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ASYMPTOTIC EXPANSION OF SECOND AND FOURTH PAINLEVE EQUATION

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ABSTRACT

Investigating the existence of an asymptotic expansion to all orders for the solution of (P II), within a certain 'range of the initial value parameters; we then give an improved (more explicit) version of the R.H. treatment for (P II) in the particular case we are interested in (i.e., α= 0, solutions bounded on the real axis). We then take advantage of this linearization to reduce the connection problem to a rather intricate-stationary phase analysis; from it we extract the amplitude part of the connection formula (presumably the phase part could also be derived in this way, but we did not manage to digit out). The form of the expansion and the integral equation also shed some light on the behaviour of the solution, in the large, and it fits especially well with the appearance of poles on the real axis when $|r|$ becomes larger than 1.

Keywords ***Asymptotic expansion, meromorphic, rational transformation, riccati equation, transcendence degree.***

I ASYMPTOTIC INTRODUCTION

In the past few years, much work has been devoted to the so called Painleve equations. Let us recall that these were first studied by Painleve and his pupils-around 1900 he recognized that they were essentially the only second-order equations (besides the elementary ones: linear, Riccati, elliptic equations,...) such that the movable singularities are poles (i.e., movable essential singularities are excluded). It was also shown that Eq. (P VI) is a particular case of the Schlesinger equations which describe the isomonodromic deformations of linear equations with regular singular points. The other equations can be obtained by a limiting procedure. Interest in the Painleve equations revived when Ablowitz, Ramani, and Segur showed that reductions of PDE's solvable by the inverse spectral transform should be of Painled type, and consequently should reduce to elementary equations (this almost never occurs) or to one of the six Painleve transcendants. Shortly after, the Japanese school undertook the generalization of the deformation theory to linear systems with irregular singular points and found various results on the Painled equations as a by-product.

The present paper deals with some properties of the second Painleve equation

$$w'' = tw + 2w^3 + \propto$$

and we shall in fact consider the special case $\propto = 0$; it is the only case where it is known that there exist solutions bounded on the real r-axis.

Flachka and Newell are the compatibility condition between two linear systems with rational coefficients and reduced it to a Riemann-Hilbert (R.H.) problem along a certain contour, the solution of which was equivalent to that of a singular integral equation. In Fokas and Ablowitz made the solution more explicit by disentangling the R. H. problem into a cascade of three R. H. problems along lines, or equivalently, a sequence of three Fredholm equations, the kernel of each one being given through the solution of the preceding one. It may be amusing to notice that they used a trick already employed by Beals and Coiffman, but which can also be found almost explicitly in Birkhoffs study of the Riemann problem. In the linearization (reduction to linear integral equations) via a R.H. problem, one of the advantages, as pointed out is that the independent variable t appears only as a parameter and the integration is performed on a spectral parameter z this is to contrast with the Gelfand-Levitan equation approach, where the integration is over t. Although the difference more or less amounts to a Fourier transform, it was suggested that the R.H. approach could be put to use in order to solve the so-called connection problem. Let us first recall its formulation; for any r $\in$ (0, 1) there exists a solution such that

$$w(t) \sim_{t=+\infty} r.1/(2\sqrt{\pi})\ t^{-\frac{1}{4}} \exp\left(-\frac{2}{3}\ t^{\frac{3}{2}}\right)$$

The problem consists in describing the asymptotic behaviour of this solution when t approaches $-\infty$; it still has the form of the Airy function, but with other values of the parameters. More precisely

$$w(t) \sim_{t=-\infty} d|t|^{-1/4} \sin\theta(t)$$

$$\theta(t) \sim \frac{2}{3}\ |t|^{3/2} - \frac{3}{4}\ d^2 \log|t| + \theta_0$$

These formulas can be found readily by a formal asymptotic expansion of the solution near $-\infty$. The global connection problem consists in finding the two functions d= d(r) (amplitude) and $\theta_0 = \theta_0(r)$ (phase). We show below that the solution does in fact admit an asymptotic expansion is the first term and we recover the amplitude formula

