DIFFRACTION OF OBLIQUELY INCIDENT ELASTIC P WAVES BY TWO EQUAL CO-AXIAL CIRCULAR CRACKS

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ASBTRACT

The two-dimensional problem of diffraction of an arbitrary incident elastic longitudinal wave by two equal co-axial circular cracksare in an infinite, isotropic and homogeneous elastic medium to solved by a simple integral equation technique. Approximate expressions for the far-field amplitudes, the scattering cross section are derived when the wavelength is large as compared to the radius of the circular cracks. By taking appropriate limits, the corresponding results for various limiting configurations are derived.

INTRODUCTION

Scattering of elastic waves by cracks is a problem of considerable importance in the field of fracture mechanics, quantitative nondestructive evaluation of materials, geophysics and seismology. Recently various two-dimensional problems of diffraction of plane acoustic wave by a semi-circular soft or rigid infinite strip have been discussed by different techniques[1-5]. Shail[6] solved the problem of diffraction of low-frequency acoustic waves by an infinite circular are soft strip by integral equation techniques. These integral equation techniques give the solution of two Fredholm integral equation of the first kind with logarithmic kernels which are derived by using the well-known solutions of Carleman integral equations[7,8].These integral equation techniques as well as their applications are quite complicated and cumbersome. Sampath and Jain[9], Jain and Jain[10-12] developed a simple independent integral equation technique to solve various two-dimensional Dirichlet as well as Neumann boundary value problems involving two equal co-axial infinite circularsare strips.These techniques have been further used to solve two-dimensional problems of diffraction of elastic P waves by (i) two equal co-axial circular are rigid strips[13], (ii) two equal parallel and coplanar Griffith cracks[14].

We present here for the first time the solution of the two-dimensional problem of diffraction of obliquely incidence low frequency elastic P waves by two equal co-axial circular cracks are embedded, in an infinite, isotropic and homogeneous elastic medium, by these simple integral equation techniques[8-12]. With use of the usual Green's function approach, the solution of this problem is first reduced to a pair of governing simultaneous

Fredholm integral equations of the first kind. When the wavelength is large as compared to the radius of circular cracks, solutions of this pair of governing integral equations is further reduced to that of a set of pairs of simultaneous Fredholm integral equations of the first kind. By making appropriate substitutions[10], the first pair of simultaneous Fredholm integral equations of the first kind of this set is solved to obtain approximate expression for the two unknown functions. Approximate expressions are derived for the far-field amplitudes, the scattering cross section. By taking appropriate limits, we derive the corresponding known solution of the problem of diffraction of obliquely incident P waves by two parallel and coplanar Griffith cracks[14]. This serves as a check on our analysis. The other two corresponding results of the limiting configurations of a circular are crack and a semi-circular crack seem to be new.

II. INCIDENT P WAVES

Consider a cylindrical polar coordinate system (r,θ, x_3) such that the two equal co-axial circular cracks are defined by the equations

$r = a$, $0 <\beta<|\theta|<\alpha<\pi$, $-\infty <x_3<\infty$ (see fig. 1),

where a is the radius of the circular crack. By normalizing all the lengths with respect to 'a', the cracks are now defined by the equation

$r = 1$, $0 <\beta< |\theta| <\alpha<\pi$, $-\infty <x_3<\infty$.

Let $u^o(\mathrm{x})$ be the displacement field (the time factor $e^{-i\omega t}$ is suppressed throughout the analysis) associated with the incident elastic P waves propagating in the infinite, isotropic and homogeneous medium occupying the whole region S of the $x_1 - x_2$ plane in the direction making angle ϕ with the positive direction of x_1 axis and is defined as:

$$u^o(\mathrm{x}) = im_1 A_0\hat{b} \exp (im_1(\mathrm{x}.\hat{b}), \hat{b} = \hat{e}_1 \cos\phi + \hat{e}_2 \sin\phi, x \in S, \tag{2.1}$$

where $\mathrm{x} = (x_1, x_2)$, $m_1^2 = (\rho_0\omega^2a^2)/(\lambda + 2u)$, A_0 is the known constant e_1 and e_2 are the unit vectors along the x_1 and x_2 axes, λ and are the lame constants of the medium.ρ_0, is the density of the medium, and ω is the frequency of the incident wave. The constant stiffness tensor $C_{ijk1}(\mathrm{x})$ of the infinite host medium is defined as:

$$C_{ijkl}(\mathrm{x}) = \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{jl}\delta_{jk}), x\in S, \tag{2.2}$$

where δ's are Kronecker deltas and the indices are 1, 2. In the absence of body forces and the cracks, $u^o(\mathrm{x})$ satisfies the equilibrium equations

$$C_{ijkl}u^o_{k, lj}(\mathrm{x}) + \mu m_2^2 u^o_1(\mathrm{x}), x\in S, \tag{2.3}$$

where$u^o_{k, lj}(\mathrm{x}) = \frac{\partial^2 u^o_k(x)}{\partial xi\, \partial xj}$;$m_2^2 = \rho_0\omega^2a^2/\mu$

