International Journal of Advance Research in Science and Engineering
Volume No.07, Special Issue No.04, March 2018
www.ijarse.com
IJARSE
ISSN: 2319-8354

# A note on differential and integral equations for the Legendre-Hermite polynomials

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*This note, intends to investigate the recurrence relation, differential equation, integro-differential equation, partial differential equation and integral equation for the Legendre-Hermite polynomials with the help of factorization method.*

***Keywords:*** **Legendre-Hermite polynomials; Recurrence relation; Differential equation; Integral equation.**

*2010 MSC:* 45J05; 65Q30; 65R20

## 1. Introduction and preliminaries

Recently, Subuhi Khan *et. al.*[4] explored the hybrid Legendre-Appell polynomials (LeAP) ${}_{S}\mathcal{A}_k(u,v)$ as the discrete Appell twist of the Legendre polynomials. The LeAP ${}_{S}\mathcal{A}_k(u,v)$ are specified by the generating equation:

$$\mathcal{A}(w)e^{vw}\mathcal{C}_0(-uw^2)=\sum_{k=0}^{\infty}{}_{S}\mathcal{A}_k(u,v)\frac{w^k}{k!}, \tag{1.1}$$

or, equivalently

$$\mathcal{A}(w)e^{(vw+D_u^{-1}w^2)}=\sum_{k=0}^{\infty}{}_{S}\mathcal{A}_k(u,v)\frac{w^k}{k!},\quad D_u^{-1}:=\int_0^u h(\sigma)d\sigma. \tag{1.2}$$

We consider the following form of the Hermite polynomials $\mathcal{H}e_k(v)$ :

$$e^{(vw-\frac{w^2}{2})}=\sum_{k=0}^{\infty}\mathcal{H}e_k(v)\frac{w^k}{k!} \tag{1.3}$$

and possess the following series form:

$$\mathrm{He}_k(v) = k!\sum_{l=0}^{[\frac{k}{2}]}\frac{(-1)^l}{l!(k-2l)!}\frac{v^{k-2l}}{2^l}. \tag{1.4}$$

Choosing $A(w) = e^{-\frac{w^2}{2}}$, we deduct a hybrid family of polynomials namely the Legendre-Hermite polynomials (LeHP) ${}_S\mathrm{He}_k(u,v)$, satisfying the following generating equation:

$$e^{(vw-\frac{w^2}{2})}C_0(-uw^2) = \sum_{k=0}^{\infty}{}_S\mathrm{He}_k(u,v)\frac{w^k}{k!}, \tag{1.5}$$

or, equivalently

$$e^{(vw-\frac{w^2}{2})}e^{D_u^{-1}w^2} = \sum_{k=0}^{\infty}{}_S\mathrm{He}_k(u,v)\frac{w^k}{k!}. \tag{1.6}$$

The process of using factorization method for attaining the differential equations of the Appell and hybrid type Appell polynomials is explored in [1, 7, 3, 5, 6]. The recurrence relations, differential equations and other results of the Appell and hybrid type Appell polynomials help in solutions of the developing problems originating in certain branches of science and engineering. Inspired by the above mentioned work, we derive these properties as well as integral equations for the Legendre-Hermite polynomials.

Let $\{r_k(u)\}_{k=0}^{\infty}$ denotes a polynomial sequence, which is of degree $k$, $(k \in \mathbb{N}_0 := \{0,1,2,\ldots\})$. The $\varphi_k^-$ and $\varphi_k^+$ are called multiplicative and derivative differential operators, if

$$\varphi_k^+\{r_k(u)\} = r_{k+1}(u), \tag{1.7}$$

and

$$\varphi_k^-\{r_k(u)\} = r_{k-1}(u) \tag{1.8}$$

holds respectively, for $\{r_k(u)\}_{k=0}^{\infty}$. The sequence of polynomials $\{r_k(u)\}_{k=0}^{\infty}$ is then called quasi-monomial. An important and essential property such as the differential equation

$$(\varphi_{k+1}^-\varphi_k^+)\{r_k(u)\} = r_k(u), \tag{1.9}$$

can be deduced using these operators. The process exploited in deriving differential equations by means of equation (1.9) is recognized as the factorization method [2, 1].

