqueness theorem, for the system (5) of linear differential equations. We have $Y^\beta(t) = 0$

Consequently,

The family $Y(t)$ is tangent to M. Finally, Let $\beta: t \to \beta(t), t \in J$ be an arbitrary curve in M and Let $Z(t)$ be an T(M) parallel family along β such that $Z(t_0) \in M_\beta(t_0)$ for some $t_0 \in J$. The set of $t \in J$such that $Z(t) \in M_\beta(t)$ is clearly closed in J. The argument above shows that this set is open in J. Thus $Z(t) \in M_\beta(t)$ for all $t \in J$ and the first half of the theorem is proved.

To prove the converse, Suppose that for each curve as above the relation $Y(0) \in M_p$ implies $Y(t) \in M_\gamma(t)$ for each t

In (γ) we have therefore

$Y^\alpha(t) = 0,\ \dot{x_\alpha}(t) = 0$ and Eq (4) follows

Now substitute (4) into (3)

Since $\Gamma bc^a = \Gamma cb^a$

We obtain

$\ddot{x_\alpha}(t) + 2\sum_{j,\beta} \Gamma j\beta^\alpha \dot{x_j}(t)\dot{x_\beta}(t) + \sum_{\beta,\gamma}^{\alpha} \dot{x_\beta}(t)\dot{x_\gamma}(t) = 0$ …(6)

Since, $\dot{x_\alpha}(t) = 0$ we conclude from the Uniqueness theorem for the non linear system (6) that $x_j(t)$ is constant i.e $x_\alpha(t) = 0$ for all tin a certain interval around 0.

The function $x_j(t)$, are differentiable consequently. A piece π' of π containing P is a curve in U. Hence a curve in M. Let π^* be the maximal-M-geodesic tangent to π at P. Parametrized by the arc length t^* measure from P. Since M is complete $t^* \rightarrow$ runs from $-\infty\ to\ \infty$.

Now $\pi^*(t) = \pi(t)$, if t is sufficiently small: moreover the set of t values for which $\pi(t) = \pi^*(t)$ is open and closed. Thus $\pi \in \pi^*$, π is curve in M

Proposition 4.5

Let U be a normal neighborhood of a point p in a Riemannian Manifold (M, g). If $q \in U$ and if $\gamma: [0,1] \rightarrow M$ is the radial geodesic such that $\gamma(0) = p$ and $\gamma(1) = q$. The γ is the Unique shortest curve in U (up to reparameterization) connect p to q.

Conclusion:

The utilization of distance function, Geodesics on Topological Manifold M and Tangent bundles T(M), with the help of coordinate transformation and exponential function. The convergence property of these functions on vector bundle as well as Riemannian manifold defined and proved the completeness property of T(M) and M, which will applicable to extended the application of measure manifold in space-time complete measure manifold.

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