The components $\tau^{o}r_1(x)$ and $\tau^{o}r_2(x)$ of the stress tensor $\tau^{o}(x)$ associated with the incident field are given by

$$\tau^{o}_{r1}(x) = \tau^{o}_{11}(x)\cos\theta + \tau^{o}_{12}(x)\sin\theta, \quad x\in S, \tag{2.4}$$

$$\tau^{o}_{r2}(x) = \tau^{o}_{21}(x)\cos\theta + \tau^{o}_{22}(x)\sin\theta, \quad x\in S, \tag{2.5}$$

where

$$\tau^{o}_{11}(x) = -A_0\,(m_1^{2}\mu/\tau^{2})\,\{\cos^{2}\phi + (1-2\tau^{2})\sin^{2}\phi\}\exp[im_1(x_1\cos\phi + x_2\sin\phi)],$$

$$\tau^{o}_{12}(x) = -A_0 m_1^{2}\mu\,\{\cos^{2}\phi\}\exp[im_1(x_1\cos\phi + x_2\sin\phi)],$$

$$\tau^{o}_{22}(x) = -A_0 m_1^{2}\mu\,\{(1-2\tau^{2})\cos^{2}\phi + \sin^{2}\phi\}\exp[im_1(x_1\cos\phi + x_2\sin\phi)],$$

$$\tau = m_1/m_2 = [\mu/(\lambda + 2\mu)]^{1/2}.$$

Let the displacement vector, the stress tensor associated with the scattered field and the total field be denoted by $u^{s}(x)$ and $\tau^{s}(x)$ and $u(x)$, $\tau(x)$ respectively.

The boundary conditions are

$$\tau_{r1}(x) = \tau^{S}_{r1}(x) + \tau^{o}_{r1}(x) = 0, \quad i=1, 2, \text{ as } x \text{ tends to the points on C}, \tag{2.6}$$

where the arc C are defined by the equation $r = 1$, $\beta< |\theta| <\alpha$, $u_1(x)$, $\tau_{r1}(x)$, i=1, 2 are continuous across $r = 1$, $0 \leq |\theta| <\beta$, $a< |\theta| \leq\pi$, (2.7)

In addition, we must satisfy the radiation condition at infinity and the appropriate edge conditions at the tips of the cracks.

Thus, $u(x)$ satisfies the distributional formula

$$\begin{aligned}\mathrm{div}[C_{ijkl}u_{k,l}(x)] &= \mathrm{div}[C_{ijkl}u_{k,l}(x)] + C_{k\alpha ilgk}(x_c)\,(\partial/\partial_{xl})\,\delta(x-x_c)\,\hat{n}_a(x_c)\\ &= \mu m_2^{2}u_i(x),\ x\in S,\end{aligned} \tag{2.8}$$

where the bar denotes the distributional derivative,

$$g_k(x_c) = u_k(x_c)\,|_{-} - u_k(x_c)|_{+},\ k = 1, 2$$

are the jumps in the components of the displacement vector across the arcs C and $\hat{n}(x_c)$ is the unit vector along the outward normal drown at the point x_c of the arcs C.

The above distributional formula incorporates all the continuity requirements of the components u_ι, $\tau_{r\iota}$, ι=1, 2, given in the above boundary conditions (2.6) and (2.7). Following the usual Green's function approach, the integral representation formulas for the components $u^{S}_{m}(x)$ of the displacement vector of the scattered field are given by

$$\begin{aligned}u^{S}_{m}(x) = \int_{-\alpha}^{-\beta}\int_{\beta}^{\alpha}[&C_{k1i1}\,g_k(x_c)\cos\theta\,G_{im,i}(x,x_c)\\ &+ C_{k2i\,i}\,g_k(x_c)\sin\theta\,G_{1m,1}[(x,x_c)]\,d\theta,\quad m = 1, 2,\ x\in D,\end{aligned}$$

$$= \mu \int_{-\alpha}^{-\beta} + \int_{\beta}^{\alpha} [g_1(x_c) \{\cos\theta \, [\left(\tfrac{1}{\tau^2}\right) G_{1m,1}(x, x_c)$$
$$+ (1/\tau^2 - 2)\, G_{2m,2}(x, x_c)] + \sin\theta\, [G_{1m,2}(x, x_c)$$
$$+ G_{2m,1}(x, x_c)]\} + g_2(x_c) \{\cos\theta\, [G_{1m,2}(x, x_c)$$
$$+ (1/\tau^2)\, G_{2m,2}(x, x_c)]\}\, d\theta, m = 1, 2, \quad (2.10)$$

where $x_c = (x_1', x_2') = (\cos\theta', \sin\theta')$ and Green's function $G_{1m}(x, x_c)$ are defined as [15, 16]

$$G_{1m}(x, x_c) = (i/4\mu m_2^2)\, [\delta_{1m}\, m_2 2\, H_0^{(1)}(m_2 R)$$
$$+ (\partial^2/\partial x_1 \partial x_m)\, [H_0^{(1)}(m_2, R) - H_0^{(1)}(m_1, R)]], 1 = 1, 2, \quad (2.11)$$

where $R = |x - x_c|$ and $H_0^{(1)}$ is the Hankel function of the first kind of order zero.

The boundary conditions (2.6), (2.7), the formulas (2.4), (2.5), (2.10) and the relations

$$\tau^S_{ri}(x) = \tau^S_{11}(x) \cos\theta + \tau^S_{12}(x) \sin\theta, \; i = 1, 2,$$
$$\tau^S_{11}(x) = (\mu/\tau^2)\, \{u^S_{1,1}(x) + (1-2\tau^2)\, u^S_{2,2}(x)\},$$
$$\tau^S_{12}(x) = \tau^S_{21}(x) = \mu\, \{\mu^S_{1,2}(x) + u^S_{2,1}(x)\},$$
$$\tau^S_{11}(x) = (\mu/\tau^2)\, \{u^S_{2,2}(x) + (1-2\tau^2)\, u^S_{1,1}(x)\},$$

