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An Integral Mean Estimate for the Polar Derivative of a Polynomial whose zeros are within a circle

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Abstract . Let $D_\alpha p(z)$ denote the polar derivative of a polynomial $p(z)$ of degree n with respect α. In this paper, we obtain certain integral inequalities for the polar derivative D_α of polynomials of the form $p(z) := a_n z^n + \sum\limits_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$ having all zeros in the disc $|z| \leq k$, $k \leq 1$. Our result refines and generalizes many prior results involving both polar as well as ordinary derivative.

Keywords: Polar derivative, Complex Polynomials, Inequalities, Zeros.

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1. Introduction

Let F_n denote the space of complex polynomials $F(z) := \sum\limits_{j=0}^{n} a_j z^j$ of degree $n \geq 1$. If $p \in F_n$, then concerning the estimate of $|p'(z)|$ on the unit circle, we have the following well-known result due to Bernstein [6].

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|. \tag{1.1}$$

If we consider the class of polynomials $f \in F_n$ having all zeros in $|z| \leq 1$, then the bound in inequality (1.1) can be considerably improved. In fact, Turán [16] proved that if $p \in F_n$ and $p(z)$ has all zeros in $|z| \leq 1$, then inequality (1.1) can be replaced by

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{1.2}$$

As an extension of (1.2), it was shown by Malik [11] that if $p \in F_n$ has all zeros in $|z| \leq k$ where $k \leq 1$, then

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{1+k} \max_{|z|=1} |p(z)|. \tag{1.3}$$

Since,

$$\left\{ \int_0^{2\pi} |p(e^{\iota\theta})|^s d\theta \right\}^{\frac{1}{s}} \to \max_{0 \leq \theta < 2\pi} |p(e^{\iota\theta})| \ as \ s \to \infty$$

which is the basic result from Analysis (for example see [14], p-70 or [15], p-91), it is an interesting problem to extend inequalities (1.1), (1.2) and (1.3) to L^s spaces. In this direction

Zygmund [17] was the first who extended inequality (1.1) to L^s, $s \geq 1$ spaces and proved the following result.

$$\left\{ \int_0^{2\pi} |p'(e^{\iota\theta})|^s d\theta \right\}^{\frac{1}{s}} \leq n \left\{ \int_0^{2\pi} |p(e^{\iota\theta})|^s d\theta \right\}^{\frac{1}{s}}. \tag{1.4}$$

Arestöv [1] proved that inequality (1.4) remains true for $0 < s < 1$ as well.
The result is best possible and equality holds for $p(z) = tz^n$, $t \neq 0$. If we let $s \to \infty$, we get inequality (1.1).
As an extension of inequality (1.3), to L^s, $s > 0$ space, we have the following inequality.

$$n\left\{\int_0^{2\pi} |p(e^{\iota\theta})|^s d\theta\right\}^{\frac{1}{s}} \leq \left\{\int_0^{2\pi} |1 + ke^{\iota\theta}|^s d\theta\right\}^{\frac{1}{s}} \left\{\int_0^{2\pi} |p'(e^{\iota\theta})|^s d\theta\right\}^{\frac{1}{s}}. \quad (1.5)$$

Inequality (1.5) was found by Aziz [2].
As a generalization of inequality (1.5) Aziz and Ahemad [3] proved that if $p \in F_n$ having all its zeros in $|z| \leq k$, $k \leq 1$, then for each $s > 0$, $p > 1$, $q > 1$ with $p^{-1} + q^{-1} = 1$,

$$n\left\{\int_0^{2\pi} |p(e^{\iota\theta})|^s d\theta\right\}^{\frac{1}{s}} \leq \left\{\int_0^{2\pi} |1 + ke^{\iota\theta}|^{qs} d\theta\right\}^{\frac{1}{qs}} \left\{\int_0^{2\pi} |p'(e^{\iota\theta})|^{ps} d\theta\right\}^{\frac{1}{ps}}. \quad (1.6)$$

Consider the class of polynomials $F_{n,\mu} := a_n z^n + \sum\limits_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$ of degree n. $F_{n,\mu}$ is a linear space and $F_{n,1} = F_n$. For these class of polynomials Aziz and Shah [5] generalized inequality (1.6) and proved that, if $p \in F_{n,\mu}$ having all zeros in $|z| \leq k$, $k \leq 1$, then for $s > 0$

$$n\left\{\int_0^{2\pi} |p(e^{\iota\theta})|^s d\theta\right\}^{\frac{1}{s}} \leq \left\{\int_0^{2\pi} |1 + k^{\mu} e^{\iota\theta}|^s d\theta\right\}^{\frac{1}{s}} \left\{\int_0^{2\pi} |p'(e^{\iota\theta})|^s d\theta\right\}^{\frac{1}{s}}. \quad (1.7)$$