$$d^2(r) = -\frac{1}{\pi}\ \log(1 - r^2)$$

We are also able to give some details on the terms of the expansion. This was first obtained by ingenious roundabout methods and, as the authors themselves pointed out, it was rather a “derivation” than a proof, since many steps rely on formal asymptotic expansions for an associated PDE which would be extremely difficult to rigorize. Using the Gelfand-Levitan equation we proved recently by Mac Leod and Clarkson that they have obtained the phase formula

$$\theta_0 = \frac{\pi}{4} - arg\{\Gamma(1 - i/2d^2)\} - \frac{3}{2}\log 2$$

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which was formally derived. Let us notice that this connection problem is physically important because it provides a rather ubiquitous model for the reflection through a nonlinear caustic. The Japanese school has also obtained using the z function, results which pertain to the connection problems associated with some of the Painleve equations, but it may be worth pointing out that, since they consider the theory from the deformation viewpoint, (P VI) appears to be the simplest equation and (P I) and (P II) the most difficult to deal with (even though they are much simpler to write down) because they correspond to the coalescence of many singularities; this explains that it cannot be found explicitly-nor be deduced in a straightforward way-from their work.

The paper is organized as follows. We first prove the existence of an asymptotic expansion to all orders for the solution of (P II), within a certain 'range of the initial value parameters; we then give an improved (more explicit) version of the R.H. treatment for (P II) in the particular case we are interested in (i.e., α= 0, solutions bounded on the real axis). We then take advantage of this linearization to reduce the con- nection problem to a-rather intricate-stationary phase analysis; from it we extract the amplitude part of the connection formula (presumably the phase part could also be derived in this way, but we did not manage to digit out). The form of the expansion and the integral equation also shed some light on the behaviour of the solution, in the large, and it fits especially well with the appearance of poles on the real axis when $|r|$ becomes larger than 1.

II ASYMPTOTIC DEVELOPMENTS

In this part, we will prove the existence of asymptotic developments of a certain form for the solution of (P II). Since we are interested in the oscillating part of the solution, it will be convenient to change t into -t, and also to perform a scaling on the unknown function. we thus write the equation

$f'' + tf = 2c^2 f^3$

where f is defined on some interval $[t_0, +\infty]$ and c is a complex number noticed that if c=0, (E_0) is a Airy equation, and for all c, cf$(-t)$ is the Painleve transcendent.

In this we settled the problem of the irreducibility of the first differential equation of Painleve. Namely we proved that no solution of the first Painleve equation is classical. So the first Painleve equation defines highly transcendental functions different from the classical functions. The proof depends on the condition introduced in which is of arithmetic nature and plays an important role in the proof of the irreducibility of the first equation.

Our framework tells us that if an ordinary algebraic differential equation of second order satisfies the condition, then no transcendental solution of the differential equation is classical.

In this paper we discuss in this framework the irreducibility of the second and fourth equations of Painleve.

$$\frac{d^2w}{dz^2} = 2w^3 + zw + \propto \qquad P_{II}$$

$$\frac{d^2w}{dz^2} = \frac{1}{2w}\left(\frac{dw}{dz}\right)^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - \propto)w + \frac{\beta}{w} \qquad P_{IV}$$

Since, for particular values of the complex parameters α, β the equations $P_{II}(\propto)$ and $P_{IV}(\propto,\beta)$ have algebraic solutions, or classical solutions rationally expressed by solutions of Riccati equations. Our objective should be the determination of all the classical solutions of the equations. To this end, we have to do the following:

(i) To show that the second and fourth equations satisfy the condition for general α, β.

(ii) To determine transcendental classical solutions for particular values of the parameters α, β for which the second and fourth equations do not satisfy the condition.

(iii) To list up all the algebraic solutions.

In their proofs, we use birational transformations between solutions of a Painleve equation. As for (iii) Murata determined algebraic solutions of second and fourth equations. (i) and (ii) were done by Noumi and Okamoto for the second and fourth equations respectively. Murata worked out with the third Painleve equation. All these works were done in the above framework. But in these works the authors checked the arithmetic condition by straightforward calculations. The calculations are hard particularly in the fourth and third equations, so that there is little hope of applying their calculations to the fifth and sixth Painleve equations. We analysed the note of Okamoto on the fourth Painleve equation and tried to simplify his argument so that we can treat the fifth and sixth equations. We succeeded in this attempt, and we are preparing papers on solutions of the third, fifth and sixth equations. The aim of the present paper is to explain our method for the second and fourth equations.

