International Journal of Advance Research in Science and Engineering
Volume No.07, Special Issue No.04, March 2018
www.ijarse.com
IJARSE
ISSN: 2319-8354

Weighted power function distribution from Azzalinis's family

Sofi Mudasir[1]*, S.P. Ahmad[2], A.A. Bhat[3]

[1,2,3]*Department of Statistics, University of Kashmir, Srinagar, (India)*

ABSTRACT

Power function distribution is a flexible life time distribution that may propose a good fit to different sets of failure data and provides more suitable information about reliability and hazard rates. In this paper, we derive a new weighted version of power function distribution which is known as weighted power function distribution based on a modified weighted version of Azzalinis's approach. The statistical properties of this model are discussed and derived, including rth moment, survival function, hazard rate function, mode, harmonic mean. Also the ordered statistics and information measures of a new model are derived and studied.

Keywords: AIC and BIC, Ordered Statistics, Shannon's Entropy, Weighted Distribution

I.INTRODUCTION

The power function distribution is a flexible life time distribution that may propose a good fit to different sets of failure data and provides more suitable information about reliability and hazard rates and thus preferred over some distributions such as exponential, lognormal and Weibull distributions. Meniconi and Barry (1996) discussed the application of Power function distribution and showed that this distribution is the best distribution to test the reliability of electrical component as compared to Exponential, lognormal and Weibull distributions by studying the reliability and hazard functions. **S**aran and Pandey (2004) have put forward the concept of record values which are found in many situations of daily life as well as in many statistical applications. By using the order statistics they have obtained the best linear unbiased estimates of the parameter of the power function distribution in terms of kth upper record values. Chang (2007) presents characterizations of the power function distribution by independence of record values. Rahman et al. (2012) obtains the Bayes estimates of the power function distribution by using different symmetric and asymmetric loss functions. Zaka and Akhter (2013) discuss different methods for estimating the parameters of Power function distribution. Sultan et al. (2014) discussed the problem of Bayesian estimation for power function distribution under different priors.

Based on a modified weighted version of Azzalinis's (1985) approach, in this paper a new generalization of the power function distribution has been proposed. If $f_0(x)$ is the probability density function (pdf) and $\overline{F}_0(x)$ is the corresponding survival function such that the cumulative distributive function (cdf) $F_0(x)$ exists, then the new weighted distribution is defined as:

$$f_w(x;\alpha,\lambda) = kf_0(x)\overline{F}_0(\lambda x) \tag{1}$$

Where k is the normalizing constant

The probability density function of the weighted power function distribution (WPFD) has been derived from the definition given in equation (1). The pdf of the power function distribution is given as

$$f(x;\alpha) = \alpha x^{\alpha-1}; \;\; 0 < x < 1 \tag{2}$$

Where $\alpha > 0$ is the shape parameter.

The cumulative distribution function corresponding to (2) is given by

$$F(x;\alpha) = x^{\alpha} \tag{3}$$

The survival function is given by

$$S(x) = 1 - x^{\alpha} \tag{4}$$

Also, $$k = \frac{2}{2-\lambda^{\alpha}} \tag{5}$$

By using the equations (1), (2), (4) and (5), we obtain the pdf of the weighted power function distribution and is given by

$$f_w(x;\alpha,\lambda) = \frac{2\alpha}{2-\lambda^{\alpha}} x^{\alpha-1}\left(1-(\lambda x)^{\alpha}\right), \alpha, \lambda > 0 \tag{6}$$

The cumulative distribution function corresponding to (6) is given by

$$F_w(x;\alpha,\lambda) = \frac{x^{\alpha}}{2-\lambda^{\alpha}}\left(2-(\lambda x)^{\alpha}\right)$$