For any complex number α consider the operator D_α which maps a polynomial $p(z)$ of degree n into

$$D_\alpha p(z) := np(z) + (\alpha - z)p'(z).$$

The operator $D_\alpha p(z)$ is known as polar differentiation of $p(z)$ with respect to α. $D_\alpha p(z)$ is a polynomial of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha \to \infty} \frac{D_\alpha p(z)}{\alpha} = p'(z).$$

As an extension of inequality (1.4) to the polar derivative of polynomial, we have the following inequality.

$$\left\{\int_0^{2\pi} |D_\alpha p(e^{\iota\theta})|^s d\theta\right\}^{\frac{1}{s}} \leq n(|\alpha| + 1)\left\{\int_0^{2\pi} |p(e^{\iota\theta})|^s d\theta\right\}^{\frac{1}{s}}, \quad (1.8)$$

The inequality (1.8) was first proved by Govil et al [9] but the proof of the theorem was not correct as was first pointed out by Aziz and Rather [4] who in the same paper have given the correct proof of the inequality (1.8) for $s \geq 1$. The inequality (1.8) is then independently proved by Rather [13] for $s > 0$. Recently, Dewan et al [7] proved that for every real or complex number α with $|\alpha| \geq k$ and for each $s > 0$,

$$n(|\alpha| - k)\left\{\int_0^{2\pi} |p(e^{\iota\theta})|^s\right\}^{\frac{1}{s}} \leq \left\{\int_0^{2\pi} |1 + ke^{\iota\theta}|^s d\theta\right\}^{\frac{1}{s}} \max_{|z|=1} |D_\alpha p(z)|. \quad (1.9)$$

In this paper, we besides extending inequality (1.9) to the class $F_{n,\mu}$ of polynomials, also find its generalization and refinement which in turn generalizes and refines many other results proved earlier.

2. Main Results.

In this section we state our main results:

Theorem 1. *Let $p \in \mathcal{P}_{n,\mu}$, $1 \le \mu \le n$, having all its zeros in $|z| \le k$, $k \le 1$, then for real or complex numbers α, β with $|\alpha| \ge A_\mu$, $|\beta| \le 1$ and $s > 0$,*

$$n(|\alpha| - A_\mu)\left\{ \int_0^{2\pi} \frac{\left|p(e^{\iota\theta}) + \frac{\beta A_\mu}{k^n} m_k\right|^s}{\left(|D_\alpha p(e^{\iota\theta})| - \frac{nA_\mu}{k^n} m_k\right)^s} d\theta \right\}^{\frac{1}{s}} \le \left\{ \int_0^{2\pi} |1 + A_\mu e^{\iota\theta}|^s d\theta \right\}^{\frac{1}{s}}, \tag{2.1}$$

where

$$A_\mu = \frac{n(|a_n| - \frac{m_k}{k^n})k^{2\mu} + \mu|a_{n-\mu}|k^{\mu-1}}{n(|a_n| - \frac{m_k}{k^n})k^{\mu-1} + \mu|a_{n-\mu}|}.$$

$$m = \min_{|z|=k} |p(z)|.$$

Next, we prove the following result.

Theorem 2. *Let $p \in F_{n,\mu}$ having all zeros in $|z| \le k$, $k \le 1$, then for every real or complex numbers α, β with $|\alpha| \ge A_\mu$, $|\beta| \le 1$, $s > 0$, $p > 1$, $q > 1$ with $p^{-1} + q^{-1} = 1$,*

$$n(|\alpha| - A_\mu)\left\{ \int_0^{2\pi} \left|p(e^{\iota\theta}) + \frac{\beta A_\mu}{k^n} m\right|^s d\theta \right\}^{\frac{1}{s}} \le \left\{ \int_0^{2\pi} |1 + A_\mu e^{\iota\theta}|^{ps} d\theta \right\}^{\frac{1}{ps}} \left\{ \int_0^{2\pi} \left(|D_\alpha p(e^{\iota\theta})| - \frac{nA_\mu}{k^n} m\right)^{qs} d\theta \right\}^{\frac{1}{qs}}, \tag{2.3}$$

where A_μ and m are same as defined in Theorem 1.

Letting $q \to \infty$ so that $p = 1$ in Theorem 2, we get the following result.