lead to the following pair of governing integral equations of this problem

$$\int_{-\alpha}^{-\beta} \int_{\beta}^{\alpha} (g_1(\theta')\, [\cos\theta K_{11}(\theta, \theta') + \sin(\theta + \theta') K_{12}(\theta, \theta')$$
$$+ \sin\theta \sin\theta'\, K_{13}(\theta, \theta')] + g_2(\theta')\, [\cos\theta \cos\theta'\, K_{12}(\theta, \theta')$$
$$+ \sin\theta \cos\theta'\, K_{13}(\theta, \theta') + \sin\theta \sin\theta'\, K_{14}(\theta + \theta')$$
$$+ \cos\theta \sin\theta'\, K_{15}(\theta + \theta')])\, d\theta'$$
$$= (A_0 m_2^2/\mu)\, \{[(1 - 2\tau^2 \sin^2\phi) \cos\theta + \tau^2 \sin^2\phi \sin\theta]$$
$$x\, [1 + \iota m_2 \tau\, (x_1 \cos\phi + x_2 \sin\phi) + 0(m_2^2)]\}, \; \beta < |\theta| < \alpha, \quad (2.13)$$

$$\int_{-\alpha}^{-\beta} \int_{\beta}^{\alpha} (g_1(\theta))[\cos\theta \cos\theta' K_{12}(\theta, \theta') + \cos\theta \sin\theta' K_{13}(\theta + \theta')$$
$$+ \sin\theta \sin\theta'\, K_{14}(\theta, \theta') + \sin\theta \sin\theta'\, K_{15}(\theta, \theta')]$$
$$+ g_2(\theta')\, [\cos\theta \cos\theta'\, K_{13}(\theta, \theta') + \sin(\theta + \theta')\, K_{14}(\theta, \theta')$$
$$+ \sin\theta \sin\theta'\, K_{18}(\theta, \theta')])\, d\theta'$$
$$= (A_0 m_2^2/\mu)\, \{[\tau^2 \sin^2\theta \cos\theta + (1 - 2\tau^2 \cos^2\phi) \sin\theta]$$

where

$$K_{11}(\theta, \theta') = (1/\tau^4)\, G_{11,11}(\theta, \theta') + (2/\tau^2)\, (1/\tau^2 - 2)\, G_{12,12}(\theta, \theta')$$
$$+ (1/\tau^2 - 2)^2\, G_{22,22}(\theta, \theta'),$$

$$K_{12}(\theta, \theta') = (1/\tau^2)\,[G_{11,\,12}(\theta, \theta') + G_{12,\,11}(\theta, \theta')]$$
$$+ (1/\tau^2 - 2)\,[G_{12,\,22}(\theta, \theta') + G_{22,\,12}(\theta, \theta')],$$
$$K_{13}(\theta, \theta') = G_{11,\,22}(\theta, \theta') + 2G_{12,\,12}(\theta, \theta') + G_{22,\,11}(\theta, \theta')$$
$$K_{14}(\theta, \theta') = (1/\tau^2 - 2)\,[G_{11,\,12}(\theta, \theta') + G_{12,\,11}(\theta, \theta')]$$
$$+ (1/\tau^2)\,[G_{12,\,22}(\theta, \theta') + G_{22,\,12}(\theta, \theta')],$$
$$K_{15}(\theta, \theta') = (1/\tau^2)\,(1/\tau^2 - 2)\,[G_{11,\,11}(\theta, \theta') + G_{22,\,22}(\theta, \theta')]$$
$$+ (1/\tau^4 + (1/\tau^2 - 2)^2)\,G_{12,\,12}(\theta, \theta'),$$
$$K_{18}(\theta, \theta') = (1/\tau^2 - 2)^2\,[G_{11,\,11}(\theta, \theta') + (2/\tau^2)\,(1/\tau^2 - 2)$$
$$x\, G_{12,\,12}(\theta, \theta') + (1/\tau^4)\,G_{22,\,22}(\theta, \theta')], \tag{2.15}$$

and

$$G_{ij,\,mk}\,(\theta, \theta') = G_{ij,\,mk}(x, x_c) = [(\partial^2/\partial x_m \partial x_k)\,G_{ij}(x, x_c)]_{r\,=\,r'\,=\,1}$$
$$i, j, m, k = 1, 2 \tag{2.16}$$

In the subsequent analysis, we shall assume that $m_j << 1$, $j = 1, 2$ and $m_1 = 0(m_2)$ so that, we can use the expansion

$$H_0^{(1)}\,(m_j R) = (2\iota/\pi)\,\{[q_j + \log 2R] - (m^2_j R^2/4)\,[q_j + \log 2R - 1]$$
$$+ 0(m_j^4)\},\quad j = 1, 2 \tag{2.17}$$

where

$$q_j = \log\,(m_j/4) + \Gamma - i\pi/2, j = 1, 2$$

and $\Gamma = 0.5772$ is Euler's constant. (2.18)

Using the expansions (2.17) and eqn. (2.11) in qn. (2.16), we obtain the following approximate expressions for $G_{ij,\,mk}\,(\theta, \theta')$;