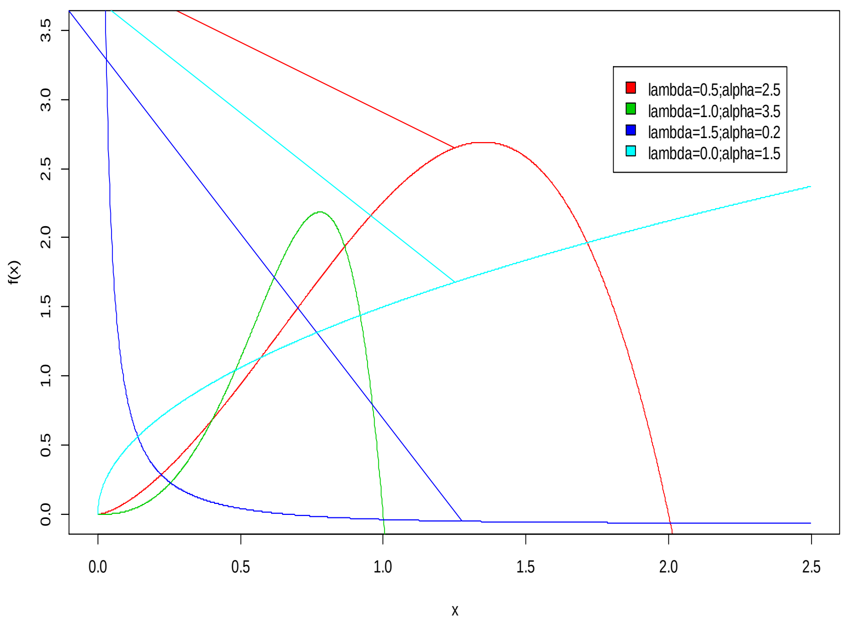

Figure 1: Probability distribution function of weighted power function distribution

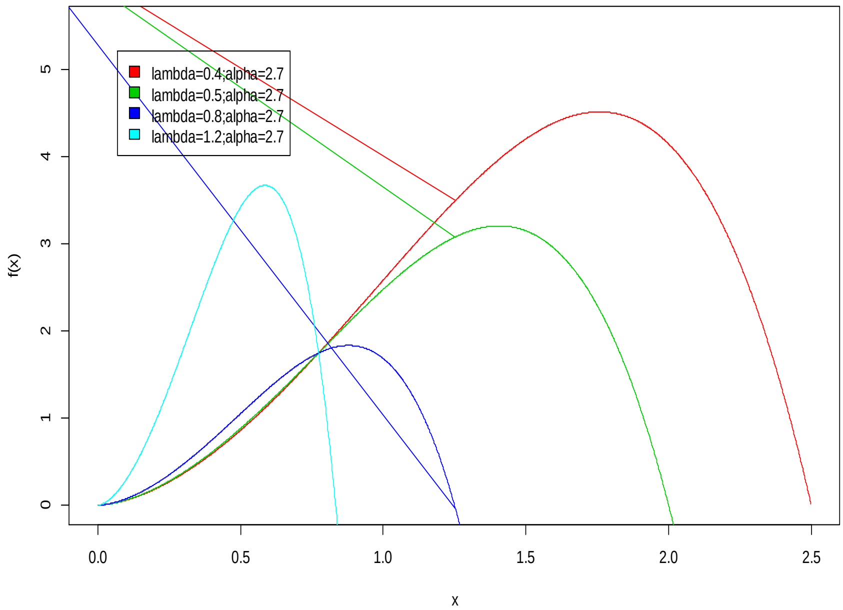

Figure 2: Probability distribution function of weighted power function distribution

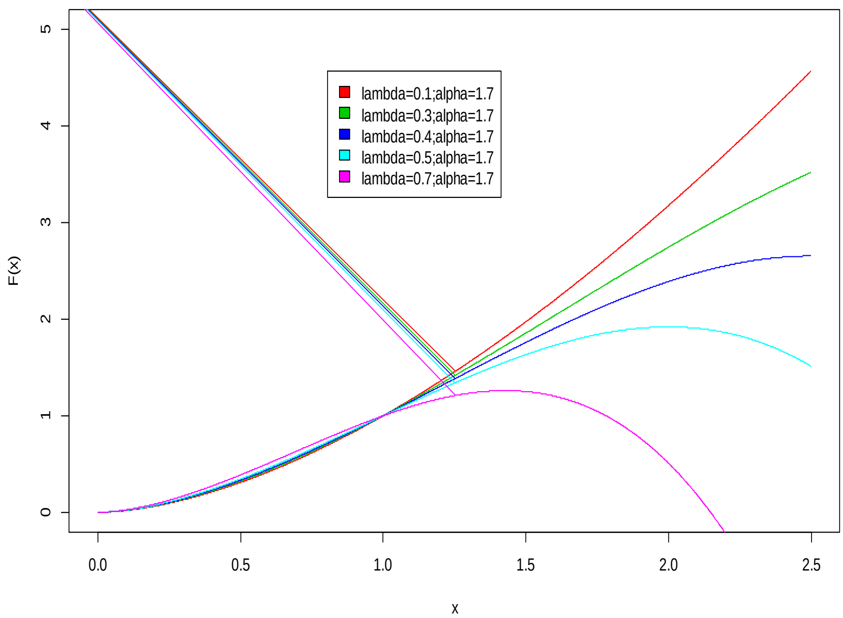

Figure 3: Cumulative distribution function of weighted power function distribution