## 2.Recurrence relations and differential equations

First of all, we acquire the recurrence relation for the LeHP ${}_S\mathrm{He}_k(u,v)$ by demonstrating the following result:

**Theorem 2.1** *The following recurrence relation holds true for the Legendre-Hermite polynomials* ${}_S\mathrm{He}_k(u,v)$*:*

$$ {}_S\mathrm{He}_{k+1}(u,v) = v\,{}_S\mathrm{He}_k(u,v) + (2D_u^{-1}-1)k\;{}_S\mathrm{He}_{k-1}(u,v). \tag{2.1}$$

*Proof.* By taking the derivatives of equation $(1.6)$ w.r.t. $w$ along both sides, it reduces to

$$\left(v - w + 2D_u^{-1}w\right)e^{vw-\frac{w^2}{2}}e^{-D_u^{-1}w} = \sum_{k=0}^{\infty} {}_S\mathrm{He}_{k+1}(u,v)\frac{w^k}{k!}. \tag{2.2}$$

Comparing the coefficients of identical exponents of $w$ along both sides of the above equation and interchanging the sides of final equation, assertion (2.1) is proved.

Now, we proceed to explore the shift operators for the LeHP ${}_S\mathrm{He}_k(u,v)$ by demonstrating the succeeding result:

**Theorem 2.2** *The following expressions holds true for the shift operators of the Legendre-Hermite polynomials* ${}_S\mathrm{He}_k(u,v)$*:*

$$ {}_v\lambda_k^- := \frac{1}{k}D_v, \tag{2.3}$$

$$ {}_u\lambda_k^- := \frac{1}{k}D_v^{-1}\,D_u, \tag{2.4}$$

$$ {}_v\lambda_k^+ := v + (2D_u^{-1}-1)D_v \tag{2.5}$$

and

$$ {}_u\lambda_k^+ := v + (2-D_u)D_v^{-1}. \tag{2.6}$$

*Proof.* Taking the derivatives of equation (1.6) w.r.t. $v$ and afterwards comparing the coefficients of identical exponents of $w$ along both sides of the final equation, we find

$$\frac{\partial}{\partial v}\{{}_S\mathrm{He}_k(u,v)\} = k\;{}_S\mathrm{He}_{k-1}(u,v), \tag{2.7}$$

Consequently, we have

$$ {}_{v}\lambda_{k}^{-}\{{}_{S}\mathrm{H}e_{k}(u,v)\} = \frac{1}{k} D_{v}\{{}_{S}\mathrm{H}e_{k}(u,v)\} = {}_{S}\mathrm{H}e_{k-1}(u,v), \tag{2.8} $$

hence, assertion (2.3) follows.

Again, taking the derivatives of equation (1.6) w.r.t. $u$ and afterwards comparing the coefficients of identical exponents of $w$ along both sides of the final equation, we find

$$ \frac{\partial}{\partial u}\{{}_{S}\mathrm{H}e_{k}(u,v)\} = k(k-1)\ {}_{S}\mathrm{H}e_{k-2}(u,v), \tag{2.9} $$

which in view of equation (2.7) can be written as

$$ \frac{\partial}{\partial u}\{{}_{S}\mathrm{H}e_{k}(u,v)\} = kD_{v}\{{}_{S}\mathrm{H}e_{k-1}(u,v)\}, $$

consequently, we have

$$ {}_{u}\lambda_{k}^{-}\{{}_{S}\mathrm{H}e_{k}(u,v)\} = \frac{1}{k} D_{v}^{-1}\ D_{u}\{{}_{S}\mathrm{H}e_{k}(u,v)\} = {}_{S}\mathrm{H}e_{k-1}(u,v), $$

hence assertion (2.4) is proved.

