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Stability of Equilibrium in Dynamical Heterogeneous Duopoly

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ABSTRACT

In the mathematical economics, Cournot Duopoly model is very much popular among researchers. Duopoly is the sub-case of oligopoly, in which mainly two firms rule the market. Inspired by this model, many researches have been made in this field. Some researchers have discussed the effect of co-operation among firms involved in R& D activities, some other have made researches on non co-operative oligopoly. John P. Laitner appraises conjectural duopoly models as an alternative to non-conjectural ones. Some researchers worked on the homogeneous duopoly models, where firms use same type of strategies to earn profit. Here, we study heterogeneous Duopoly model, where players use heterogeneous strategies against each other. There exists three main types of strategies-naive, bounded rational and adaptive. In this paper, there is one model with linear cost and linear demand function, then there are two nonlinear Duopoly models, in one of which demand function is linear and cost function is nonlinear and then demand function is non linear and cost function of both the firms is linear. Existence and stability of Boundary and Nash equilibrium points is checked with the help of Jacobian matrix and Eigen values.

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Assumptions

Models have been formulated under assumption that goods produced are homogeneous, demand function is iso-elastic and both player use different strategies.

Linear Duopoly Model

The underlying assumption that in Duopoly, players are dealing in homogeneous goods which are perfect substitutes quantity supplied be x_i , where $i = 1,2$. Inverse Demand function is given by $Y = a - bX$, where a and b are positive constants. $X = x_1 + x_2$ is the total supply. Cost function is $C_i = c_i x_i$

So, Profit Function for the i[th] firm is

$$\pi_i = Yx_i - C_i$$

$$= x_i(a - bX) - c_i x_i \qquad i = 1,2$$

$$i.e.\ \pi_1 = x_1[a - b(x_1 + x_2)] - c_1 x_1$$

$$\pi_2 = x_2[a - b(x_1 + x_2)] - c_2 x_2$$

Each player wants to maximize his profit. So, in order to find profit maximizing quantity, the marginal profit is given by:

$$\frac{\partial \pi_1}{\partial x_1} = a - 2bx_1 - bx_2 - c_1$$

$$\frac{\partial \pi_2}{\partial x_2} = a - bx_1 - 2bx_2 - c_2$$

For $\frac{\partial \pi_1}{\partial x_1} = 0, \frac{\partial \pi_2}{\partial x_2} = 0$, equations are

$$a - c_1 - bx_2 - 2bx_1 = 0$$
$$a - c_2 - bx_1 - 2bx_2 = 0$$

Solving first equation gives

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$$x_1 = \frac{a - c_1 - bx_2}{2b} \tag{1}$$

By using the concept of maxima minima, we find that profit is the maximum for this value of x_1. This is reaction function for the first firm. Similarly, reaction function for the second firm is given by

$$x_2 = \frac{1}{2b}\left(a - bx_1 - c_2\right) \tag{2}$$

The general reaction function is

$$x_i = \frac{1}{2b}(a - b\sum_{\substack{j=1 \\ j\neq i}}^{2} x_j - c_i)$$

The first player is taken to be boundedly rational, second to be naïve player. Denote by $x_i(t)$ and $x_i(t+1)$, the output of the player i at the time t and $t+1$ respectively. The first player being boundedly rational makes his output decisions on the basis of the expected marginal profit. The dynamical equation of the first player is $x_1(t+1) = x_1(t) + \alpha\, x_1(t)\dfrac{\partial \pi_1}{\partial x_1(t)}$, $t = 0,1,2,3...$, $where\ \alpha > 0$ is the speed of adjustment. (3)

$$i.e.\ x_1(t+1) = x_1(t) + \alpha\, x_1(t)(a - 2bx_1 - bx_2 - c_1)\ \text{using(1)} \tag{4}$$

Also, the dynamical equation of the naive player is

$$x_2(t+1) = \frac{1}{2b}\left(a - bx_1 - c_2\right) \qquad \text{using (2)} \tag{5}$$

Boundary, Nash Equilibrium Points and their Stability

The equilibrium point of the Duopoly game is obtained by the nonnegative fixed point of the system of nonlinear equations (4) and (5).For finding fixed points it is needed to find $x_i(t+1) = x_i(t)$, $i = 1,2$ in each of (4) and (5), So, system of equations is given by