$$G_{11,\,11}(\theta, \theta') = -(1/(4\pi\mu R_1^2))\,[2\tau^2\cos(\theta + \theta') - (1 - \tau^2)\cos 2(\theta + \theta') + 0(m_2)],$$
$$G_{11,\,12}(\theta, \theta') = -(1/(4\pi\mu R_1^2))\,[(1 - \tau^2)\sin(\theta + \theta') - (1 - \tau^2)\sin 2(\theta + \theta') + 0(m_2)],$$
$$G_{11,\,22}(\theta, \theta') = -(1/(4\pi\mu R_1^2))\,[-2\cos(\theta + \theta') + (1 - \tau^2)\cos 2(\theta + \theta') + 0(m_2)],$$
$$G_{12,\,11}(\theta, \theta') = -(1/(4\pi\mu R_1^2))\,[-(1 - \tau^2)\sin(\theta + \theta') - (1 - \tau^2)\cos 2\,(\theta + \theta') + 0(m_2)],$$
$$G_{12,\,12}(\theta, \theta') = -(1/(4\pi\mu R_1^2))\,[(1 - \tau^2)\cos 2(\theta + \theta') + 0(m_2)],$$
$$G_{12,\,22}(\theta, \theta') = -(1/(4\pi\mu R_1^2))\,[-(1 - \tau^2)\sin(\theta + \theta') + (1 - \tau^2)\sin 2(\theta + \theta') + 0(m_2)],$$
$$G_{22,\,11}(\theta, \theta') = -(1/(4\pi\mu R_1^2))\,[2\cos(\theta + \theta') + (1 - \tau^2)\cos 2(\theta + \theta') + 0(m_2)],$$
$$G_{22,\,12}(\theta, \theta') = -(1/(4\pi\mu R_1^2))\,[(1 + \tau^2)\sin(\theta + \theta') + (1 - \tau^2)\sin 2(\theta + \theta') + 0(m_2)],$$

$$G_{22,22}(\theta, \theta') = -(1/(4\pi\mu R_1^2))\,[-2\tau^2\cos(\theta+\theta') - (1-\tau^2)\cos 2(\theta+\theta') + 0(m_2)], \tag{2.19}$$

where

$$R_1^2 = 4\sin^2(\theta-\theta')/2.$$

When, we substitute the values of $K_{\iota j}(\theta, \theta')$, j = 1–6, from eqn. (2.15) in the integral equations (2.13) and (2.14), we get

$$\int_{-\alpha}^{-\beta}\int_{\beta}^{\alpha}[g_1(\theta')\,M_1(\theta,\theta') + g_2(\theta')M_2(\theta,\theta')]d\theta'$$

$$= A_0 m_2^2\,[1-2\tau^2\sin^2\phi)\cos\theta + \tau^2\sin^2\phi\sin\theta]\,\{1 + \iota m_2\tau\,(x_1\cos\phi + x_2\sin\phi) + 0(m_2)\},\ \beta<|\theta|<\alpha, \tag{2.20}$$

$$\int_{-\alpha}^{-\beta}\int_{\beta}^{\alpha}[g_1(\theta')\,L_1(\theta,\theta') + g_2(\theta')M_1(\theta,\theta')]d\theta'$$

$$= A_0 m_2^2\,[\tau^2\sin^2\phi\cos\theta + (1-2\tau^2\cos^2\phi)\sin\theta]\,\{1 + \iota m_2\tau\,(x_1\cos\phi + x_2\sin\phi) + 0(m_2)\},\ \beta<|\theta|<\alpha, \tag{2.21}$$

where

$$M_1(\theta, \theta') = \frac{(1-\tau^2)}{4\pi\sin^2\frac{1}{2}(\theta-\theta')}\,[\sin(\theta+\theta')\,(1-\cos(\theta-\theta')] + 0(m_2),$$

$$M_2(\theta, \theta') = \frac{(1-\tau^2)}{4\pi\sin^2\frac{1}{2}(\theta-\theta')}\,[1+\cos(\theta+\theta') - \cos(\theta+\theta')\cos(\theta-\theta')] + 0(m_2),$$

$$L_1(\theta, \theta') = \frac{(1-\tau^2)}{4\pi\sin^2\frac{1}{2}(\theta-\theta')}\,[1-\cos(\theta-\theta') + \cos(\theta-\theta')\cos(\theta+\theta')] + 0(m_2). \tag{2.22}$$

The above expressions and the expansion of the right-hand member of the integral equations (2.20) and (2.21) suggest that the unknown density functions $g_1(\theta')$, ι=1, 2 in the above integral equations must be of the form

$$g_1(\theta') = A_0 m_2^2\,[g_1(0)(\theta') + m_2 g_1^{(1)}(\theta') + m_2^2 g_1^{(2)}(\theta') + 0(m_2^3)\},\ \iota=1, 2. \tag{2.23}$$

Now we substitute the above expansions (2.23) in the integral equations (2.20) and (2.21), equate the coefficients of equal power of m_2 on both sides, and obtain an infinite set of pairs of simultaneous Fredholm integral equations of the first kind. We consider here the following first pair of this set involving the unknown density functions $g_1^{(0)}(\theta')$, i=1, 2;

$$\int_{\beta}^{\alpha}[g_{11}^{(0)}(\theta)\left\{\frac{1-\cos\theta'}{(\cos\theta-\cos\theta')^2} - \cos\theta\ \cos\theta'\right\} + g_{12}^{(0)}(\theta)\left\{\frac{\sin\theta\ \sin\theta}{(\cos\theta-\cos\theta')^2} + \sin\theta\ \sin\theta'\right\}$$

$$- g_{21}^{(0)}(\theta')\sin\theta\ \cos\theta'\} - g_{22}^{(0)}(\theta')\cos\theta\ \sin\theta']\,d\theta'$$

$= -(\pi/(1-\tau^2))\,[\tau^2 \sin^2\phi \cos\theta + (1-2\tau^2\cos^2\phi)\sin\theta],$ (2.24)

$$\int_\beta^\alpha [-g_{11}^{(0)}(\theta')\sin\theta\cos\theta' - g_{12}^{(0)}(\theta')\cos\theta\sin\theta' + g_{21}^{(0)}(\theta')\left\{\frac{1-\cos\theta\cos\theta'}{(\cos\theta-\cos\theta')^2} + \cos\theta\cos\theta'\right\}$$

$$g_{22}^{(0)}(\theta')\left\{\frac{\sin\theta\ \sin\theta'}{(\cos\theta-\cos\theta')^2} - \sin\theta\ \sin\theta'\right\}]\,d\theta'$$

