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Coupled Coincidence Point in Metric Spaces

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Abstract*:*
In this research article we use SCC-Map and --contraction type T-coupling to prove the theorem for coupled coincidence point in metric space. we give example to illustrate our result. .
Keywords : Coupled Fixed Point; Coupled Coincidence Point; --Contraction Type T-Coupling; SCC-Map.
2010MSC : 47H10, 54H25,

I. INTRODUCTION AND MATHEMATICAL PRELIMINARIES

T. Gnana Bhaskar and V. Lakshmikantham [9] introduced the concept of coupled fixed point of mapping $F : X \times X \to X$. Lakshmikantham V. and Ćirić L. [13] introduced coupled coincidence point. Then results on existence of coupled fixed point and coupled coincidence points appeared in many papers [1, 2, 4, 5, 8, 10, 13, 14, 15, 16, 17, 18, 19] . Choudhury et al.[7] introduced concept of coupling and proved the existence and uniqueness of strong coupled fixed point for couplings using Kannan type contractions for complete metric spaces. We generalize Choudhury result and prove coupled coincidence point in metric spaces. Finally we give an example to support our result.

Definition 1.1. (Coupled Fixed Point) [9]. An element $(x, y) \in X \times X$, where X is any non-empty set, is called a coupled fixed point of the mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.

Definition 1.2. (Strong Coupled Fixed Point) [7]. An element $(x, y) \in X \times X$, where X is any non-empty set, is called a strong coupled fixed point of the mapping $F : X \times X \to X$ if (x, y) is coupled fixed point and $x = y$; that is if $F(x, x) = x$.

Definition 1.3. (Coupled Banach Contraction Mapping) [9]. Let (X, d) be a metric space. A mapping $F : X \times X \to X$ is called coupled Banach contraction if there exists $k \in (0, 1)$ s.t.

$\forall\ (x, y), (u, v) \in X \times X$, the following inequality is satisfied:

$$d(F(x, y), F(u, v)) \leq \frac{k}{2}[d(x, u) + d(y, v)].$$

Definition 1.4. (Cyclic Mapping) [11]. Let A and B be two non-empty subsets of a given set X. Any function $f : X \to X$ is said to be cyclic (with respect to A and B) if

$$f(A) \subset B \quad and \quad f(B) \subset A.$$

Definition 1.5. (Coupling) [7]. Let (X, d) be a metric space and $A\ and\ B$ be two non-empty subsets of X. Then a function $F : X \times X \to X$ is said to be a coupling with respect to $A\ and\ B$ if

$$F(x, y) \in B\ and\ F(y, x) \in A$$

whenever $x \in A$ and $y \in B$.

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Definition 1.6. (Coupled Coincidence Point of F and g) [13]. An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x, y) = g(x)$ and $F(y, x) = g(y)$.

Definition 1.7. [12]. A function $\psi : [0, \infty) \to [0, \infty)$ is called an altering distance function, if the following properties are satisfied:
(i) ψ is monotone increasing and continuous,
(ii) $\psi(t) = 0$ iff $t = 0$.

Lemma 1.8. [3]. Let $\phi \in \Phi$ and $\{u_n\}$ be a given sequence such that $u_n \to 0^+$ as $n \to \infty$. Then $\phi(u_n) \to 0^+$ as $n \to \infty$. Also $\phi(0) = 0$.
Where Φ is set of all functions $\phi : [0, \infty) \to [0, \infty)$ satisfying :
(i) ϕ is non-decreasing,
(ii) $\phi(t) < t \quad for\ all \quad t > 0$,
(iii) $\lim_{r \to t^+} \phi(r) < t \quad for\ all \quad t > 0$.

Definition 1.9. [3]. Let A and B be two non-empty subsets of a partial metric space (X, p). A coupling $F : X^2 \to X$ is said a ϕ-contraction type coupling with respect to A and B if there exists $\phi \in \Phi$ such that

$$p(F(x, y), F(u, v)) \leq| \phi(max\{p(x, u), p(y, v)\}),$$

for any $x, v \in A$ and $y, u \in B$.