III SECOND PAINLEVE EQUATION

The second Painleve equation is equivalent to the following system $S_{II}(\propto)$ of ordinary differential equations of first order

$$S_{II}(\propto) \quad \frac{dq}{dt} = p - q^2 - \frac{t}{2} \quad \frac{dp}{dt} = 2pq + \propto + \frac{1}{2}$$

where $\propto$ is a complex parameter. In fact, if we eliminate the unknown p from $S_{II}(\propto)$, we get the second Painleve equation

$$P_{II}(\propto) \quad \frac{d^2 q}{dt^2} = 2q^3 + tq + \propto$$

So the second Painleve equation $P_{II}(\propto)$ and the system $S_{II}(\propto)$ are parametrized by the complex line C.

We review birational transformations of solutions of the system $S_{II}(\propto)$associated with a group of complex affine transformations of the complex line C. We define affine transformations s, t_+, t_- of C by $s(\propto) = -1 - \propto, t_-(\propto) = \propto - 1, t_+(\propto) = \propto + 1 \ for \ \propto \epsilon$C. Let G be the subgroup generated by them in the group of complex affine transformations of the complex line C. Then the group G is isomorphic to the semidirect product of a cyclic group (s) of order two and a group (t_+, t_-). Since the latter group is isomorphic to the additive group of the integers Z, we find $G \cong \frac{Z}{2K} \propto Z$, so that it is isomorphic to the affine Weyl group of the root system of type. Let C_D be the subset of C that consists of all the complex numbers a satisfying the following conditions:

$$-\frac{1}{2} \leq \Re(\propto) \leq 0$$

$$\Im(\propto) \geq 0 \text{ if } \Re(\propto) = 0 \ or - \frac{1}{2}$$

where $\Re(\propto)$and $\Im(\propto)$ denote the real and imaginary parts respectively of a complex number v. We see that C_D is a fundamental region of C for the group G.

For $\propto \epsilon$C, let Σ(α) be the set of solutions (p, q) of the system $S_{II}(\propto)$. Here we assume that a solution is meromorphic over a complex domain. We set $\sum = U_{\propto} \sum(\propto)$(disjoint union). We define rational transformations $s_*, (t_-)_*, (t_+)_*$of the set Σ as follows

we define $s_*(p,q)$ ϵΣ(—1 — α) by

$$s_*(p,q) = \left(p, q + \frac{\propto + \frac{1}{2}}{p}\right) \quad if \ \propto \neq -\frac{1}{2}$$

and

$$s_*(p,q) = (p,q) \quad if \ \propto = -\frac{1}{2}$$

We define $(t_-)_*(p,q)$ ϵΣ($\propto -1$) by

$$(t_-)_*(p,q) = \left(-p + 2\left(q + \frac{\alpha+\frac{1}{2}}{p}\right)^2 + t, -q - \frac{\alpha+\frac{1}{2}}{p}\right) \quad \text{if} \ \propto \neq -\frac{1}{2},$$

and

$$(t_-)_*(p,q) = (-p + 2q^2 + t, -q) \quad if \ \propto = -\frac{1}{2}$$

The definitions of $s_*, (t_-)_*, (t_+)_*$ are well-defined by the following facts for $(p,q) \epsilon \sum \alpha$

p $\neq 0 \ if \ \alpha \neq -\frac{1}{2};$

$$p - 2q^2 - t \neq 0 \ if \ \alpha \neq -\frac{1}{2}.$$

In fact, the assertion

is trivial. If $p - 2q^2 - t = 0$, we have $0 = \left(\frac{d}{dt}\right)(p - 2q^2 - t) = \alpha - \frac{1}{2}$, which proves the assertion (ii).

Since we have $s_*, (t_-)_*, (t_+)_*$ = 1, where 1 denotes the identity transformation of Σ, we see that the mappings $s_*, (t_-)_*, (t_+)_*$ define birational transformations. Let G_* be the subgroup generated by $(t_+)_*$ and s_* in the group of all bijections of the set Σ. The group G_* consists of birational transformations of Σ that respect the natural fibration $\pi: \sum \to C$denned byπ: Σ(a) ∋ (p, q) →$\propto \epsilon$ C. Hence we have a surjective group morphism φ of G_* onto G such that $\varphi(s_*) = s, \varphi(t_0)t$. Since $s_*(t_+)_* s_* = (t_-)_*$ and $s_*(t_-)_* s_* = (t_+)_*$, φ is an isomorphism of G_* onto G.

