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EXISTENCE AND NON-EXISTENCE OF ECCENTRICITY CORDIAL LABELING OF GRAPHS

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ABSTRACT. Let G be a (p,q)-connected graph. We define a map Ψ from the set of all vertices of G to the set $\{0,1\}$ in such a way that, a vertex x has the label 1 if the eccentricity of that vertex is odd, and 0 otherwise. For each edge $e = xy$, we put the label as the absolute difference of labels of its end vertices. We can say Ψ, a eccentricity cordial labeling if $|\Psi_{ev}(0) - \Psi_{ev}(1)| \leq 1$ where $\Psi_{ev}(0)$, $\Psi_{ev}(1)$ respectively denote the number of vertices and edges together labelled by 0 and by 1. A graph which satisfies these labeling conditions will be called as eccentricity cordial graph. In this manuscript, we have studied some classes of graphs and their eccentricity cordial nature.

1. Introduction

Let $G = (V, E)$ be a (p,q) graph. Throughout this paper we have considered only simple, connected, and undirected graphs. The number of vertices of G is called the order of G and the number of edges of G is called the size G. Graph labeling has wide range of applications, see [2]. The graph labeling problem was introduced by Rosa called graceful labeling [4] in the year 1967. In 1980, Cahit [1] introduced the cordial labeling of graphs. Motivated by this labeling we introduce here a new type of graph labeling, called eccentricity cordial labeling. Also here we discuss the nature of this labeling on some standard graphs. Let x be any real number. Then the symbol $\lfloor x \rfloor$ stands for the largest integer less than or equal to x and $\lceil x \rceil$ stands for the smallest integer greater than or equal to x. Terms and definitions not defined here are used in the sense of Harary [3].

2. Preliminaries

Definition 2.1. [3] The distance $d(u,v)$ between two points u and v in a connected graph G is the length of a shortest path joining them.

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Definition 2.2. [3] The eccentricity $e(v)$ of a vertex v in a connected graph G is $\max d(u, v)$ for all u in G.

Definition 2.3. The Cartesian product of two graphs G_1 and G_2 is the graph $G_1 \Box G_2$ with the vertex set $V_1 \Box V_2$ and two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent whenever $[u_1 = v_1$ and u_2 adj $v_2]$ or $[u_2 = v_2$ and u_1 adj $v_1]$.

Definition 2.4. The graph $B_m = S_m \Box K_2$ is called a book graph, where S_m is the star with $m + 1$ vertices.

Definition 2.5. The notation $C(n; k)$ denotes the cycle C_n with k cords sharing a common endpoint called the apex. The graph graph $C(n; n-3)$ is called a shell. Notice that the shell $C(n; n-3)$ is the same as the fan $F_n = P_n + K_1$. A shell-butterfly graph SB_n is a one-point union of two shells $C(n; n-3)$ of with two pendent edges at the apex.

Example 2.1. A shell-butterfly graph SB_7 is given below.

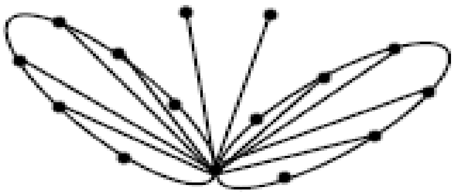

Fig.1

Definition 2.6. Let $C_n^{(t)}$ denote the one-point union of t cycles of length n. The graph $C_3^{(t)}$ is called a *friendship graph*.

Definition 2.7. *Jelly fish* graphs $J(m, n)$ obtained from a cycle C_4 : $uxvyu$ by joining x and y with an edge and appending m pendent edges to u and n pendent edges to v.

Definition 2.8. The graph obtained by joining two disjoint cycles $u_1u_2 \dots u_nu_1$ and $v_1v_2 \dots v_nv_1$ with an edge u_1v_1 is called *dumbbell graph* Db_n.

Definition 2.9. The triangular snake T_n is obtained from the path P_n by replacing each edge of the path by a triangle C_3.

Definition 2.10. The *armed crown* AC_n is obtained from the cycle C_n with $V(AC_n) = V(C_n) \cup \{v_i, w_i : 1 \leq i \leq n\}$ and $E(AC_n) = E(C_n) \cup \{u_iw_i, w_iv_i : 1 \leq i \leq n\}$.