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$$x_1(a-2bx_1-bx_2-c_1)=0 \tag{6}$$

$$a-2bx_2-bx_1-c_2=0 \tag{7}$$

Equation (6) gives either $x_1=0$ or $x_1=\dfrac{a-bx_2-c_1}{2b}$

If $x_1=0$, then (7) gives $x_2=\dfrac{a-c_2}{2b}$

Also, for $x_1=\dfrac{a-bx_2-c_1}{2b}$, equation (7) gives

$$a-2bx_2-b\left(\frac{a-bx_2-c_1}{2b}\right)-c_2=0$$

$$i.e.\ \ x_2=\frac{a+c_1-2c_2}{3b}$$

$$i.e.\, x_1=\frac{a-b\left(\dfrac{a+c_1-2c_2}{3b}\right)-c_1}{2b}$$

$$i.e.\, x_1=\frac{a-2c_1+c_2}{3b}$$

$$\text{So,}\quad E_1=\left(0,\frac{a-c_2}{2b}\right) \tag{8}$$

$$\text{and}\qquad E_2=\left(\frac{a+c_2-2c_1}{3b},\frac{a+c_1-2c_2}{3b}\right) \tag{9}$$

Where, E_1 is boundary equilibrium point and E_2 is Nash Equilibrium point. To check the stability of the equilibrium point, The Jacobian matrix of the system of equations given by (4) and (5) at the equilibrium points is calculated first, then nature of Eigen values of this Jacobian matrix at the equilibrium points will determine the stability of equilibrium points. Jacobian matrix is given by

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$$J = \begin{bmatrix} \dfrac{\partial x_1'}{\partial x_1} & \dfrac{\partial x_1'}{\partial x_2} \\ \dfrac{\partial x_2'}{\partial x_1} & \dfrac{\partial x_2'}{\partial x_2} \end{bmatrix}$$

$$i.e \quad J = \begin{bmatrix} 1+\alpha\left(a-4bx_1-bx_2-c_1\right) & -\alpha bx_1 \\ -\dfrac{1}{2} & 0 \end{bmatrix}$$

At the boundary equilibrium point E_1, the Jacobian matrix is

$$J(E_1) = \begin{bmatrix} 1+\alpha\left(a-b\left(\dfrac{a-c_2}{2b}\right)-c_1\right) & 0 \\ -\dfrac{1}{2} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+\alpha\left(\dfrac{a-2c_1+c_2}{2}\right) & 0 \\ -\dfrac{1}{2} & 0 \end{bmatrix}$$

Let β be the Eigen values of $J(E_1)$. Then Eigen values will be obtained if :

$$\begin{vmatrix} 1+\alpha\left(\dfrac{a-2c_1+c_2}{2}\right)-\beta & 0 \\ -\dfrac{1}{2} & -\beta \end{vmatrix} = 0$$

$$i.e.if \ -\beta\left\{1+\alpha\left(\frac{a-2c_1+c_2}{2}\right)-\beta\right\} = 0$$

Eigen values of $J(E_1)$ are $\beta_1 = 1+\alpha\dfrac{a-2c_1+c_2}{2}$,

$\beta_2 = 0$. So, $|\beta_1| > 1$ is not unique.

Then E_1 is unstable fixed point of discrete dynamical system in (4) and (5)

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Similarly, at the Nash equilibrium point E_2, the Jacobian matrix is,

$$i.e\ \ J = \begin{bmatrix} 1+\alpha\left(a-4b\left(\frac{a+c_2-2c_1}{3b}\right)-b\left(\frac{a+c_1-2c_2}{3b}\right)-c_1\right) & -\alpha b\left(\frac{a+c_2-2c_1}{3b}\right) \\ -\frac{1}{2} & 0 \end{bmatrix}$$

(10)

Let γ be Eigen values of $J(E_2)$. Eigen values are obtained by taking:

$$\begin{vmatrix} 1+\alpha\left(a-\frac{4}{3}(a+c_2-2c_1)-\frac{1}{3}(a+c_1-2c_2)-c_1\right)-\gamma & -\frac{\alpha}{3}(a+c_2-2c_1) \\ -\frac{1}{2} & -\gamma \end{vmatrix} = 0$$

$$i.e.\ \begin{vmatrix} 1-\frac{2\alpha}{3}(a+c_2-2c_1)-\gamma & -\frac{\alpha}{3}(a+c_2-2c_1) \\ -\frac{1}{2} & -\gamma \end{vmatrix} = 0$$

$$\Rightarrow \left\{1-\frac{2\alpha}{3}(a+c_2-2c_1)-\gamma\right\}(-\gamma)-\frac{\alpha}{6}(a+c_2-2c_1)=0$$

$$\Rightarrow \gamma^2-\gamma\left(1-\frac{2\alpha}{3}(a+c_2-2c_1)-\frac{\alpha}{6}(a+c_2-2c_1)\right)=0$$