II.BASIC PROPERTIES OF WEIGHTED POWER FUNCTION DISTRIBUTION

In this section we have studied the basic properties of weighted power function distribution. We first evaluate the rth moment of the proposed distribution as under:

$$\mu_r' = E(x^r) = \int_0^1 x^r f_w(x;\alpha,\lambda)dx$$

$$\Rightarrow \qquad \mu_r' = \frac{2\alpha}{2-\lambda^\alpha}\left[\frac{1}{\alpha+r} - \frac{\lambda^\alpha}{2\alpha+r}\right] \tag{7}$$

If r=1 in eq. (7), we get the mean of our model and is given by

$$\mu_1' = \frac{2\alpha}{2-\lambda^\alpha}\left[\frac{1}{\alpha+1} - \frac{\lambda^\alpha}{2\alpha+1}\right]$$

If r=2, in eq. (7), we get

$$\mu_2' = \frac{2\alpha}{2-\lambda^\alpha}\left[\frac{1}{\alpha+2} - \frac{\lambda^\alpha}{2\alpha+2}\right]$$

Now, variance denoted by μ_2 is given by

$$\mu_2 = \mu_2' - (\mu_1')^2$$

$$= \frac{2\alpha}{2-\lambda^\alpha}\left[\left(\frac{1}{\alpha+2} - \frac{\lambda^\alpha}{2\alpha+2}\right) - \frac{2\alpha}{2-\lambda^\alpha}\left(\frac{1}{\alpha+1} - \frac{\lambda^\alpha}{2\alpha+2}\right)^2\right]$$

The survival function is given by

$$S_w(x) = 1 - F_w(x;\alpha,\lambda)$$

$$= 1 - \frac{x^\alpha}{2-\lambda^\alpha}\left(2 - (\lambda x)^\alpha\right)$$

And the hazard rate function is

$$h_w(x) = \frac{f_w(x;\alpha,\lambda)}{S_w(x)}$$

$$= \frac{2\alpha x^{\alpha-1}\left(1-(\lambda x)^\alpha\right)}{2\lambda^\alpha - \left(2x^\alpha - \lambda^\alpha x^{2\alpha}\right)}$$

Moment Generating Function

The moment generating function of density (6) can readily be obtained as

$$M_X(t) = E(e^{tx})$$

$$\Rightarrow \qquad M_X(t) = \int_0^1 e^{tx} f_w(x;\alpha,\lambda)dx$$

$$\Rightarrow \qquad M_X(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\frac{2\alpha}{2-\lambda^\alpha}\left[\frac{1}{\alpha+r}-\frac{\lambda^\alpha}{2\alpha+r}\right]$$

MODE OF WEIGHTED POWER FUNCTION DISTRIBUTION

The mode of the weighted power function distribution is obtained by finding the first derivative of $\log\left(f_w(x;\alpha,\lambda)\right)$ with respect to x and equating to zero.

Therefore, the mode at $x=x_0$ is given by

$$\frac{d}{dx}\left[\log\left(\frac{2\alpha}{2-\lambda^\alpha}x^{\alpha-1}\left(1-(\lambda x)^\alpha\right)\right)\right]=0$$

$$\Rightarrow \qquad x_0=\left(\frac{\alpha-1}{2\alpha\lambda^\alpha-\lambda^\alpha}\right)^{\frac{1}{\alpha}}$$

HARMONIC MEAN

The harmonic mean denoted by H is given as

$$\frac{1}{H}=\int_0^1\frac{1}{x}f_w(x;\alpha,\lambda)dx$$

$$\Rightarrow \qquad \frac{1}{H}=\frac{2\alpha}{2-\lambda^\alpha}\int_0^1 x^{\alpha-2}\left(1-(\lambda x)^\alpha\right)dx$$

$$\Rightarrow \qquad \frac{1}{H}=\frac{2\alpha}{2-\lambda^\alpha}\left[\frac{1}{\alpha-1}-\frac{\lambda^\alpha}{2\alpha-1}\right]$$

$$\Rightarrow \qquad H=\frac{2-\lambda^\alpha}{2\alpha}\left[\frac{(2\alpha-1)(\alpha-1)}{(2\alpha-1)-\lambda^\alpha(\alpha-1)}\right]$$