$= -(\pi/(1-\tau^2))\,[(1-2\tau^2\sin^2\phi)\cos\theta + \tau^2\sin^2\phi\sin\theta],$ (2.25)

where we have used the relations[9–11]

$g_1^{(0)}(\theta') = g_{11}^{(0)}(\theta') + g_{12}^{(0)}(\theta'),\ \iota=1, 2,$ (2.25a)

and $g_{11}^{(0)}(\theta')$, $g_{12}^{(0)}(\theta')$ are even and odd degree parts of the functions $g_1^{(0)}(\theta')$, i=1,2

After integrating by parts, the integrals occurring in the left-hand members of the two simultaneous integral equations (2.24) and (2.25), we get

$$\int_\beta^\alpha (\mathrm{I}_{11}^{(0)}(\theta')\left[\frac{1}{\cos\theta'-\cos\theta} - \cos\theta\right]\sin\theta' + \sin\theta\,\mathrm{I}_{12}^{(0)}(\theta')\left[\frac{1}{\cos\theta'-\cos\theta} - \cos\theta'\right]$$

$-\sin\theta\ \mathrm{I}_{21}^{(0)}(\theta')\sin\theta' + \cos\theta\ \mathrm{I}_{22}^{(0)}(\theta')\cos\theta')\,d\theta'$

$= -(\pi/(1-\tau^2))\,[(\tau^2\sin^2\phi\cos\theta + (1-2\tau^2\cos^2\phi)\cos\theta],\ \beta<\theta<\alpha,$ (2.26)

$$\int_\beta^\alpha (-\sin\theta\ \mathrm{I}_{11}^{(0)}(\theta')\sin\theta' + \cos\theta\,\mathrm{I}_{12}^{(0)}(\theta')\cos\theta' + \mathrm{I}_{21}^{(0)}(\theta')\sin\theta'\ [\frac{1}{\cos\theta'-\cos\theta} + \cos\theta]$$

$$\mathrm{I}_{22}^{(0)}(\theta')\sin\theta\left[\frac{1}{\cos\theta'-\cos\theta} + \cos\theta'\right]d\theta'$$

$= -(\pi/(1-\tau^2))\,[(\tau^2\sin^2\phi\sin\theta + (1-2\tau^2\sin^2\phi)\cos\theta],\ \beta<\theta<\alpha,$ (2.27)

where we have used

$\mathrm{I}_{ij}^{(0)}(\theta') = (d/d\theta')\,[g_{ij}^{(0)}(\theta')],\ i, j = 1, 2$ (2.28)

and the edge conditions

$$\int_\beta^\alpha \mathrm{I}_{ij}^{(0)}(\theta')\,d\theta' = 0,\ \ i,j = 1,2 \qquad (2.29)$$

which readily follow from the edge conditions $g(\pm\alpha) = 0$, $g_1(\pm\beta) = 0$.

Finally, to solve the above pair of simultaneous linear integral equations (2.26) and (2.27), we make the substitutions[8-11]

$\cos\theta' = A\cos y + B,\ \ \cos\theta = A\cos x + B,$ (2.30)

where

$A = ½(\cos\beta - \cos\alpha)$, $B = ½(\cos\beta + \cos\alpha)$, $0 < x, y < \pi$, when $\beta<\theta, \theta' <\alpha$.

The above substitutions and the right-hand sides of the integral equations (2.27) and (2.28) suggest that the solutions $I_{ij}^{(0)}(\theta')$, $i, j = 1, 2$, of the above integral equations are of the forms

$$I_{11}^{(0)}(\theta') = (C_i + D_i \cos y + E_i \cos^2 y)/(A \sin y),\ i = 1, 2, \tag{2.31}$$

$$I_{12}^{(0)}(\theta') = \sin\theta' + (F_i + G_i \cos y + H_i \cos^2 y)/(A \sin y),\ i = 1, 2, \tag{2.32}$$

where the constants C_i, D_i, E_i, F_i, G_i H_i 'S (i= 1, 2) are still to evaluated. In order to evaluate these constants in the solutions (2.31) and (2.32), we make the above substitutions (2.30) – (2.32) in the integral equations (2.27) and (2.28) and equate the constant terms, coefficients of sinθ, cosθ, sin2θ on both sides of the eqns. (2.27) and (2.28) respectively and get

$$D_1/A - 2BE_1/A^2 = 0, \qquad D_2/A - 2BE_2/A^2 = 0,$$

$$H_1 = 0, \quad H_2 = 0$$

$$2E_1/A_2 - C_1 + BF_2 + AG_2/2 = \tau^2 \sin^2\phi/(1-\tau^2),$$

$$2E_2/A_2 + C_2 + BF_1 + AG_1/2 = (1 - 2\tau^2 \sin^2\phi)/(1-\tau^2),$$

$$G_1/A - 2BH_1/A^2 - BF_1 - AG_1/2 - C_2 = (1 - 2\tau^2 \cos^2\phi)/(1-\tau^2),$$

$$G_2/A - 2BH_2/A^2 - BF_2 - AG_2/2 - C_1 = \tau^2 \sin^2\phi)/(1-\tau^2), \tag{2.33}$$