Definition 2.11. For a (p,q)-connected graph, we construct a map $f : V(G) \to \{0,1\}$ by

$$f(u) = \begin{cases} 0 & \text{if } e(u) \text{ is even} \\ 1 & \text{otherwise} \end{cases}$$

For each edge uv we assign $|f(u) - f(v)|$. The graph G is said to have an eccentricity cordial labeling if $|f_{ev}(0) - f_{ev}(1)| \leq 1$, where $f_{ev}(i)$, $i \in \{0,1\}$ denote the number of vertices and edges labelled by i, $i \in \{0,1\}$. In this case, G is called as eccentricity cordial graph.

3. Main Results

Theorem 3.1. The book graph B_m is not eccentricity cordial, for all m.

Proof. Let $V(B_m) = \{u,v\} \cup \{u_i, v_i : 1 \leq i \leq m\}$ be the vertex set of B_m and $E(B_m) = \{uu_i, vv_i, u_iv_i : 1 \leq i \leq m\} \cup \{uv\}$ be the edge set of B_m. The eccentricities of each vertex of the book graph is given below:

$$e(x) = \begin{cases} 3 & \text{if } \ x = u_i \text{ or } x = v_i \\ 2 & \text{if } \ x = u \text{ or } x = v \end{cases}$$

Using this data, we conclude that, the vertices u_i, v_i have the label 1, and the vertices u, v have the label 0. Then the edges uu_i, vv_i, where $1 \leq i \leq m$, are labelled by 1 whereas u_iv_i where $1 \leq i \leq m$, are labelled by 0. Also the edge uv has the label 0. Thus, we have the total number of vertices and edges labelled by 0 is $n+2$ and that by 1 is $4n$. This gives the non-existence of eccentricity cordial labeling. □

Theorem 3.2. Shell-butterfly graph SB_n is not eccentricity cordial.

Proof. Let $V(SB_n) = \{u,v,w\} \cup \{u_i, v_i : 1 \leq i \leq n-1\}$ and $E(SB_n) = \{uv, uw\} \cup \{uu_i, uv_i : 1 \leq i \leq n-1\} \cup \{u_iu_{i+1}, v_iv_{i+1} : 1 \leq i \leq n-2\}$. Note that the eccentricities of the vertex u is 1 and all the remaining vertices have eccentricity 2. It follows that u recieves the label 1 and the other vertices receive the label 0. It is easy to check that the number of vertices and edges labelled by 1 is $2n+3$ and by 0 is $4n-4$. It follows that shell-butterfly graph SB_n is not eccentricity cordial. □

Theorem 3.3. Friendship graph $C_3^{(t)}$ is eccentricity cordial if and only if $t \geq 2$.

Proof. Let u be the vertex obtained by joining t copies of the cycle C_3. Let the i^{th} cycle be $C_i : uu_1^iu_2^iu$. As eccentricity of u is 1 and that of remaining vertices is 2, they receive the labels 1, 0 respectively. Here the edges $u_1^iu_2^i$ $(1 \leq i \leq t)$ contributes t number of zeros and the edges $u_1u_1^i, u_1u_2^i$ contribute each t number of zeros. It follows that

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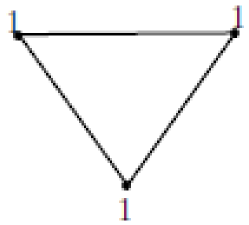
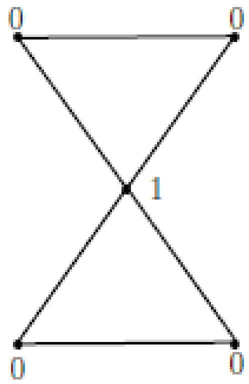
the total number of vertices and edges labelled by 0 is $3t$ and that by 1 is $2t + 1$. Thus $|\Psi_{ev}(0) - \Psi_{ev}(1)| = t - 1$. Hence $t = 2$ or $t = 1$. Conversely suppose $t = 1$ or $t = 2$, then the following figures illustrates the existence of eccentricity cordial labeling of both the cases.





Theorem 3.4. Dumbbell graph Db_n is not eccentricity cordial for all n.