Eigen values of the above Jacobian matrix are roots of characteristic equation

$$\gamma^2+A_1\gamma+A_2=0,$$

$$A_1=-\left(1-\frac{2\alpha}{3}(a+c_2-2c_1)\right), A_2=-\frac{\alpha}{6}(a+c_2-2c_1)$$

(11)

Discriminate of above quadratic equation is given by

$$D=A_1^2-4A_2$$
$$=\left(1+\frac{2\alpha}{3}(a+c_2-2c_1)\right)^2+4\frac{\alpha}{6}(a+c_2-2c_1)$$

Clearly, $D>0$, which means Eigen values of Nash equilibrium are real.

Now Nash Equilibrium is locally stable if and only if

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$$
\begin{aligned}
&(i)\left|A_2\right|<1, \\
&(ii)\ 1-A_1+A_2>0 \\
&(iii)\ 1+A_1+A_2>0
\end{aligned}
\tag{12}
$$

The first condition is $\left|\frac{\alpha}{6}\left(a+c_2-2c_1\right)\right|<1$ which means $\alpha<\frac{6}{a-2c_1+c_2}$ (13)

Then second condition $1-A_1+A_2>0$ becomes $1+\left(1-\frac{2\alpha}{3}\left(a+c_2-2c_1\right)\right)-\frac{\alpha}{6}\left(a+c_2-2c_1\right)>0$

which gives $2-\frac{5\alpha\left(a+c_2-2c_1\right)}{6}>0$

i.e. $\frac{5\alpha\left(a+c_2-2c_1\right)}{6}<2$

i.e. $\alpha<\frac{12}{a+c_2-2c_1}$ (14)

Also, $1+A_1+A_2=1-\left(1-\frac{2\alpha}{3}\left(a+c_2-2c_1\right)\right)-\frac{\alpha}{6}\left(a+c_2-2c_1\right)$

$$=\frac{3\alpha}{6}\left(a+c_2-2c_1\right)$$

$$>0$$

So, third condition is satisfied.

From (13) and (14), it is clear that Nash equilibrium is stable if $\alpha<\frac{12}{a+c_2-2c_1}$

Duopoly Model with Linear Demand and Non-Linear Cost Function

Here it is assumed that in Duopoly, players are dealing in homogeneous goods which are perfect substitutes quantity supplied be x_i , where $i=1,2$. Inverse Demand function is given by $Y=a-bX$, where a and b are positive constants. $X=x_1+x_2$ is the total supply. Non-linear cost function is $C_i=c_ix_i^2$

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So, Profit Function for the i[th] firm is

$$\pi_i = Yx_i - C_i$$
$$= x_i(a - bX) - c_i x_i^2 \quad , i = 1,2.$$

Each player wants to maximize his profit. So, in order to find profit maximizing quantity, it is found that marginal profit

$$\frac{\partial \pi_1}{\partial x_1} = a - 2bx_1 - bx_2 - 2c_1 x_1 \tag{15}$$

For $\frac{\partial \pi_1}{\partial x_1} = 0$

$$x_1 = \frac{a - bx_2}{2(b + c_1)}$$

Also, $\frac{\partial \pi_2}{\partial x_2} = a - bx_1 - 2bx_2 - 2c_2 x_2$ gives

$$x_2 = \left(\frac{a - bx_1}{2(b + c_2)} \right) \tag{16}$$

Further investigation shows that for this value of x_1 and x_2 profit is the maximum. The general reaction function is

$$x_i = \frac{1}{2(b + c_i)}\left(a - b\sum_{\substack{j=1 \\ j \neq i}}^{2} x_j\right)$$

(17)

The first player is taken to be boundedly rational, second to be naïve player. Denote by $x_i(t)$ and $x_i(t+1)$, the output of the player i at the time t and $t+1$ respectively. The first player being boundedly rational makes his output decisions on the basis of the expected marginal profit. The dynamical equation of the first player is

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$x_1(t+1)=x_1(t)+\alpha\, x_1(t)\dfrac{\partial\pi_1}{\partial x_1(t)}$, $t=0,1,2,3...$, *where* $\alpha>0$ is the speed of adjustment.

$$i.e. x_1(t+1)=x_1(t)+\alpha\, x_1(t)(a-2(b+c_1)x_1-bx_2) \text{ using (14)} \tag{18}$$