Substituting the values of the density functions $I_{ij}^{(0)}(\theta')$, $i, j = 1, 2$ from relations (2.31) and (2.32) in the edge conditions (2.29), we get

$$C_1J_0 + D_1J_1 + E_1J_2 = 0, \quad \int_0^{\pi} (F_1 + G_1 \cos y + H_1 \cos 2y)\, dy = 0,$$

where

$$J_n = \int_0^{\pi} \frac{\cos nt}{[1-(A\cos t + B)^2]^{1/2}}\, dt,\ n = 0, 1, 2, \ldots . \tag{2.33a}$$

After solving the above set of eqns. (2.33) and (2.33a), we get the required values of the constants;

$$E_1 = \frac{(\tau^2 \sin 2\phi)A^2}{(1-\tau^2)\left(2 + A^2 + \frac{2ABJ_1}{J_0} + \frac{A^2J_2}{J_0}\right)},$$

$$E_2 = \frac{[(1-2\tau^2 \sin 2\phi) - A^2(1-\tau^2)]A^2}{(1-\tau^2)\ (2 + A^2 + \frac{2ABJ_1}{J_0} + A^2J_2/J_0)},$$

$$C_1 = (2BJ_1/J_0 + AJ_2/J_0)E_1/A,\ D_1 = (2B/A)E_1,\ i = 1, 2,$$

$$G_1 = 2A - 2E_2/A,\ \ G_2 = 2E_1/A,\ H_1 = F_1 = 0,\ i = 1, 2. \tag{2.34}$$

Thus the eqns. (2.31) – (2.34) yield the following required expressions for $I_{ij}^{(0)}(\theta')$, ι, j = 1, 2;

$$I_{11}^{(0)}(\theta') = \frac{[1-A^2 - \frac{2ABJ_1}{J_0} - \frac{A^2J_2}{J_0} - 2B\cos\theta + \cos2\theta}{A^2[(\cos\beta-\cos\theta')(\cos\theta'-\cos\alpha)]^{1/2}}\ E_\iota,\ \iota=1, 2 \tag{2.35}$$

$$I_{12}^{(0)}(\theta') = \frac{(\cos\theta'-B)\sin\theta'}{A[(\cos\beta-\cos\theta')(\cos\theta'-\cos\alpha)]^{1/2}}\ G_\iota,\ \iota=1, 2, \tag{2.36}$$

where the constants E_ι and G_ι's ($i = 1, 2$) are given by eqns. (2.34).

Substituting the expressions of $I_{ij}^{(0)}(\theta')$, $i, j = 1, 2$ from eqns. (2.35) and (2.36) in the relations (2.28), (2.25a) and (2.23), we obtain the values of $g_\iota(\theta')$ up to the order $0(m_2^2)$. Fortunately, we do not require the values of $g_\iota(\theta')$, $i = 1, 2$, for deriving the expressions for the various physical quantities of interest in this problem. These can be readily derived by using the values of $I_{ij}^{(0)}(\theta')$, $\iota, j = 1, 2$, given by the relations (2.35) and (2.36).

III. Physical Quantities

A. FAR-FIELD AMPLITUDES AND SCATTERING CROSS SECTION

Using the polar coordinate (r, θ), where $x_1 = r\cos\theta$, $x_2 = r\sin\theta$ and the asymptotic formulas

$$H_0(m_jR) \approx (2/(\pi m_j r))^{1/2}\exp(im_jr-\pi/4))\exp(-\iota m_j\cos(\theta-\theta')),\ \text{as } r\to\infty, \tag{3.1}$$

where, $R = |x - x_c|$, $x_c = (\cos\theta', \sin\theta')$ in the integral representation formulas (2.10) for the components $u^S{}_1(x)$, we get

$$u^S{}_1(x) = (2/(\pi m_1 r))^{1/2} h_{\iota1}(\theta')\exp(im_1r-\pi/4))$$
$$+ (2/(\pi m_2 r))^{1/2} h_{\iota2}(\theta')\exp(i(m_2r-\pi/4)),\ i = 1, 2 \text{ as } r\to\infty, \tag{3.2}$$

where

$$h_{11}(\theta) = -\left(\frac{m_1}{4}\right)\cos\theta \int_{-\alpha}^{-\beta}\int_{\beta}^{\alpha}\{g_1(\theta')\,[(1-2\tau^2\sin^2\theta)\cos\theta'$$
$$+ \tau^2\sin^2\theta\sin\theta'] + g_2(\theta')\,[(1-2\tau^2\cos^2\theta)\sin\theta'$$
$$+ \tau^2\sin^2\theta\cos\theta']\}\exp(-im_1\cos(\theta-\theta'))\,d\theta', \tag{3.3}$$

$$h_{21}(\theta) = \tan\theta\, h_{11}(\theta), (3.4)$$

$$h_{12}(\theta) = (m_2/4)\sin\theta\int_{-\alpha}^{-\beta}\int_{\beta}^{\alpha}\{g_1(\theta')\sin(\theta'-2\theta) + g_2(\theta')\cos(\theta'-2\theta)\}$$
$$x\exp(-im_2\cos(\theta-\theta'))\,d\theta', \tag{3.5}$$

$$h_{22}(\theta) = -\cot\theta\, h_{12}(\theta). \tag{3.6}$$

Therefore,

$$u_r^s(r, \theta) \approx (2/(\pi m_1 r))^{1/2} f_1(\theta') \exp(i(m_1 r - \pi/4)),\ r\to\infty, \tag{3.7}$$

$$u_\theta^s(r, \theta) \approx (2/(\pi m_2 r))^{1/2} f_2(\theta') \exp(i(m_2 r - \pi/4)),\ r\to\infty, \tag{3.8}$$