Theorem 1.10. [3]. Let A and B be two non-empty closed subsets of a complete partial metric space (X, p). Let $F : X^2 \to X$ be a ϕ-contraction type coupling with respect to A and B. Then $A \cap B \neq \emptyset$ and F has a unique strong coupled fixed point in $A \cap B$.

II. MAIN RESULT

Our main result is divided into two subsections. In the first subsection we introduce SCC-Map, ϕ-contraction type T-coupling (with respect to A and B) and prove existence theorem of coupled coincidence point.

2.1 Coupled Coincidence Point of --Contraction Type T-Coupling in Metric Spaces.

Before proving the main theorem of this subsection we introduce some definitions.

Definition 2.1.1. (SCC-Map). Let A and B be any two non-empty subsets of a metric space (X, d) and $T : X \to X$ be a self map on X. Then T is said to be $SCC - map$ (with respect to A and B), if
(i) $T(A) \subseteq A$ and $T(B) \subseteq B$,
(ii) $T(A)$ and $T(B)$ are closed in X.

Remark 2.1.2. The identity map is not $SCC - Map$ in general. Identity map is $SCC - Map$ (with respect to A and B) whenever A and B are closed subsets of X, i.e. Identity map can't be considered as $SCC - Map$ with respect to open sets.

Definition 2.1.3. (ϕ-Contraction Type T-Coupling). Let A and B be any two non-empty subsets of metric space (X, d) and $T : X \to X$ is $SCC - Map$ on X (with respect to A and B). Then a coupling $F : X \times X \to X$ is said to be ϕ-Contraction Type T-Coupling (with respect to A and B), if

$$d(F(x,y),F(u,v)) \le \phi(max\{d(Tx,Tu),d(Ty,Tv)\})$$

for any $x,v \in A$ and $y,u \in B$ and $\phi \in \Phi$ defined in Lemma 1.8.

Note 2.1.4. If A and B are two non-empty subsets of a metric space (X,d) and $F : X \times X \to X$ is a coupling with respect to A and B. Then by definition of coupling for $a \in A$ and $b \in B$, we have $F(a,b) \in B$ and $F(b,a) \in A$.

Now let (a,b) be the coupled fixed point of F, then $F(a,b) = a$ and $F(b,a) = b$. But in general this is absurd because $F(a,b) \in B$ *and* $a \in A$. Similarly $F(b,a) \in A$ *and* $b \in B$. This is only possible for $a, b \in A \cap B$.

The most important fact to be noted is that for any coupling $F : X \times X \to X$ (with respect to A and B), where A and B be any two non-empty subsets of metric space (X,d), if we investigate for coupled fixed point (x,y) in product space $A \times B$, then we should directly investigate in product subspace $(A \cap B) \times (A \cap B)$. Similarly for strong coupled fixed point, we should investigate it in $A \cap B$.

Theorem 2.1.5. Let A and B be any two complete subspaces of a metric space (X,d) and $T : X \to X$ is $SCC - Map$ on X (w.r.t. A and B). Let $F : X \times X \to X$ be a ϕ-contraction type T-coupling (with respect to A and B), then
(i) $T(A) \cap T(B) \neq \emptyset$,
(ii) F and T have atleast one coupled coincidence point in $A \times B$.

Proof. Since F is ϕ-contractive type T-coupling (with respect to A and B), we have

$$d(F(x,y),F(u,v)) \le \phi(max\{d(Tx,Tu),d(Ty,Tv)\}) \tag{1}$$

where $x,v \in A$ and $y,u \in B$ and $\phi \in \Phi$.
As A and B are non-empty subsets of X and F is ϕ-contraction type T-coupling (with respect to A and B), then for $x_0 \in A$ and $y_0 \in B$ we define sequences $\{x_n\}$ and $\{y_n\}$ in A and B respectively, such that

$$Tx_{n+1} = F(y_n, x_n) \quad and \quad Ty_{n+1} = F(x_n, y_n). \tag{2}$$

We claim $Tx_n \neq Ty_{n+1}$ and $Ty_n \neq Tx_{n+1}$ $\forall$ n.
If possible suppose for some n, $Tx_n = Ty_{n+1}$ and $Ty_n = Tx_{n+1}$. Then by using (2), we have