Second player is naive player, the dynamical equation of the naive player is

$$x_2(t+1)=\frac{1}{2(b+c_2)}(a-bx_1)\text{using (14)} \tag{19}$$

Boundary, Nash Equilibrium Points and their Stability

Equations (18) and (19) collectively represent the discrete Dynamic system of duopoly game with heterogeneous competitors when cost function is nonlinear. The equilibrium point of the duopoly game is obtained by the non-negative fixed point of the system of nonlinear equations (18) and (19).Taking $x_i(t+1)=x_i(t)$, $i=1,2.$ in each of (18) and (19),

$$x_1(a-2(b+c_1)x_1-bx_2)=0 \tag{20}$$

$$a-bx_1-2(b+c_2)x_2=0 \tag{21}$$

From(18), either $x_1=0$ or $a-2(b+c_1)x_1-bx_2=0$

For $x_1=0$, equation (19) gives $x_2=\dfrac{a}{2(b+c_2)}$

For Solving $a-2(b+c_1)x_1-bx_2=0$ and

$$a-bx_1-2(b+c_2)x_2=0$$

Multiply first equation by '$2(b+c_2)$', second by 'b' and subtract, which gives

$$2a(b+c_2)-4(b+c_1)(b+c_2)x_1-ab+b^2x_1=0$$

$$\Rightarrow (ab+2ac_2)+\left[-3b^2-4b(c_1+c_2)-4c_1c_2\right]x_1=0$$

$$\Rightarrow x_1=\frac{a(b+2c_2)}{3b^2+4b(c_1+c_2)+4c_1c_2}$$

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Now, multiply first equations by 'b', second by '$2(b+c_2)$' and subtract, which gives

$$ab-b^2x_2-2a(b+c_1)+4(b+c_1)(b+c_2)x_2=0$$

$$x_2\left[3b^2+4b(c_1+c_2)+4c_1c_2\right]=-ab+2a(b+c_1)$$

$$\Rightarrow x_2=\frac{a(b+2c_1)}{3b^2+4b(c_1+c_2)+4c_1c_2}$$

For these two values of x_1 , we get two equilibrium points:

$$E_1=\left(0,\frac{a}{2(b+c_1)}\right) \text{ and } E_2=\left(\frac{a(b+2c_2)}{3b^2+4b(c_1+c_2)+4c_1c_2},\frac{a(b+2c_1)}{3b^2+4b(c_1+c_2)+4c_1c_2}\right)$$

(22)

Where, E_1 is boundary equilibrium point and E_2 is Nash Equilibrium point. To check the stability of the equilibrium point, the Jacobian matrix of the system of equations given by (18) and (19) at the equilibrium points is first calculated, then nature of Eigen values of this Jacobian matrix at the equilibrium points will determine the stability of equilibrium points.

Jacobian matrix is given by $$J=\begin{bmatrix}1+\alpha(a-4(b+c_1)x_1-bx_2) & -\alpha bx_1\\ -\dfrac{b}{2(b+c_2)} & 0\end{bmatrix} \qquad (23)$$

At the boundary equilibrium point E_1, the Jacobian matrix is

$$J(E_1)=\begin{bmatrix}1+\alpha\left(a-\dfrac{ab}{2(b+c_1)}\right) & 0\\ -\dfrac{b}{2(b+c_2)} & 0\end{bmatrix} \qquad (24)$$

Let λ_1 be the Eigen value of $J(E_1)$, The Eigen values of the $J(E_1)$ are given by:

$$\begin{vmatrix}1+\alpha a\dfrac{b+2c_1}{2(b+c_1)}-\lambda_1 & 0\\ -\dfrac{b}{2(b+c_2)} & -\lambda_1\end{vmatrix}=0$$

$$i.e.\left(1+\alpha a\frac{b+2c_1}{2(b+c_1)}-\lambda_1\right)\lambda_1=0$$

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$\lambda_{11} = 1 + \dfrac{\alpha a(b+2c_1)}{2(b+c_1)}$ and $\lambda_{2,3} = 0$.