$$\begin{aligned} f_1(\theta) &= h_{11}(\theta)\cos\theta + h_{21}(\theta)\sin\theta \\ &= -(\tau m^2/4)\,[(1-2\tau^2\sin^2\theta)\,p_{11}(\theta) + (1-2\tau^2\cos^2\theta)\,q_{21}(\theta) \\ &\quad + \tau^2\sin^2\theta\,(p_{21}(\theta) + q_{11}(\theta))], \end{aligned} \tag{3.9}$$

$$\begin{aligned} f_2(\theta) &= h_{22}(\theta)\cos\theta + h_{12}(\theta)\sin\theta \\ &= -(m^2/4)\,[\sin^2\theta(q_{22}(\theta) - p_{12}(\theta)) + \cos^2\theta\,(p_{22}(\theta) + q_{12}(\theta))], \end{aligned} \tag{3.10}$$

$$p_{ij}(\theta) = \int_{-\alpha}^{-\beta}\int_{\beta}^{\alpha} g_i(\theta')\cos\theta' \exp\left(-im_j\cos(\theta-\theta')\right) d\theta',\ i, j = 1,2, \tag{3.11}$$

$$p_{ij}(\theta) = \int_{-\alpha}^{-\beta}\int_{\beta}^{\alpha} g_i(\theta')\sin\theta' \exp\left(-im_j\cos(\theta-\theta')\right) d\theta',\ i, j = 1,2, \tag{3.12}$$

We get the following values of $p_{ij}(\theta)$ and $q_{ij}(\theta)$, $(i, j = 1, 2)$, after using the relations (2.23), (2.25a), the expansion of $\exp(-im_j \cos(\theta - \theta'))$, and the substitution of the values of $I_{ij}^{(0)}(\theta')$ after integrating by parts of the integrals occurring in the right hand side of eqns. (3.11) and (3.12);

$$p_{ij}(\theta) = A_0 m_2^2\,[-2\pi\, C_i + 0(m_j))],\ i, j = 1, 2, \tag{3.13}$$

$$p_{ij}(\theta) = A_0 m_2^2\,[\pi\, A\, G_i + 0(m_j))],\ i, j = 1, 2, \tag{3.14}$$

where C_i, G_i, $(i = 1, 2)$ are given by eqns. (2.34) and we have used the edge conditions $g_\iota(0)(\underline{+}\alpha) = 0$, $g_\iota(0)(\underline{+}\beta) = 0$.

Far-Field Amplitude

Finally, we obtain the required approximate expressions for the far-filed amplitudes $f_1(\theta)$ after substituting the values of above integrals (3.13) and (3.14) in eqns. (3.9) and (3.10);

$$\begin{aligned} f_1(\theta) = \frac{A_0\,\pi\, m_2^3\,\tau\, A}{2}\, &\{(1-\tau_2)\,(A + 2BJ_1/J_0 + AJ_2/J_0)(P_1/Q_1) \\ &-(\tau^2\cos^2\theta)\,(A - 2BJ_1/J_0 - AJ_2/J_0)\,(P_1/Q_1) \\ &-((\tau^2\sin^2\theta)/Q_2)\,[(A - 2BJ_1/J_0 - AJ_2/J_0)P_2 \\ &-2A^2\,(1 - 2ABJ_1/J_0 - A2J_2/J_0)], + 0(m_2)\}, \end{aligned} \tag{3.15}$$

$$\begin{aligned} f_2(\theta) = \frac{A_0\,\pi\, m_2^3\,\tau\, A}{2}\, &\{\sin^2\theta\,(A + 2BJ_1/J_0 + AJ_2/J_0)(P_1/Q_1) \\ &-(\cos^2\theta/Q_2)\,[(A - 2BJ_1/J_0 - AJ_2/J_0)P_2 \\ &-2A^2\,(1 - 2ABJ_1/J_0 - A2J_2/J_0)], + 0(m_2)\}, \end{aligned} \tag{3.16}$$

where

$$P_1 = \tau^2 \sin^2\phi/(1-\tau^2),\ P_2 = (1-2\tau^2 \sin^2\phi)/(1-\tau^2), \tag{3.17}$$

$$Q_1 = 2 + A^2 + 2AB\, J_1/J_0 + A_2J_2/J_1, \tag{3.18}$$

$$Q_2 = 2 - A^2 - 2AB\, J_1/J_0 - A_2J_2/J_1, \tag{3.19}$$

Scattering Cross section:

The value of the scattering cross section is obtained by substituting the above values of far-field amplitudes $f_1(\theta)$ and $f_2(\theta)$ in the formula

$$\Sigma_p = \frac{2a}{\pi m_1^3 |A_0|^2} \int_0^{2\pi} [|f_1(\theta)|^2 + \tau^2 |f_2(\theta)|^2]\, d\theta, \tag{3.20}$$

which readily yields[17]

$$\begin{aligned}\Sigma_p = \frac{\pi\ a\ m_2^3 A^2}{2\ \tau}\ &\{2(1-\tau^2)^2 (A + 2BJ_1/J_0 + AJ_2/J_0)^2 (P_1^2/Q_1^2)\\ &+ (1+\tau^4)\ \{(A - 2BJ_1/J_0 - AJ_2/J_0)^2 (P_1^2/Q_1^2)\\ &+ [(A - 2BJ_1/J_0 - AJ_2/J_0)P_2\\ &- 2A^2 (1 - 2ABJ_1/J_0 - A^2J_2/J_0)]^2\, Q_2^2\} + 0(m_2)\}.\end{aligned} \tag{3.21}$$

As far as the author know, the above results seem to be new.

IV. LIMITING RESULTS

CASE-I. ACircular Crack: Calculation of Far-field Amplitude and Scattering Cross section:

When, $\beta \to 0$, we derive from eqns. (3.21) the following results for the corresponding problem of an infinite circular crack

r=1, $-\alpha < \theta < \alpha$, $-\infty < x_3 < \infty$.