Proof. The proof is divided into two cases according to the nature of n.
Case 1. $n \equiv 0 \pmod 2$.
In this case, the eccentricity of each vertex is as follows:

$$e(u_i) = e(v_i) \begin{cases} \frac{n}{2} + i & \text{if } 1 \leq i \leq \frac{n}{2} + 1 \\ \frac{3n+4}{2} - i & \text{if } \frac{n}{2} + 2 \leq i \leq n \end{cases}$$

Note that for $n \equiv 2 \pmod 4$ each odd indexed vertex has even eccentricity and an even indexed vertex has odd eccentricity. Thus they are labeled by 0 and 1 respectively. Similarly for, $n \equiv 0 \pmod 4$, each odd indexed vertex has eccentricity odd and even indexed vertex has eccentricity even. So they are labeled by 1 and 0 respectively. It follows that the number of vertices and edges labeled by 0 is $n + 1$ and the number of vertices and edges labeled by 1 is $3n$. Hence in this case Db_n is not an eccentricity cordial graph.
Case 2. $n \equiv 1 \pmod 2$.
Here, the eccentricity of each vertex is as follows:

$$e(u_i) = e(v_i) \begin{cases} \frac{n-1}{2} + i & \text{if } 1 \leq i \leq \frac{n+1}{2} \\ \frac{3n+3}{2} - i & \text{if } \frac{n+3}{2} \leq i \leq n \end{cases}$$

A similar argument shows that, there doesn't exist an eccentricity cordial labeling. Thus, Db_n is not an eccentricity cordial graph, for all values of n.

□

Theorem 3.5. The jelly fish graph $J(m, n)$ is not a eccentricity cordial graph, for all values of m, n.

Proof. Let $u_i (1 \leq i \leq m)$ be the appended vertices to u and $v_j (1 \leq i \leq n)$ be the appended vertices to v. The pendent vertices u_i, v_j of this graph have eccentricity 4. The vertices u and v have eccentricity 3. Also x, y have eccentricity 2. It follows that u_i, v_j, x and y receive the label 0 and the u, v receive the label 1. This implies the number of vertices and edges labeled by 1 and by 0 differ by 3. Hence $J(m, n)$ is not a eccentricity cordial graph. □

Theorem 3.6. Triangular Snake T_n, is eccentricity cordial iff $n \leq 4$.

Proof. Let $V(T_n) = \{u_i, v_j : 1 \leq i \leq n, 1 \leq j \leq n - 1\}$ and $E(T_n) = \{u_i u_{i+1}, u_i v_i, v_i u_{i+1} : 1 \leq i \leq n - 1\}$. The existence of eccentricity cordial labeling on T_2, T_3 and T_4 are displayed in figure 3

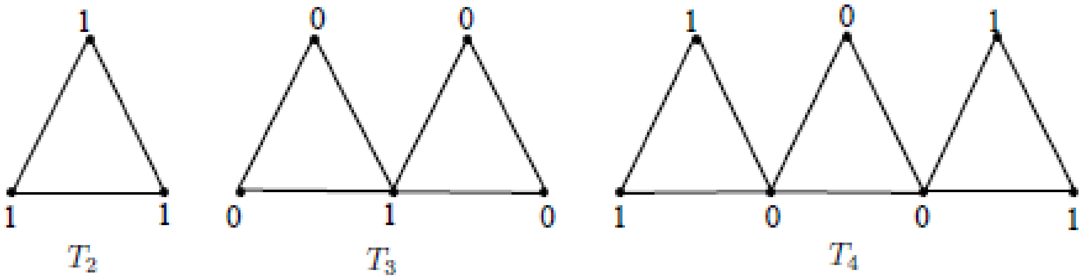


FIGURE 1

Conversely suppose $n > 4$, we divide the proof into two cases due to the nature of n.

Case 1. n is even.