So, $|\lambda_1| > 1$and $|\lambda_{2,3}| < 1$. Then $E_1$is saddle point of discrete dynamical system in (18) and (19)

Similarly, at the Nash equilibrium point E_2 , the Jacobian matrix is

$$J(E_2) = \begin{bmatrix} 1+\alpha\left(a - \dfrac{4a(b+c_1)(b+2c_2)}{3b^2+4b(c_1+c_2)+4c_1c_2} - \dfrac{ab(b+2c_1)}{3b^2+4b(c_1+c_2)+4c_1c_2}\right) & -\alpha ab\dfrac{b+2c_2}{3b^2+4b(c_1+c_2)+4c_1c_2} \\ -\dfrac{b}{2(b+c_2)} & 0 \end{bmatrix}$$

Eigen values x of $J(E_2)$ are given by:

$$\begin{vmatrix} 1+\alpha\left(a - \dfrac{4a(b+c_1)(b+2c_2)}{3b^2+4b(c_1+c_2)+4c_1c_2} - \dfrac{ab(b+2c_1)}{3b^2+4b(c_1+c_2)+4c_1c_2}\right) - x & -\alpha ab\dfrac{b+2c_2}{3b^2+4b(c_1+c_2)+4c_1c_2} \\ -\dfrac{b}{2(b+c_2)} & -x \end{vmatrix} = 0$$

On solving above determinant, equation obtained is given by:

$$x^2 - x\left\{1+\alpha\left(a - \frac{4a(b+c_1)(b+2c_2)}{3b^2+4b(c_1+c_2)+4c_1c_2} - \frac{ab(b+2c_1)}{3b^2+4b(c_1+c_2)+4c_1c_2}\right)\right\} - \frac{\alpha b^2 a(b+2c_2)}{2(b+c_2)(3b^2+4b(c_1+c_2)+4c_1c_2)} = 0$$

Eigen values of the above Jacobian matrix are roots of characteristic equation

$$x^2 + A_1 x + A_2 = 0, \tag{25}$$

where

$$A_1 = -\left\{1+\alpha\left(a + \frac{-4a(b+c_1)(b+2c_2) - ab(b+2c_1)}{3b^2+4b(c_1+c_2)+4c_1c_2}\right)\right\},\ A_2 = -\frac{\alpha b^2 a(b+2c_2)}{2(b+c_2)(3b^2+4b(c_1+c_2)+4c_1c_2)}$$

Eigen values of Nash equilibrium are real if equation (25) has real roots, which is possible if Discriminate = $A_1^2 - 4A_2 > 0$

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$$i.e.\left\{1+\alpha\left(a+\frac{-4a(b+c_1)(b+2c_2)-ab(b+2c_1)}{3b^2+4b(c_1+c_2)+4c_1c_2}\right)\right\}^2+4\frac{\alpha b^2 a(b+2c_2)}{2(b+c_2)(3b^2+4b(c_1+c_2)+4c_1c_2)}>0$$

, which is true.

Hence, Real Eigen values of Nash equilibrium points will be obtained.

Now Nash Equilibrium is asymptotically stable if all Eigen values given in eq. (25) has magnitude less than one. Which is possible if f

$(i)\ 1-A_1+A_2>0$

$(ii) 1+A_1+A_2>0$

$(iii)\ |A_2|<1$

Second condition becomes

$$-\alpha a\left(1-\frac{4(b+c_1)(b+2c_2)}{3b^2+4b(c_1+c_2)+4c_1c_2}-\frac{b(b+2c_1)}{3b^2+4b(c_1+c_2)+4c_1c_2}\right)-\frac{\alpha ab^2(b+2c_2)}{2(b+c_2)(3b^2+4b(c_1+c_2)+4c_1c_2)}$$

$$i.e.-\alpha a+\frac{4\alpha a(b+c_1)(b+2c_2)}{3b^2+4b(c_1+c_2)+4c_1c_2}+\frac{\alpha ab(b+2c_1)}{3b^2+4b(c_1+c_2)+4c_1c_2}-\frac{\alpha ab^2(b+2c_2)}{2(b+c_2)(3b^2+4b(c_1+c_2)+4c_1c_2)}$$

$$i.e.-\alpha a+\frac{\alpha a(b+2c_2)}{3b^2+4b(c_1+c_2)+4c_1c_2}\left[4(b+c_1)-\frac{b^2}{2(b+c_2)}\right]+\frac{\alpha ab(b+2c_1)}{3b^2+4b(c_1+c_2)+4c_1c_2}$$

$$i.e.-\alpha a+\frac{\alpha a(b+2c_2)}{3b^2+4b(c_1+c_2)+4c_1c_2}\left[\frac{7b^2+8b(c_1+c_2)+8c_1c_2}{2(b+c_2)}\right]+\frac{\alpha ab(b+2c_1)}{3b^2+4b(c_1+c_2)+4c_1c_2}$$

$$i.e.\frac{\alpha a}{2(b+c_2)(3b^2+4b(c_1+c_2)+4c_1c_2)}\left[\begin{array}{l}-2(b+c_2)(3b^2+4b(c_1+c_2)+4c_1c_2)+(b+2c_2)(7b^2+8b(c_1+c_2)+8c_1c_2)\\+2b(b+2c_1)(b+c_2)\end{array}\right]$$

$$i.e.\frac{\alpha a}{2(b+c_2)(3b^2+4b(c_1+c_2)+4c_1c_2)}\left[3b^3+10b^2c_2+8b(c_1+c_2)c_2+8c_1c_2^2+4b^2c_1+4bc_1c_2\right]$$