Corollary 2.1. Let $p \in F_{n,\mu}$ having all zeros in $|z| \le k$, $k \le 1$, then for every real or complex numbers α, β with $|\alpha| \ge A_\mu$, $|\beta| \le 1$, $s > 0$,

$$n(|\alpha| - A_\mu)\left\{ \int_0^{2\pi} \left|p(e^{\iota\theta}) + \frac{\beta A_\mu}{k^n} m\right|^s d\theta \right\}^{\frac{1}{s}} \le \left\{ \int_0^{2\pi} |1 + A_\mu e^{\iota\theta}|^s d\theta \right\}^{\frac{1}{s}} \left\{ \max_{|z|=1} |D_\alpha p(z)| - \frac{nA_\mu}{k^n} m \right\}. \tag{2.4}$$

Also if we let $p \to \infty$ so that $q = 1$ in Theorem 2, then under the same hypothesis as Corollary 2.1, we get the following inequality.

$$n(|\alpha| - A_\mu)\left\{ \int_0^{2\pi} \left|p(e^{\iota\theta}) + \frac{\beta A_\mu}{k^n} m\right|^s d\theta \right\}^{\frac{1}{s}} \le (1 + A_\mu)\left\{ \int_0^{2\pi} \left(|D_\alpha p(e^{\iota\theta})| - \frac{nA_\mu}{k^n} m\right)^s d\theta \right\}^{\frac{1}{s}}, \tag{2.5}$$

where A_μ and m are same as defined in Theorem 1.

If we divide both sides of inequality (2.3) by $|\alpha|$ and Let $|\alpha| \to \infty$, we get the following result.

Corollary 2.2. Let $p \in F_{n,\mu}$ having all zeros in $|z| \le k$, $k \le 1$, then for every real or complex number β with $|\beta| \le 1$, $s > 0$, $p > 1$, $q > 1$ with $p^{-1} + q^{-1} = 1$,

$$n\left\{ \int_0^{2\pi} \left| p(e^{\iota\theta}) + \frac{\beta A_\mu}{k^n} m \right|^s d\theta \right\}^{\frac{1}{s}} \le \left\{ \int_0^{2\pi} |1 + A_\mu e^{\iota\theta}|^{ps} d\theta \right\}^{\frac{1}{ps}} \left\{ \int_0^{2\pi} |p'(e^{\iota\theta}|^{qs} d\theta \right\}^{\frac{1}{qs}}. \qquad (2.6)$$

Letting $s \to \infty$ in inequality (2.3) and choosing argument of β suitably with $|\beta| = 1$, we get the following refinement of a result due to Dewan, Singh and Lal [8].

Corollary 2.3. Let $p \in F_{n,\mu}$ having all zeros in $|z| \le k$, $k \le 1$, then for every real or complex number α with $|\alpha| \ge A_\mu$,

$$\max_{|z|=1} |D_\alpha p(z)| \ge \frac{n(|\alpha| - A_\mu)}{1 + A_\mu} \left\{ \max_{|z|=1} |p(z)| + \frac{A_\mu}{k^n} \min_{|z|=k} |p(z)| \right\} + \frac{nA_\mu}{k^n} \min_{|z|=k} |p(z)|.$$

Equivalently,

$$\max_{|z|=1} |D_\alpha p(z)| \ge \frac{n}{1 + A_\mu} \left\{ (|\alpha| - A_\mu) \max_{|z|=1} |p(z)| + (|\alpha| + 1) \frac{A_\mu}{k^n} \min_{|z|=k} |p(z)| \right\}. \qquad (2.7)$$

3. Lemmas

For the proof of these theorems we need the following two lemmas which are due to Mir et al [12].

Lemma 1. If $p \in F_{n,\mu}$ having all its zeros in $|z| \le k$ where $k \le 1$ and $q(z) = z^n \overline{p(\frac{1}{\bar{z}})}$, then for $|z| = 1$,

$$|q'(z)| \le A_\mu |p'(z)| - \frac{nA_\mu}{k^n} m$$

and

$$\frac{\mu |a_{n-\mu}|}{n(|a_n| - \frac{m}{k^n})} \le k^\mu,$$

where $m = \min_{|z|=k} |p(z)|$ and A_μ is same as defined in Theorem 1.

Lemma 2. If $p \in F_{n,\mu}$ having all its zeros in $|z| \le k$ where $k \le 1$, then

$$A_\mu \le k^\mu,$$

where A_μ is as defined in Theorem 1.