III. ORDERED STATISTICS

Let $X_{(1)}, X_{(2)}$ and $X_{(r)}$ denote the smallest, second smallest and the r^{th} smallest of $\{X_1, X_2, ..., X_n\}$. Then the random variables $X_{(1)}, X_{(2)}, ..., X_{(n)}$ are called the ordered statistics of the sample $\{X_1, X_2, ..., X_n\}$. The probability density function of the r^{th} order statistics $X_{(r)}$ is given as:

$$f_{X_{(r)}}(x)=\frac{n!}{(r-1)!(n-r)!}f(x)\left[F_X(x)\right]^{r-1}\left[1-F_X(x)\right]^{n-r}$$

For r=1,2,…, n

The probability density function of the r^{th} order statistic of the weighted power function distribution is given as

$$f_{X_{(r)}}(x)=\frac{n!}{(r-1)!(n-r)!}\frac{2\alpha}{2-\lambda^{\alpha}}x^{\alpha-1}\left(1-(\lambda x)^{\alpha}\right)\left[\frac{1}{2-\lambda^{\alpha}}\left(2x^{\alpha}-\lambda^{\alpha}x^{2\alpha}\right)\right]^{r-1}\left[1-\frac{1}{2-\lambda^{\alpha}}\left(2x^{\alpha}-\lambda^{\alpha}x^{2\alpha}\right)\right]^{n-r}$$

$$=\frac{2\alpha}{\left(2-\lambda^{\alpha}\right)^{r}}\frac{n!}{(r-1)!(n-r)!}x^{\alpha-1}\left(1-(\lambda x)^{\alpha}\right)\left(2x^{\alpha}-\lambda^{\alpha}x^{2\alpha}\right)^{r-1}\left[1-\frac{1}{2-\lambda^{\alpha}}\left(2x^{\alpha}-\lambda^{\alpha}x^{2\alpha}\right)\right]^{n-r} \tag{8}$$

Now, by putting r=1, n in equation (8), we get the probability density function of the smallest $X_{(1)}$ and the largest $X_{(n)}$ ordered statistics as

$$f_{X_{(1)}}(x)=\frac{2\alpha n}{2-\lambda^{\alpha}}x^{\alpha-1}\left(1-(\lambda x)^{\alpha}\right)\left[1-\frac{2x^{\alpha}-\lambda^{\alpha}x^{2\alpha}}{2-\lambda^{\alpha}}\right]^{n-1}$$

$$f_{X_{(n)}}(x)=\frac{2\alpha n}{\left(2-\lambda^{\alpha}\right)^{n}}x^{\alpha-1}\left(1-(\lambda x)^{\alpha}\right)\left(2x^{\alpha}-\lambda^{\alpha}x^{2\alpha}\right)^{n-1}$$

IV. SHANNON'S ENTROPY OF WEIGHTED POWER FUNCTION DISTRIBUTION

For deriving the Shannon's entropy of probability distribution, we need the following definition that more details can be found in Thomas J.A. et al. (1991). The oblivious generalizations of the definition of entropy for a probability density function $f(x)$ defined on the real line as

$$H\left[f(x)\right]=-\int_{-\infty}^{\infty}\log\left\{f(x)\right\}f(x)dx=E\left[-\log f(x)\right] \tag{9}$$

Provided the integral exists.

From equation (6), we have

$$\log\left\{f_w(x;\alpha,\lambda)\right\}=\log\left(\frac{2\alpha}{2-\lambda^{\alpha}}\right)+(\alpha-1)\log x+\log\left(1-(\lambda x)^{\alpha}\right) \tag{10}$$

Substitute the value of equation (10) in equation (9), we get

$$H\left[f_w(x;\alpha,\lambda)\right]=-\log\left(\frac{2\alpha}{2-\lambda^{\alpha}}\right)-(\alpha-1)E(\log x)-E\left(\log\left(1-(\lambda x)^{\alpha}\right)\right) \tag{11}$$

Now, $E(\log x)=\frac{2\alpha}{2-\lambda^{\alpha}}\int_{0}^{1}\log(x)x^{\alpha-1}\left(1-(\lambda x)^{\alpha}\right)dx$

$$\Rightarrow \quad E(\log x)=\frac{2\alpha}{2-\lambda^{\alpha}}\left[\int_0^1 \log(x)x^{\alpha-1}dx-\lambda^{\alpha}\int_0^1 x^{2\alpha-1}\log(x)dx\right]$$