So

$$1+A_1+A_2=\frac{\alpha a}{2(b+c_2)(3b^2+4b(c_1+c_2)+4c_1c_2)}\left[3b^3+10b^2c_2+8b(c_1+c_2)c_2+8c_1c_2^2+4b^2c_1+4bc_1c_2\right]>0$$

Also, from third condition $|A_2|<1$

$$if\ |A_2|-1<0$$

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$$if\ \ \alpha a\left[\frac{b^3+2b^2c_2}{2(b+c_2)(3b^2+4b(c_1+c_2)+4c_1c_2)}\right]-1<0$$

$$i.f\ \alpha a\left[\frac{b^3+2b^2c_2}{2(b+c_2)(3b^2+4b(c_1+c_2)+4c_1c_2)}\right]<1$$

$$if\ \ \alpha<\frac{2(b+c_1)(3b^2+4b(c_1+c_2)+4c_1c_2)}{a(b^3+2b^2c_2)}$$

From first condition, $1-A_1+A_2>0$

$$i.e.\ 1+1+\alpha\left(a-\frac{4a(b+c_1)(b+2c_2)}{3b^2+4b(c_1+c_2)+4c_1c_2}-\frac{ab(b+2c_1)}{3b^2+4b(c_1+c_2)+4c_1c_2}\right)-\frac{\alpha b^2a(b+2c_2)}{2(b+c_2)(3b^2+4b(c_1+c_2)+4c_1c_2)}>0$$

$$i.e.\ 2+a\alpha\left(1-\frac{4(b+c_1)(b+2c_2)}{3b^2+4b(c_1+c_2)+4c_1c_2}-\frac{b(b+2c_1)}{3b^2+4b(c_1+c_2)+4c_1c_2}\right)-\frac{\alpha b^2a(b+2c_2)}{2(b+c_2)(3b^2+4b(c_1+c_2)+4c_1c_2)}>0$$

$$i.e.\ a\alpha\left(1-\frac{4(b+c_1)(b+2c_2)}{3b^2+4b(c_1+c_2)+4c_1c_2}-\frac{b(b+2c_1)}{3b^2+4b(c_1+c_2)+4c_1c_2}\right)-\frac{\alpha b^2a(b+2c_2)}{2(b+c_2)(3b^2+4b(c_1+c_2)+4c_1c_2)}>-2$$

$$i.e.\ a\alpha\left(1-\frac{4(b+c_1)(b+2c_2)}{3b^2+4b(c_1+c_2)+4c_1c_2}-\frac{b(b+2c_1)}{3b^2+4b(c_1+c_2)+4c_1c_2}\right)-\frac{\alpha b^2a(b+2c_2)}{2(b+c_2)(3b^2+4b(c_1+c_2)+4c_1c_2)}>-2$$

$$i.e.2(b+c_2)a\alpha(3b^2+4b(c_1+c_2)+4c_1c_2-4b^2-8bc_2-4bc_1-8c_1c_2-b^2-2bc_1)-\alpha b^2a(b+2c_2)>-4(b+c_2)(3b^2+4b(c_1+c_2)+4c_1c_2)$$

$$i.e.\ 2(b+c_2)a\alpha(-2b^2-2bc_1-4bc_2-4c_1c_2)-\alpha b^2a(b+2c_2)>-4(b+c_2)(3b^2+4b(c_1+c_2)+4c_1c_2)$$

$$i.e.\ 2(b+c_2)a\alpha[-2b(b+c_1)-4c_2(b+c_1)]-\alpha b^2(b+2c_2)>-4(b+c_2)(3b^2+4b(c_1+c_2)+4c_1c_2)$$

$$i.e.\ -4(b+c_2)a\alpha[(b+c_1)(b+2c_2)]-\alpha b^2a(b+2c_2)>-4(b+c_2)(3b^2+4b(c_1+c_2)+4c_1c_2)$$

$$i.e.\alpha a(b+2c_2)[4(b+c_1)(b+c_2)+b^2]<4(b+c_2)(3b^2+4b(c_1+c_2)+4c_1c_2)$$

$$\alpha a(b+2c_2)[4b^2+4bc_1+4bc_2+4c_1c_2+b^2]<4(b+c_2)(3b^2+4b(c_1+c_2)+4c_1c_2)$$