4. Proofs of Theorem.

Proof of Theorem 1. Let $q(z) = z^n \overline{p(\frac{1}{\bar{z}})}$, then $p(z) = z^n \overline{q(\frac{1}{\bar{z}})}$. Therefore for $|z| = 1$,

$$|q'(z)| = |np(z) - zp'(z)|, \qquad |p'(z)| = |nq(z) - zq'(z)|. \qquad (4.1)$$

Now for every real or complex number β with $|\beta| \le 1$, we have by use of Lemma 1 and (4.1) for $|z| = 1$,

$$\left| q'(z) + \frac{n\bar{\beta} A_\mu z^{n-1}}{k^n} m \right| \le |q'(z)| + \frac{nA_\mu}{k^n} m \le A_\mu |p'(z)| = A_\mu |nq(z) - zq'(z)| \qquad (4.2)$$

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and

$$|D_\alpha p(z)| \geq |np(z) + (\alpha - z)p'(z)| \geq (|\alpha| - A_\mu)|p'(z)| + \frac{nA_\mu}{k^n}m. \tag{4.3}$$

Inasmuch as $p(z)$ has all zeros in $|z| \leq k$ where $k \leq 1$, by Guass-Lucas Theorem all zeros of $p'(z)$ also lie in $|z| \leq k \leq 1$. Therefore $z^{n-1}\overline{p'(\frac{1}{\bar{z}})} = nq(z) - zq'(z)$ does not vanish in $|z| < 1$. Hence,

$$w(z) = \frac{z\Big(q'(z) + \frac{n\overline{\beta}A_\mu z^{n-1}}{k^n}m\Big)}{A_\mu(nq(z) - zq'(z))}$$

is analytic in $|z| \leq 1$ and by inequality (4.2), $|w(z)| \leq 1$ for $|z| = 1$. Furthermore $w(0) = 0$. Thus the function $1 + A_\mu w(z)$ is subordinate to the function $1 + A_\mu z$ for $|z| = 1$. Hence by well-known property of subordination ([10] ,p-422), we have for $s > 0$

$$\int_0^{2\pi} |1 + A_\mu w(e^{\iota\theta})|^s d\theta \leq \int_0^{2\pi} |1 + A_\mu e^{\iota\theta}|^s d\theta. \tag{4.4}$$

Now,

$$1 + A_\mu w(z) = \frac{n\Big(q(z) + \frac{\overline{\beta}A_\mu z^n}{k^n}m\Big)}{nq(z) - zq'(z)}.$$

Therefore for $|z| = 1$, we have by use of equation (4.1)

$$n\Big|q(z) + \frac{\overline{\beta}A_\mu z^n}{k^n}m\Big| = |1 + A_\mu w(z)||p'(z)|.$$

Which further implies

$$n\Big|z^n\overline{p\Big(\frac{1}{\bar{z}}\Big)} + \frac{\overline{\beta}A_\mu z^n}{k^n}m\Big| = |1 + A_\mu w(z)||p'(z)|.$$

Hence for $|z| = 1$, we get

$$n\Big|p(z) + \frac{\beta A_\mu}{k^n}m\Big| = |1 + A_\mu w(z)||p'(z)|.$$

Thus with the help of inequality (4.3), we obtain

$$n(|\alpha| - A_\mu)\Bigg\{\int_0^{2\pi} \frac{\Big|p(e^{\iota\theta}) + \frac{\beta A_\mu}{k^n}m_k\Big|^s}{\Big(|D_\alpha p(e^{\iota\theta})| - \frac{nA_\mu}{k^n}m_k\Big)^s} d\theta\Bigg\}^{\frac{1}{s}} \leq \Bigg\{\int_0^{2\pi} |1 + A_\mu w(e^{\iota\theta})|^s d\theta\Bigg\}^{\frac{1}{s}},$$

which gives the required result with the help of (4.4).

Proof of Theorem 2. As in Theorem 1 we have

$$n^s(|\alpha| - A_\mu)^s \int_0^{2\pi} \Big|p(e^{\iota\theta}) + \frac{\beta A_\mu}{k^n}m\Big|^s d\theta \leq \int_0^{2\pi} |1 + A_\mu w(e^{\iota\theta})|^s \Big(D_\alpha p(e^{\iota\theta})| - \frac{nA_\mu}{k^n}m\Big)^s d\theta$$

Since $p > 1$ $q > 1$ and $p^{-1} + q^{-1} = 1$, therefore by Holders Inequality, we obtain

$$n(|\alpha| - A_\mu)\Bigg\{\int_0^{2\pi} \Big|p(e^{\iota\theta}) + \frac{\beta A_\mu}{k^n}m\Big|^s d\theta\Bigg\}^{\frac{1}{s}} \leq \Bigg\{\int_0^{2\pi} |1 + A_\mu w(e^{\iota\theta})|^{ps} d\theta\Bigg\}^{\frac{1}{ps}} \Bigg\{\Big(D_\alpha p(e^{\iota\theta})| - \frac{nA_\mu}{k^n}m\Big)^{qs} d\theta\Bigg\}^{\frac{1}{qs}}$$

which proves theorem 2 completely.

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