The value of the scattering cross section is obtained by substituting the above values of far-field amplitudes $f_1(\theta)$ and $f_2(\theta)$ as derived for earlier case.

$$\Sigma_{p*} = [\Sigma_p]_{\beta=0} = \frac{\pi\ a\ m_2^3\ A^2}{2\tau} \{2(1-\tau^2)^2 [\sin^2\left(\frac{\alpha}{2}\right)$$

$+ 2 \cos^2 (\alpha/2) J_1' / J_0' + \sin^2(\alpha/2) J_2'/J_0']^2 P_1^2/Q_1'^2$

$+ (1+\tau^4) [(\sin^2(\alpha/2) - 2\cos^2(\alpha/2) J_1'/J_0'$

$- \sin^2(\alpha/2) J_2'/J_0')^2 (P_1^2/Q_1'^2 + P_2^2/Q_2'_2)$

$- (4 \sin^2 (\alpha/2) /Q_2'^2) [(\sin^2(\alpha/2) - 2\cos^2 (\alpha/2) J_1'/J_0'$

$- \sin^2 (\alpha/2) J_2'/J_0') (1 - ½ \sin^2\alpha J_1'/J_0'$

$- \sin^4 (\alpha/2) J_2'/J_0')] + 4 \sin^4 (\alpha/2) [1 - ½ \sin^2\alpha J_1'/J_0'$

$$- \sin^4 (\alpha/2) J_2'/J_0']^2 + 0(m_2)] \tag{4.1}$$

where,

$$J_n' = \int_0^{\pi} \frac{\cos(nt)}{\left[1-\left(\sin^2\left(\frac{\alpha}{2}\right)\cos\tau + \cos^2\left(\frac{\alpha}{2}\right)\right)2\right]^{\frac{1}{2}}} dt, \tag{4.2}$$

$$Q_1' = 2+\sin^4\left(\frac{\alpha}{2}\right) + \frac{1}{2}\sin^2\alpha J_1' / J_0' + \sin^4\left(\frac{\alpha}{2}\right) J_2' / J_0', \tag{4.3}$$

$$Q_2' = 4 - Q_1', \tag{4.4}$$

CASE-II. An Infinite Semi-Circular Crack: Calculation of Far-field Amplitude and Scattering Cross section:

When $\alpha \dashrightarrow \frac{\pi}{2}$ in the above results(4.1), we get the corresponding problem of an infinite semi-circular crack occupying the region[17]

$r=1, -\frac{\pi}{2} < |\theta| < \frac{\pi}{2}, -\infty < x < \infty.$

As far as the authors know, even the above results for the two limiting configurations also seem to be new.

Case-III. Two equal Griffith cracks:

We let β, α, and $a \to \infty$ such that $a\beta \to a_2$, $a\alpha \to a_1$, $A_0 a^2/a^2_1 \to B_0$, ($c = \frac{a2}{a1} < 1$, when we assume that $0 < \alpha + \beta < \pi$), in the eqns. (3.21), and obtain the following corresponding result for the scattering cross section is obtained for two equal and parallel Griffith cracks;

$a_2 < |x_2| < a_1, x_1 = \infty, -\infty < x_3 < \infty$

$$\Sigma_p{}^{**} = \frac{\pi^2 \, a1 \, M2^3}{32\tau} \{[P2^2 (1+\tau^4)$$

$$+ (3\tau - 4\tau^2 + 3)\ P_1^2]\ (1+c^2-2E/F)^2 + 0(M_2)\} \quad (4.5)$$

Where

$$M_2 = \rho\omega^2 a^2{}_1/\mu,$$

Bo is known constant and F=F $\left(\frac{\pi}{2}, \sqrt{(1-c^2)}\right)$, E=E $\left(\frac{\pi}{2}, \sqrt{(1-c^2)}\right)$ are the complete elliptical integrals of first and second kind respectively. The above results agree with the known results[14, 17], for two equal parallel coplanar Griffith cracks; b $<|x_1| < a$, $x_2=0$, $-\infty<x_3<\infty$, when we interchange the values for P_1 and P_2, and change the values of a_1 to a, a_2 to be and $\phi$to $\left(\frac{\pi}{2} - \gamma\right)$. This serves as a check on our analysis presented here.

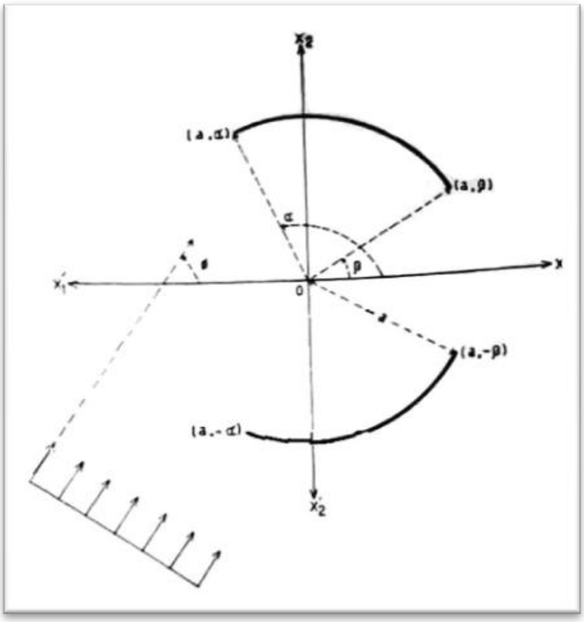

Fig. 1. Section of the two equal circular cracks in the $x_1 - x_2$ plane.

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