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$$i.e.\alpha < \frac{4(b+c_2)(3b^2+4bc_1+4bc_2+4c_1c_2)}{a(b+2c_2)(5b^2+4bc_1+4bc_2+4c_1c_2)}$$

Duopoly Model with Non-Linear Demand and Linear Cost Function

Under above mentioned assumptions let quantity supplied be x_i , where $i=1,2$. $X = x_1 + x_2$ is the total supply of goods in the market. Iso- elasic inverse demand function is given by $Y = \frac{1}{X}$. Here, cost function $C_i = c_i x_i \;\; i=1,2.$ is linear. So, Profit Function for the 1st and 2nd firms are:

$$\pi_1 = Yx_1 - C_1$$

$$= \frac{x_1}{(x_1+x_2)} - c_1 x_1$$

$$\pi_2 = Yx_2 - C_2$$

$$= \frac{x_2}{(x_1+x_2)} - c_2 x_2$$

As mentioned above , in order to find profit maximizing level of output, the marginal profit and value of output is found for which

$$\frac{\partial \pi_i}{\partial x_i} = 0, i=1,2.$$

For $\frac{\partial \pi_1}{\partial x_1} = 0$

$$\frac{x_1 + x_2 - x_1}{(x_1+x_2)^2} - c_1 = 0$$

$$\Rightarrow \frac{x_2}{(x_1+x_2)^2} - c_1 = 0 \qquad \Rightarrow \frac{(x_1+x_2)^2}{x_2} = \frac{1}{c_1} \qquad \Rightarrow x_1 = \frac{\sqrt{x_2}}{\sqrt{c_1}} - x_2$$

Also

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$$\frac{\partial \pi_2}{\partial x_2} = 0$$
$$\Rightarrow \frac{x_1 + x_2 - x_2}{(x_1 + x_2)^2} - c_2 = 0$$
$$\Rightarrow \frac{x_1}{(x_1 + x_2)^2} - c_2 = 0$$
$$\Rightarrow \frac{(x_1 + x_2)^2}{x_1} = \frac{1}{c_2}$$
$$\Rightarrow x_2 = \frac{\sqrt{x_1}}{\sqrt{c_2}} - x_1$$

As assumed in above model, first player is boundedly rational, second is naïve player, So , output of first player at time *(t+1)* is given by $x_1(t+1) = x_1(t) + \alpha\, x_1(t)\frac{\partial \pi_1}{\partial x_1(t)}$ $, t = 0,1,2,3...$ $, where\ \alpha > 0$ is the speed of adjustment.

$$x_1(t+1) = x_1(t) + \alpha x_1(t)\left[\frac{x_2}{(x_1 + x_2)^2} - c_1\right] \tag{26}$$

Dynamical equation of naïve player is given by $x_2(t+1) = \sqrt{\frac{x_1(t)}{c_2}} - x_1(t)$ (27)

Equations (26) and (27) represents the two dimensional discrete dynamical system of the firms.

Boundary and Nash Equilibrium Points and their Stability

To find equilibrium point, the nonnegative fixed point of the system of nonlinear equations is calculated as given by (26) and (27).Taking $x_i(t+1) = x_i(t)$ $, i = 1,2$ in (26) and (27), equations become

$$x_1(t)\left[\frac{x_2(t)}{(x_1(t) + x_2(t))^2} - c_1\right] = 0 \tag{28}$$

$$\sqrt{\frac{x_1(t)}{c_2}} - x_1(t) = x_2(t) \tag{29}$$

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Either $x_1 = 0$ or $\frac{x_2}{(x_1+x_2)^2} = c_1$ (Using equation (28))

Either $x_1 = 0$ or $\frac{\sqrt{x_2}}{\sqrt{c_1}} = x_1 + x_2$ (30)

Substituting value of $x_1 = 0$ in equation (29) , $x_2 = 0$. (31)