On solving the above integral, we get

$$E(\log x)=\frac{\lambda^{\alpha}-4}{2\alpha(2-\lambda^{\alpha})} \tag{12}$$

Also, $$E\left(\log\left(1-(\lambda x)^{\alpha}\right)\right)=\frac{2\alpha}{2-\lambda^{\alpha}}\int_0^1 \log\left(1-(\lambda x)^{\alpha}\right)x^{\alpha-1}\left(1-(\lambda x)^{\alpha}\right)dx$$

$$=\frac{1}{2-\lambda^{\alpha}}\left(\frac{\left(2\lambda^{\alpha}-\lambda^{2\alpha}-1\right)}{\lambda^{\alpha}}\log(1-\lambda^{\alpha})+\frac{\lambda^{\alpha}-2}{2}\right) \tag{13}$$

On substituting the value of equations (12) and (13) in equation (11), we get

$$H\left[f_w(x;\alpha,\lambda)\right]=-\log\left(\frac{2\alpha}{2-\lambda^{\alpha}}\right)-\frac{1}{2-\lambda^{\alpha}}\left[\frac{(\alpha-1)(\lambda^{\alpha}-4)}{2\alpha}+\frac{\left(2\lambda^{\alpha}-\lambda^{2\alpha}-1\right)}{\lambda^{\alpha}}\log(1-\lambda^{\alpha})+\frac{\lambda^{\alpha}-2}{2}\right]$$

This is the required Shannon's entropy for weighted power function distribution.

V. ENTROPY ESTIMATION OF WEIGHTED POWER FUNCTION DISTRIBUTION

In order to introducing an approach for model selection, we remember Akaike and Bayesian information criterion based on entropy estimation.

Suppose that we have a statistical model of some data. Let L be the likelihood function for the model. Let K be the number of estimated parameters in the model. Then the AIC value of the model is the following

$$AIC=2K-2Log(L) \tag{14}$$

Given a set of candidate models for the data, the preferred model is the one which has the minimum AIC value.

The BIC was developed by Gideon E. Schwarz and published in a (1978) paper, where he gave a Bayesian argument for adopting it. The BIC is formally defined as

$$BIC=KLog(n)-2Log(L) \tag{15}$$

Where, n is the number of observations or equivalently the sample size.

Now, likelihood function of equation (6) is given as

$$L(x;\alpha,\lambda)=\left(\frac{2\alpha}{2-\lambda^{\alpha}}\right)^{n}\prod_{i=1}^{n}x_i^{\alpha-1}\prod_{i=1}^{n}\left(1-(\lambda x_i)^{\alpha}\right)$$

By taking log on both sides, we get

$$Log\left(L(x;\alpha,\lambda)\right)=n\log\left(\frac{2\alpha}{2-\lambda^{\alpha}}\right)+(\alpha-1)\sum_{i=1}^{n}\log x_i+\sum_{i=1}^{n}\log\left(1-(\lambda x_i)^{\alpha}\right)$$

$$\Rightarrow \quad \frac{-Log\left(L(x;\alpha,\lambda)\right)}{n} = -\log\left(\frac{2\alpha}{2-\lambda^{\alpha}}\right) - (\alpha-1)E(\log x) - E\left(\log\left(1-(\lambda x_i)^{\alpha}\right)\right) \tag{16}$$

On comparing equations (11) and (16), we get

$$\log L = -nH\left[f_w(x;\lambda,\beta,\theta)\right]$$

Thus from equations (14) and (15), we have

$$AIC = 2K + 2nH\left[f_w(x;\alpha,\lambda)\right]$$

$$BIC = K\log n + 2nH\left[f_w(x;\alpha,\lambda)\right]$$

VI.MAXIMUM LIKELIHOOD ESTIMATION

Consider a random sample of size n, consisting of values $x_1, x_2, ..., x_n$ from the weighted power function distribution with the probability density function given in equation (6), then the log-likelihood function can be immediately written as

$$Log\left(L(x;\alpha,\lambda)\right) = n\log(2\alpha) - n\log(2-\lambda^{\alpha}) + (\alpha-1)\sum_{i=1}^{n}\log x_i + \sum_{i=1}^{n}\log\left(1-(\lambda x_i)^{\alpha}\right) \tag{17}$$