Since g is C(t)-birational, a solution (p, q) is classical (resp. algebraic, rational) if and only if the solution g(p, q) is classical (resp. algebraic, rational). Now let us state our main result for the second Painleve equation.

IV THEOREM

For every integer $\alpha \in Z$ there exists a unique rational solution of the system $S_{II}(\alpha)$.

(ii) For every $\alpha \in \frac{1}{2}+Z$, there exists a unique one-parameter family of classical solutions of $S_{II}(\alpha)$, of which each solution is rationally written by a solution of a Riccati equation

$$\frac{dq}{dt} = -q^2 - \frac{t}{2},$$

(iii) Let (p, q) be a solution of $S_{II}(\alpha)$ different from those mentioned in (i) and (ii). Then neither the function p nor the function q is classical, and the transcendence degree of C (t,p,q) over C(t) equals two.

Using the birational transformations introduced above, we can explicitly write the solutions (p,q) in the assertions (i) and (ii). In fact, if (p, q) is a rational solution of $S_{II}(\alpha)$ for $\alpha \in Z$, then we have

$$(p,q) = (t_+)_*^{\alpha}\left(\frac{t}{2},0\right).$$

If (p, q) is one of the classical solutions of $S_{II}(\alpha)$ $\left(\alpha \in \frac{1}{2}+Z\right)$ in (ii), then we have

$$(p,q) = (t_+)_*^{\alpha+\frac{1}{2}}(0,q_r),$$

where q_r is a solution of the Riccati equation (1).

Let us introduce a new unknown u by

$$q = \frac{d}{dt}(\log u).$$

Substituting it into (1), we obtain the Airy differential equation

$$\frac{d^2u}{dt^2} + \frac{t}{2}u = 0$$

Hence all the classical solutions of $S_{II}(\alpha)$ for $\alpha \in \frac{1}{2}+Z$ are rationally generated from Airy functions. We explain here how we prove the theorem. Let K be an ordinary differential overfield of C(t) with derivation δ and let K[p,q] be the polynomial ring over K in two variables p and q. According to §1, we introduce a derivation X(α) on K[p, q] by

$$X(\alpha) = \delta + \left(p + q^2 - \frac{t}{2}\right)\frac{\partial}{\partial q} + \left(2pq + \alpha + \frac{1}{2}\right)\frac{\partial}{\partial p}.$$

To prove the theorem, we may assume that the parameter α belongs to the fundamental domain C_0 by the operation of G. The proof consists of the following three parts:

Non-classical solutions. If there exists an X (α) - invariant curve defined over $K\ (\alpha \in C_0)$ for any differential extension K/C(t), then we have a $\alpha = -1/2$. We conclude, for $\alpha \in C_0$such that $\alpha \neq -1/2$, every solution of $S_{II}(\alpha)$ is non-classical if it is not algebraic.

Classical solutions. For every X (-l/2) - invariant polynomial F in K[p, q] and not in K, there exists an integer i > 0 such that $(F) = (p^i)$So every transcendental classical solution of $S_{II}(-1/2)$ is defined by the Riccati equation.

Algebraic solutions. The system $S_{II}(\alpha)$ $(\alpha \in C_0)$ has a rational solution if and only if α = 0. The solution (p, q) = (t/2, 0) is a unique rational solution of $S_{II}(0)$ (In particular the assumption "if it is not algebraic" in (I) is always satisfied.).