For $x_1 \neq 0$, equation (29) and (30) gives

$$\frac{\sqrt{x_2(t)}}{\sqrt{c_1}} = x_1(t) + x_2(t) \text{ and}$$

$$\frac{\sqrt{x_1(t)}}{\sqrt{c_2}} = x_1(t) + x_2(t) \quad (32)$$

$$i.e. \sqrt{\frac{x_2(t)}{c_1}} = \sqrt{\frac{x_1(t)}{c_2}} \quad (33)$$

$$i.e. x_2(t) = \frac{c_1}{c_2} x_1(t)$$

Substituting value from (33) in (32), equation obtained is $\sqrt{\frac{x_1(t)}{c_2}} = x_1(t) + \frac{c_1}{c_2} x_1(t)$

i.e. $\sqrt{\frac{x_1(t)}{c_2}} = x_1(t)\left(\frac{c_1+c_2}{c_2}\right)$

On squaring both sides, $\frac{x_1(t)}{c_2} = x_1^2(t)\left(\frac{c_1+c_2}{c_2}\right)^2$,

$$x_1(t) = \frac{c_2}{(c_2+c_1)^2}$$

From eq. (33) $x_2(t) = \frac{c_1}{(c_2+c_1)^2}$

$$E_3 = \left(\frac{c_2}{(c_2+c_1)^2}, \frac{c_1}{(c_2+c_1)^2}\right) \quad (34)$$

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E_3 is Nash Equilibrium point.

Again to check the stability of the equilibrium point, the Jacobian matrix of the system of equations given by (26) and (27) is calculated at the equilibrium points, Jacobian matrix is given by

$$J = \begin{bmatrix} 1+\alpha\left[\dfrac{x_2^2 - x_1x_2}{(x_1+x_2)^3} - c_1\right] & \alpha x_1\left[\dfrac{x_1 - x_2}{(x_1+x_2)^3}\right] \\ \dfrac{1}{2\sqrt{x_1c_2}} - 1 & 0 \end{bmatrix} \tag{35}$$

At the Nash equilibrium point E_4, the Jacobian matrix is $J(E_4) = \begin{bmatrix} 1 & \dfrac{\alpha c_1(c_2 - c_1)}{c_1 + c_2} \\ \dfrac{c_1 - c_2}{2c_2} & 0 \end{bmatrix}$

Let Eigen values of the above Jacobin matrix be λ, which are given by

$$\begin{vmatrix} 1-\lambda & \dfrac{\alpha c_1(c_2 - c_1)}{c_1 + c_2} \\ \dfrac{c_1 - c_2}{2c_2} & -\lambda \end{vmatrix} = 0$$

Eigen values of the above Jacobian matrix are roots of characteristic equation

$\lambda^2 + A_1\lambda + A_2 = 0,$, where $A_1 = -1$, $A_2 = \dfrac{\alpha c_1(c_1 - c_2)^2}{2c_2(c_1 + c_2)}$

Eigen values of Nash equilibrium are real if discriminate > 0

$i.e.$ if $A_1^2 + 4A_2 > 0$

$i.e.$ if $1 + 4\dfrac{\alpha c_1(c_1 - c_2)^2}{2c_2(c_1 + c_2)} > 0$ which is true.

Nash equilibrium is asymptotically stable if all eigen values has magnitude less than one. Necessary and sufficient condition for local stability of Nash equilibrium point is given by

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1. $1 - A_1 + A_2 > 0$
2. $1 + A_1 + A_2 > 0$
3. $|A_2| < 1$

Now Nash equilibrium is locally stable if and only if

From first condition,

$$1 - A_1 + A_2 > 0$$
$$i.e. 1 + 1 + \frac{\alpha c_1 (c_1 - c_2)^2}{2c_2 (c_1 + c_2)} > 0$$

$$i.e. 2 + \frac{\alpha c_1 (c_1 - c_2)^2}{2c_2 (c_1 + c_2)} > 0 \quad i.e. \alpha > -\frac{4c_2 (c_1 + c_2)}{c_1 (c_1 - c_2)^2}$$

From second condition, $1 + A_1 + A_2 > 0$

$$i.e. \frac{\alpha c_1 (c_1 - c_2)^2}{2c_1 (c_1 + c_2)} > 0,$$ which is true.

From third condition,

$$\frac{\alpha c_1 (c_1 - c_2)^2}{2c_1 (c_1 + c_2)} < 1$$

Conclusion

The present study discusses stability condition of the two types of equilibrium points- Boundedly rational and Nash equilibrium points of linear and non-linear duopoly models. Models are formulated under the assumption that two players use heterogeneous strategies. One player is considered boundedly rational, second is naive. Dynamical equations of two heterogeneous players are formulated. For checking the stability conditions, Jacobian matrices are used. The eigen values of these Jacobian matrices determine the stability of equilibrium points. It is seen that Boundary equilibrium point is saddle point and Nash equilibrium is locally stable under certain conditions. These conditions are derived in all the three cases.

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