$$Tx_n = Ty_{n+1} = F(x_n, y_n) \;\; and \;\; Ty_n = Tx_{n+1} = F(y_n, x_n).$$

Which shows that (x_n, y_n) is a coupled coincidence point of F and T, so we are done in this case. Thus we assume

$$Tx_n \neq Ty_{n+1} \;\; and \;\; Ty_n \neq Tx_{n+1} \;\; \forall \;\; n.$$

Now we define a sequence $\{D_n\}$ by

$$D_n = max\{d(Tx_n, Ty_{n+1}), d(Ty_n, Tx_{n+1})\}. \tag{3}$$

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Then

$$D_n > 0 \quad \forall \quad n. \tag{4}$$

Now by using (1) and (2), we get

$$\begin{aligned} d(Tx_n, Ty_{n+1}) &= d(F(y_{n-1}, x_{n-1}), F(x_n, y_n)) \\ &\leq \phi(max\{d(Ty_{n-1}, Tx_n), d(Tx_{n-1}, Ty_n)\}) \end{aligned} \tag{5}$$

and

$$\begin{aligned} d(Ty_n, Tx_{n+1}) &= d(F(x_{n-1}, y_{n-1}), F(y_n, x_n)) \\ &\leq \phi(max\{d(Tx_{n-1}, Ty_n), d(Ty_{n-1}, Tx_n)\}). \end{aligned} \tag{6}$$

Using $\phi(t) < t \ \forall \ t > 0$, then from (3), (4), (5) and (6), we have

$$\begin{aligned} 0 &< max\{d(Tx_n, Ty_{n+1}), d(Ty_n, Tx_{n+1})\} \\ &\leq \phi(max\{d(Tx_{n-1}, Ty_n), d(Ty_{n-1}, Tx_n)\}) \\ &= D_{n-1}. \end{aligned} \tag{7}$$

Thus $D_n < D_{n-1} \ \forall \ n$. This show that $\{D_n\}$ is monotonic decreasing sequence of non-negative real numbers, therefore $\exists \ s \geq 0$, s.t.

$$\lim_{n\to+\infty} D_n = \lim_{n\to+\infty} max\{d(Tx_n, Ty_{n+1}), d(Ty_n, Tx_{n+1})\} = s. \tag{8}$$

Suppose $s > 0$, letting $n \to +\infty$ in (7), using (4) and Lemma 1.8, we have

$$\begin{aligned} 0 < s &\leq \lim_{n\to+\infty} \phi(max\{d(Tx_{n-1}, Ty_n), d(Ty_{n-1}, Tx_n)\}) \\ &= \lim_{t\to s^+} \phi(t) < s. \end{aligned}$$

Which is a contradiction, therefore $s = 0$.
So

$$\lim_{n\to+\infty} max\{d(Tx_n, Ty_{n+1}), d(Ty_n, Tx_{n+1})\} = 0.$$

i.e.

$$\lim_{n\to+\infty} d(Tx_n, Ty_{n+1}) = 0 \ and \lim_{n\to+\infty} d(Ty_n, Tx_{n+1}) = 0. \tag{9}$$

i.e.

$$\lim_{n\to+\infty} d(Tx_n, Ty_{n+1}) = 0 \ and \lim_{n\to+\infty} d(Ty_n, Tx_{n+1}) = 0. \tag{9}$$

Now we prove $\lim_{n\to\infty} d(Tx_n, Ty_n) = 0$.
Let us define a sequence $\{R_n\}$ by $R_n = d(Tx_n, Ty_n)$. If $R_{n_0} = 0$ for some n_0, then $Tx_{n_0} = Ty_{n_0}$ and so $Tx_{n_0+1} = Ty_{n_0+1}$, by induction we have
$d(Tx_n, Ty_n) = 0 \ \forall \ n \geq n_0$. Thus $\lim_{n\to\infty} d(Tx_n, Ty_n) = 0$.
Now we assume $R_n > 0 \ \ \forall \ \ n$, then by using (1),(2) and definition of ϕ, we have

$$
\begin{aligned}
R_n = d(Tx_n, Ty_n) &= d(F(y_{n-1}, x_{n-1}), F(x_{n-1}, y_{n-1})) \\
&\leq \phi(max\{d(Ty_{n-1}, Tx_{n-1}), d(Tx_{n-1}, Ty_{n-1})\}) \\
&= \phi(d(Tx_{n-1}, Ty_{n-1})) \\
&= \phi(R_{n-1}) \\
&< R_{n-1}.
\end{aligned}
$$