Next, to obtain the raising operator ${}_{v}\lambda_{k}^{+}$, we make use of equation

$$ {}_{S}\mathrm{H}e_{k-1}(u,v) = \frac{1}{k} D_{v}\{{}_{S}\mathrm{H}e_{k}(u,v)\}, \tag{2.10} $$

in recurrence relation (2.1) and in view of the fact that

$$ {}_{v}\lambda_{k}^{+}\{{}_{S}\mathrm{H}e_{k}(u,v)\} = {}_{S}\mathrm{H}e_{k+1}(u,v), $$

we find

$$ {}_{v}\lambda_{k}^{+} := \left(v + 2D_{u}^{-1}D_{v} - D_{v}\right)\{{}_{S}\mathrm{H}e_{k}(u,v)\} = {}_{S}\mathrm{H}e_{k+1}(u,v), \tag{2.11} $$

which proves assertion (2.5).

Next, for the raising operator ${}_{u}\lambda_{k}^{+}$, we make use of the following equation

$$ {}_{S}\mathrm{H}e_{k-1}(u,v) = \frac{1}{k} D_{v}^{-1}\ D_{u}\{{}_{S}\mathrm{H}e_{k}(u,v)\} \tag{2.12} $$

in recurrence relation (2.1) and in view of the fact that

$$ {}_{u}\lambda_{k}^{+}\{{}_{S}\mathrm{H}e_{k}(u,v)\} = {}_{S}\mathrm{H}e_{k+1}(u,v), $$

we find

$$ {}_{u}\lambda_{k}^{+} := \left(v + 2D_{v}^{-1} - D_{u}D_{v}^{-1}\right)\{{}_{S}He_{k}(u,v)\} = {}_{S}\mathrm{H}e_{k+1}(u,v), \tag{2.13} $$

which proves assertion (2.6).

**Theorem 2.3** *The following differential equation holds true for the Legendre-Hermite polynomials* ${}_{S}\mathrm{H}e_{k}(u,v)$

*:*

$$\left(vD_v + 2uD_u - D_v^2 - k\right)_S\mathrm{He}_k(u,v) = 0. \tag{2.14}$$

*Proof.*Consider the following factorization relation:

$$_v\lambda_{k+1}^- \ _v\lambda_k^+\{_S\mathrm{He}_k(u,v)\} =_S \mathrm{He}_k(u,v). \tag{2.15}$$

Inserting the shift operators given by expressions (2.3) and (2.5) in the l.h.s. of above equation, we find

$$\left(vD_v + 2D_u^{-1}\ D_v^2 - D_v^2 - k\right)_S\mathrm{He}_k(u,v) = 0,$$

which in view of relation [6, Eq.(1.4)]

$$D_v^2 = D_u\ u\ D_u,$$

yields assertion (2.14).

**Theorem 2.4** *The following integro-differential equation is satisfied by the Legendre-Hermite polynomials* $_S\mathrm{He}_k(u,v)$*:*

$$\left(vD_u - D_v^{-1}\ D_u^2 + 2D_uD_v^{-1} - (k+1)D_v\right)_S\mathrm{He}_k(u,v) = 0. \tag{2.16}$$

*Proof.* Inserting the expressions (2.3) and (2.6) in the succeeding factorization relation:

$$_u\lambda_{k+1}^- \ _u\lambda_k^+\{_S\mathrm{He}_k(u,v)\} =_S \mathrm{He}_k(u,v), \tag{2.17}$$

assertion (2.16) follows.