Here the eccentricity of the each vertex is given below:

$$e(u_i) = \begin{cases} n-i & \text{if} \quad 1 \leq i \leq \frac{n}{2} \\ i-1 & \text{if} \quad \frac{n}{2}+1 \leq i \leq n \end{cases}$$

and

$$e(v_i) \begin{cases} n-i & \text{if} \quad 1 \leq i \leq \frac{n}{2} \\ i & \text{if} \quad \frac{n}{2}+1 \leq i \leq n-1 \end{cases}$$

So, the vertices $u_2, u_4, u_6, \ldots u_{\frac{n}{2}}, u_{fracn+22}, u_{fracn+62}, u_{fracn+102}, \ldots, u_{n-1}$ are labelled by 0. The remaining vertices of the path P_n are labelled by 1. Also the vertices $v_1, v_3, v_5, \ldots, v_{n-1}$ are labelled by 1 and the remaining unlabelled $v_i^{'s}$ are labelled by 0. It follows that $n-2$ edges of the path are labelled by 1 and the edge joining $u_{\frac{n}{2}}$ and $u_{fracn+22}$ received the label 0. On the other hand the edges joining u_i and v_i, $1 \leq i \leq \frac{n}{2}$ received the label 0 and the edges joining u_i and v_i, $\frac{n}{2}+1 \leq i \leq n-1$ received the label 1. Similarly, we can identify the labels of the remaining edges of T_n. It follows that $f(1) = 3n-4$ and $f(0) = 2n$. This implies the total number of vertices and edges labelled by 1 and that by 0 differ by atleast 2.

Case 2. n is odd.

In this case, the eccentricity of the each vertex is given below:

$$e(u_i) = \begin{cases} n-i & \text{if} \quad 1 \leq i \leq \frac{n+1}{2} \\ i-1 & \text{if} \quad \frac{n+1}{2}+1 \leq i \leq n \end{cases}$$

and

$$e(v_i) \begin{cases} n-i & \text{if} \quad 1 \leq i \leq \frac{n-1}{2} \\ i & \text{if} \quad \frac{n+1}{2} \leq i \leq n-1 \end{cases}$$

As in case 1, we can find the labels of vertices and edges, then we get the number of vertices and edges labelled by 1 is $3n-3$ and the number of vertices and edges labelled by 0 is $2n-1$. Therefore, the number of vertices and edges labelled by 1 and labelled by 0 differ by atleast 3. This shows that, for odd values of n, T_n is not eccentricity cordial.

Theorem 3.7. The armed crown AC_n is eccentricity cordial if and only if $n \equiv 0, 1 \pmod 4$.

Proof. Let the vertex set and edge set of AC_n are as follows: $V(AC_n) = \{u_i, v_i, w_i : 1 \leq i \leq n\}$ and $E(AC_n) = \{u_n u_1\} \cup \{uu_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i v_i, v_i w_i : 1 \leq i \leq n\}$. Note that, if n is even then

$$e(u) \begin{cases} \frac{n+4}{2} & \text{if} \quad u = u_i \\ \frac{n+6}{2} & \text{if} \quad u = v_i \\ \frac{n+8}{2} & \text{if} \quad u = w_i \end{cases}$$

and if n is odd then

$$e(u) \begin{cases} \frac{n+3}{2} & \text{if} \quad u = u_i \\ \frac{n+5}{2} & \text{if} \quad u = v_i \\ \frac{n+7}{2} & \text{if} \quad u = w_i \end{cases}$$

The labels of the vertices are differ according to their nature. So we can split up the values of n into the following four cases.

Case 1. $n \equiv 0 \pmod 4$.
Here, all $u_i'^s$ are labelled by 0, all the $v_i'^s$ are labelled by 1 and all the $w_i'^s$ are labelled by 0 and that by 1 are equal to $3n$. So their difference is zero.

Case 2. $n \equiv 1 \pmod 4$.
The labels are given to the vertices of AC_n similar to that of case 1. Here also the difference between the total number of vertices and edges with different labels is zero.

Case 3. $n \equiv 2 \pmod 4$.
Here all $u_i'^s$ are labelled by 1, all the $v_i'^s$ are labelled by 0 and all the $w_i'^s$ are labelled by 1. One can easily check that the number of vertices and edges labelled by zero is $2n$ and that by 1 is $4n$. It follows that, their difference is greater that > 6.

Case 4. $n \equiv 3 \pmod 4$.
A similar arguement to case 3, shows that there doesn't exist an eccentricity cordial labeling of AC_n.

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