Thus $\{R_n\}$ is a monotonic decreasing sequence of non-negative real numbers. Therefore $\exists\, r \geq 0$, s.t.

$$\lim_{n\to\infty} R_n = r^+.$$

Assume $r > 0$ and proceeding similarly by using $\lim_{z\to m^+} \phi(z) < m \;\; \forall \;\; m > 0$ as above we will obtain a contradiction, so $r = 0$, i.e.

$$\lim_{n\to\infty} d(Tx_n, Ty_n) = 0. \tag{10}$$

Using triangular inequality, (9) and (10), we have

$$\lim_{n\to\infty} d(Tx_n, Tx_{n+1}) \leq \lim_{n\to\infty} [d(Tx_n, Ty_{n+1}) + d(Ty_{n+1}, Tx_{n+1})] = 0. \tag{11}$$

$$\lim_{n\to\infty} d(Ty_n, Ty_{n+1}) \leq \lim_{n\to\infty} [d(Ty_n, Tx_{n+1}) + d(Tx_{n+1}, Ty_{n+1})] = 0. \tag{12}$$

Now we show that $\{Tx_n\}$ and $\{Ty_n\}$ are Cauchy sequences in $T(A)$ and $T(B)$ respectively. Assume either $\{Tx_n\}$ or $\{Ty_n\}$ is not a Cauchy sequence, i.e.

$$\lim_{n,m\to+\infty} d(Tx_m, Tx_n) \neq 0 \quad or \quad \lim_{n,m\to+\infty} d(Ty_m, Ty_n) \neq 0.$$

Then $\exists\, \varepsilon > 0$, for which we can find subsequences of integers $\{m(k)\}$ and $\{n(k)\}$ with $n(k) > m(k) > k$, s.t.

$$max\{d(Tx_{m(k)}, Tx_{n(k)}), d(Ty_{m(k)}, Ty_{n(k)})\} \geq \varepsilon. \tag{13}$$

Further corresponding to $m(k)$ we can choose $n(k)$ in such a way that it is smallest integer with $n(k) > m(k)$ and satisfy (13), then

$$max\{d(Tx_{m(k)}, Tx_{n(k)-1}), d(Ty_{m(k)}, Ty_{n(k)-1})\} < \varepsilon. \tag{14}$$

Now by using triangular inequality, (1) and (2), we have

$$
\begin{aligned}
d(Tx_{n(k)}, Tx_{m(k)}) &\leq d(Tx_{n(k)}, Ty_{n(k)}) + d(Ty_{n(k)}, Tx_{m(k)+1}) \\
&\quad + d(Tx_{m(k)+1}, Tx_{m(k)}) \\
&= d(Tx_{n(k)}, Ty_{n(k)}) + d(Tx_{m(k)+1}, Tx_{m(k)}) \\
&\quad + d(F(x_{n(k)-1}, y_{n(k)-1}), F(y_{m(k)}, x_{m(k)})) \\
&\leq d(Tx_{n(k)}, Ty_{n(k)}) + d(Tx_{m(k)+1}, Tx_{m(k)}) \\
&\quad + \phi(max\{d(Tx_{n(k)-1}, Ty_{m(k)}), d(Ty_{n(k)-1}, Tx_{m(k)}\}) \\
&= d(Tx_{n(k)}, Ty_{n(k)}) + d(Tx_{m(k)+1}, Tx_{m(k)}) \\
&\quad + \phi(max\{d(Tx_{m(k)}, Ty_{n(k)-1}), d(Ty_{m(k)}, Tx_{n(k)-1})\}).
\end{aligned} \tag{15}
$$