Taking the partial derivatives of the log-likelihood function in (17) with respect to the parameters α and λ yields:

$$\frac{\partial Log\left(L(x;\alpha,\lambda)\right)}{\partial\alpha} = \frac{n}{\alpha} + \frac{n\lambda^{\alpha}\log\lambda}{2-\lambda^{\alpha}} + \sum_{i=1}^{n}\log x_i - \sum_{i=1}^{n}\frac{(\lambda x_i)^{\alpha}\log(\lambda x_i)}{1-(\lambda x_i)^{\alpha}}$$

$$\frac{\partial Log\left(L(x;\alpha,\lambda)\right)}{\partial\lambda} = \frac{n\alpha\lambda^{\alpha-1}}{2-\lambda^{\alpha}} - \alpha\lambda^{\alpha-1}\sum_{i=1}^{n}\frac{x_i^{\alpha}}{1-(\lambda x_i)^{\alpha}}$$

Eq

uating the above equations to zero, we get the normal equations

$$\frac{n}{\alpha} + \frac{n\lambda^{\alpha}\log\lambda}{2-\lambda^{\alpha}} + \sum_{i=1}^{n}\log x_i - \sum_{i=1}^{n}\frac{(\lambda x_i)^{\alpha}\log(\lambda x_i)}{1-(\lambda x_i)^{\alpha}} = 0 \tag{18}$$

$$\frac{n\alpha\lambda^{\alpha-1}}{2-\lambda^{\alpha}} - \alpha\lambda^{\alpha-1}\sum_{i=1}^{n}\frac{x_i^{\alpha}}{1-(\lambda x_i)^{\alpha}} = 0 \tag{19}$$

To find the maximum likelihood estimators of α and λ, we have to solve the equations (18) and (19) simultaneously.

VII. FISHER INFORMATION MATRIX

For the two parameters of WPFD all the second order derivatives of the log-likelihood function exist. Thus the inverse dispersion matrix is given by:

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\lambda} \end{pmatrix} \sim \mathrm{N}\left(\begin{pmatrix} \alpha \\ \lambda \end{pmatrix}, \begin{pmatrix} \hat{V}_{\alpha\alpha} & \hat{V}_{\alpha\lambda} \\ \hat{V}_{\lambda\alpha} & \hat{V}_{\lambda\lambda} \end{pmatrix} \right)$$

$$V^{-1} = -E\begin{pmatrix} \hat{V}_{\alpha\alpha} & \hat{V}_{\alpha\lambda} \\ \hat{V}_{\lambda\alpha} & \hat{V}_{\lambda\lambda} \end{pmatrix}$$

Where $\hat{V}_{\alpha\alpha} = \frac{\partial^2 L}{\partial\alpha\partial\alpha}$, $\hat{V}_{\alpha\lambda} = \frac{\partial^2 L}{\partial\alpha\partial\lambda}$, $\hat{V}_{\lambda\lambda} = \frac{\partial^2 L}{\partial\lambda\partial\lambda}$, $\hat{V}_{\lambda\alpha} = \frac{\partial^2 L}{\partial\lambda\partial\alpha}$

By deriving the inverse dispersion matrix, the asymptotic variances and covariances of the maximum likelihood estimators for α and λ are obtained.

REFERENCES

[1] M. Meniconi, and D.M. Barry, The Power Function Distribution: A Useful and Simple Distribution to Assess Electrical Component Reliability, *Journal of Microelectron Reliability, Vol. 36,* (1996), 1207-1212.

[2] J. Saran and A.Pandey, Estimation of Parameters of a Power Function Distribution and Its Characterization by Kth Record Values, *Journal Statistica, vol. 14,* (2004), 523-536.

[3] S.K. Chang, Characterizations of the Power Function Distribution by the Independence of Record Values, *Journal of the Chungcheong Mathematical Society, 20 (2),* (2007),139-146.

[4] H. Rahman, M. K. Roy, and A.K. Baizid, Bayes Estimation under Conjugate Prior for the Case of Power Function Distribution, *American Journal of Mathematics and statistics, vol. 2,* (2012), 44-48.

[5] A. Zaka, and A.S. Akhter, Methods for estimating the parameters of the Power Function Distribution, *Pakistan Journal of Statistics and Operational Research, vol. 9,* (2013), 213-224.

[6] R. Sultan, H. Sultan, and S.P. Ahmad, Bayesian Analysis of Power Function Distribution under Different Priors, *Journal of Statistics Application and Probability, vol. 3,* (2014), 239-249.

[7] A. Azzalini, A class of distributions which includes the normal ones, *Scandinavian Journal of Statistics, vol. 12,* (1985),171-178.