**Remark 2.1.** Differentiating equation (2.16) $k$ -times with respect to $v$, the following consequence of Theorem 2.4 is deduced:

**Corollary 2.5** *The following partial differential equation is satisfied by the Legendre-Hermite polynomials* $_S\mathrm{He}_k(u,v)$ *:*

$$\left(vD_v^k\ D_u + kD_v^{k-1}\ D_u - D_v^{k-1}\ D_u^2 + 2D_uD_v^{k-1} - (k+1)D_v^{k+1}\right)_S\mathrm{He}_k(u,v) = 0. \tag{2.18}$$

We deduce the integral equation for the LeHP $_S\mathrm{He}_k(u,v)$ in the following section.

## 3. Integral equation

In an integral equation an unknown function appears under an integral sign. Very recently, these equations for the Appell polynomials $\mathsf{A}_k(u)$, 2-iterated Appell polynomials $\mathsf{A}_k^{[2]}(u)$ and for some of their components are established in [3]. Further stressing the significance of integral equations, here we deduce the integral equation for the polynomials considered in previous section.

The integral equation for LeHP ${}_{S}\mathsf{H}_k(u,v)$ is derived by proving the following result:

**Theorem 3.1** *The Legendre-Hermite polynomials* ${}_{S}\mathsf{H}_k(u,v)$ *satisfies the following homogeneous Volterra integral equation:*

$$\rho(p) = (2uD_u - k)k!\sum_{l=0}^{[\frac{k}{2}]}\frac{(-1)^l}{l!(k-2l)!}\frac{v^{k-2l}}{2^l}$$

$$+(v+2uD_u-k)k!\sum_{l=0}^{[\frac{k-1}{2}]}\frac{(-1)^l}{l!(k-1-2l)!}\frac{v^{k-1-2l}}{2^l}+\left(\int_0^v((2uD_u-k)(v-\xi)+v)\rho(\xi)d\xi\right). \tag{3.1}$$

*Proof.* In view of equations (1.5) or (or (1.6)) and (1.4), the initial conditions are attained as follows:

$${}_{S}\mathsf{H}e_k(0,v) = \mathsf{H}e_k(v) = k!\sum_{l=0}^{[\frac{k}{2}]}\frac{(-1)^l}{l!(k-2l)!}\frac{v^{k-2l}}{2^l} \tag{3.2}$$

and

$$\frac{d}{dv}{}_{S}\mathsf{H}e_k(0,v) = k\mathsf{S}e_{k-1}(v) = k!\sum_{l=0}^{[\frac{k-1}{2}]}\frac{(-1)^l}{l!(k-1-2l)!}\frac{v^{k-1-2l}}{2^l}, \tag{3.3}$$

respectively.

Consider

$$D_v^2\{{}_{S}\mathsf{H}e_k(u,v)\} = \rho(p). \tag{3.4}$$

Integrating the last equation and using equations (3.2) and (3.3), we have

$$D_v{}_{S}\mathsf{H}e_k(u,v) = \int_0^v\phi(\xi)d\xi + k!\sum_{l=0}^{[\frac{k-1}{2}]}\frac{(-1)^l}{l!(k-1-2l)!}\frac{v^{k-1-2l}}{2^l}, \tag{3.5}$$

$$ {}_{S}He_k(u,v)=\int_0^v \phi(\xi)d\xi^2 + +k!\sum_{l=0}^{[\frac{k-1}{2}]}\frac{(-1)^l}{l!(k-1-2l)!}\frac{v^{k-1-2l}}{2^l}+k!\sum_{l=0}^{[\frac{k}{2}]}\frac{(-1)^l}{l!(k-2l)!}\frac{v^{k-2l}}{2^l}. \quad (3.6) $$

Use of expressions (3.6) in equation (2.14) yields assertion (3.1).

The differential equations play a significant role in economics, biology, physics, and engineering. In the present investigation, the differential equations for Legendre-Hermite polynomials are derived by using factorization method. Moreover the inclusion of integral equation for these polynomials is a bonus of this article. To extend this approach to other hybrid families may be taken as further research investigations.

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