V FOURTH PAINLEVE EQUATION

The following system $S_{IV}(v)$of ordinary differential equations

$$S_{IV}(v) \qquad \frac{dq}{dt} = 2pq - q^2 - 2tq + 2(v_1 - v_2)$$

$$\frac{dp}{dt} = 2pq - p^2 - 2tp + 2(v_1 - v_3)$$

where $v = v_1, v_2, v_3$ denotes an arbitrary vector on a complex plane V in C^3 defined by $v_1 + v_2 + v_3 = 0.$ The system $S_{IV}(v)$ is equivalent to the fourth Painleve equation:

$$\frac{d^2q}{dt^2} = \frac{1}{2q}\left(\frac{dq}{dt}\right)^2 + \frac{3}{2}q^3 + 4tq^2 + 2(t^2 - \alpha)q + \frac{\beta}{q}$$

In fact if we eliminate the unknown p from $S_{IV}(v)$, we get $P_{IV}(\alpha, \beta)$ under the relations:

$$\alpha = 3v_3 + 1$$

$$\beta = -2(v_2 - v_1)^2.$$

In order to state birational transformations of solutions of the system $S_{IV}(v)$ associated with a group of affine transformations of the complex plane V. We define three affine transformations s_1, s_2, t_- of V by $s_{IV}(v) = (v_3, v_2, v_1), t_-(v) = v\_(\frac{1}{3})(-1, -1, 2).$ We have $s_1^2 = s_2^2 = 1, t_- s_1 = s_1 t_{-1},$ where 1 denotes the identity transformation of V. Let G be the subgroup generated by s_1, s_2, t_- in the group of all complex affine transformations of V. We can also choose by s_1, s_2, z_0 as generators of the group G. Let H be the subgroup of G generated by s_1, s_2, s_0. Let Γ be the subset of V that consists of all the vectors $(v) = (v_3, v_2, v_1)$ satisfying the following conditions:

$\Re(v_2 - v_1) \geq 0$

$\Re(v_1 - v_3) \geq 0$

$\Re(v_3 - v_2 + 1) \geq 0$

$\Im(v_2 - v_1) \geq 0$ if $\Re(v_2 - v_1) = 0$

$\Im(v_1 - v_3) \geq 0$ if $\Re(v_1 - v_3) = 0$

$\Im(v_3 - v_2) \geq 0$ if $\Re(v_3 - v_{2+1}) = 0$

Here $\Im(v)$ and $\Re(v) = 0$ denote the real and imaginary parts respectively of a complex number v.

LEMMA

The subset Γ is a fundamental region of V for the group

Proof:

We set $V' = V \cap R^3$

and

$\Gamma' = \Gamma \cap R^3$.

The subset Γ is a fundamental region of the real vector space V for the group H, because the set Γ' is the closure of an alcove of the affine Weyl group H. We set

$\tilde{\Gamma} = \{v \in V \mid \Re(v_2 - v_1) \geq 0, \Re(v_1 - v_3) \geq 0, \Re(v_3 - v_2 + 1) \geq 0\}$

We have $\tilde{\Gamma} \supset \Gamma$, and the interior of $\tilde{\Gamma}$ agrees with that of Γ. Every H orbit on V contains a point of Γ, because Γ' is a fundamental region of V'. We show that the intersection of each H-orbit Ω and the subset Γ consists of one point. It is easy to see that this fact proves the lemma. The difference between Γ' and Γ consists of boundary. So we may assume that the H-orbit Ω contains a point of boundary. For example, let us analyse what happens on a boundary stratum

$B_3 = \{v \in \tilde{\Gamma} \mid \Re(v_2 - v_1) = 0, \Re(v_1 - v_3) = 0, \Re(v_3 - v_2 + 1) \neq 0\}$

The stratum B_3 is s_i invariant. If we set

$B_3^+ = \{v \in B_3 \mid \Im(v_2 - v_1) > 0\},$

$B_3^0 = \{v \in B_3 \mid \Im(v_2 - v_1) = 0\},$

$B_3^- = \{v \in B_3 \mid \Im(v_2 - v_1) < 0\},$

then we have a decomposition $B_3 = B_3^+ \cup B_3^0 \cup B_3^-$ (disjoint union).

VI CONCLUSION

To provide a complete classification and unified structure of the special properties which the Painlevé equations and Painlevé σ-equations possess the presently known results are rather fragmentary and non-systematic. Develop algorithmic procedures for the classification of equations with the Painlevé property. Develop software for numerically studying the Painlevé equations which utilizes the fact that they are integrable equations solvable using isomonodromy methods. To produce a general theorem on uniform asymptotics for linear systems to cover all those systems which arise as isomonodromy problems of the Painlevé equations.

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