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Similarly by using triangular inequality, (1) and (2), we can show that

$$\begin{aligned} d(Ty_{n(k)}, Ty_{m(k)}) &\leq d(Ty_{n(k)}, Tx_{n(k)}) + d(Ty_{m(k)+1}, Ty_{m(k)}) \\ &\quad +\phi(max\{d(Ty_{m(k)}, Tx_{n(k)-1}), d(Tx_{m(k)}, Ty_{n(k)-1})\}). \end{aligned} \tag{16}$$

From (13), (15) and (16), we have

$$\begin{aligned} \varepsilon &\leq max\{d(Tx_{n(k)}, Tx_{m(k)}), d(Ty_{n(k)}, Ty_{m(k)})\} \\ &\leq d(Tx_{n(k)}, Ty_{n(k)}) + max\{d(Tx_{m(k)+1}, Tx_{m(k)}), d(Ty_{m(k)+1}, Ty_{m(k)})\} \\ &\quad +\phi(max\{d(Tx_{m(k)}, Ty_{n(k)-1}), d(Ty_{m(k)}, Tx_{n(k)-1})\}). \end{aligned} \tag{17}$$

By triangular inequality , we have

$$d(Tx_{m(k)}, Ty_{n(k)-1}) \leq d(Tx_{m(k)}, Ty_{m(k)}) + d(Ty_{m(k)}, Ty_{n(k)-1}). \tag{18}$$

$$d(Ty_{m(k)}, Tx_{n(k)-1}) \leq d(Ty_{m(k)}, Tx_{m(k)}) + d(Tx_{m(k)}, Tx_{n(k)-1}). \tag{19}$$

From (14), (18) and (19), we have

$$max\{d(Tx_{m(k)}, Ty_{n(k)-1}), d(Ty_{m(k)}, Tx_{n(k)-1})\}.$$

$$\begin{aligned} &\leq d(Tx_{m(k)}, Ty_{m(k)}) + max\{d(Ty_{m(k)}, Ty_{n(k)-1}), d(Tx_{m(k)}, Tx_{n(k)-1})\} \\ &< d(Tx_{m(k)}, Ty_{m(k)}) + \varepsilon. \end{aligned} \tag{20}$$

Since ϕ is non-decreasing, we have from (20)

$$\phi(max\{d(Tx_{m(k)}, Ty_{n(k)-1}), d(Ty_{m(k)}, Tx_{n(k)-1})\}) < \phi(d(Tx_{m(k)}, Ty_{m(k)}) + \varepsilon). \tag{21}$$

Now using (21) in (17), we get

$$\begin{aligned} \varepsilon &< d(Tx_{n(k)}, Ty_{n(k)}) + max\{d(Tx_{m(k)+1}, Tx_{m(k)}), d(Ty_{m(k)+1}, Ty_{m(k)})\} \\ &\quad +\phi(d(Tx_{m(k)}, Ty_{m(k)}) + \varepsilon). \end{aligned} \tag{22}$$

Letting $k \to \infty$ in (22) and using (10), (11), (12) and property (iii) of ϕ in Lemma 1.8, we get

$$\begin{aligned} \varepsilon &< \lim_{k\to\infty} \phi(d(Tx_{m(k)}, Ty_{m(k)}) + \varepsilon) \\ &= \lim_{d(Tx_{m(k)}, Ty_{m(k)})+\varepsilon \to \varepsilon^+} \phi(d(Tx_{m(k)}, Ty_{m(k)}) + \varepsilon) \\ &< \varepsilon. \end{aligned} \tag{23}$$

Which is a contradiction. Thus $\{Tx_n\}$ and $\{Ty_n\}$ are Cauchy sequences in $T(A)$ and $T(B)$ respectively. But $T(A)$ and $T(B)$ are closed subsets of a complete subspaces A and B resp. Hence $\{Tx_n\}$ and $\{Ty_n\}$ are convergent in $T(A)$ and $T(B)$ respectively.
So $\exists$ $u \in T(A)$ and $v \in T(B)$, s.t.

$$Tx_n \to u \quad and \quad Ty_n \to v. \tag{24}$$

Using (10) in above, we get

$$u = v. \tag{25}$$

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Therefore $u = v \in T(A) \cap T(B)$, thus $T(A) \cap T(B) \neq \emptyset$. This proves part (i).
Now, as $u \in T(A)$ and $v \in T(B)$, therefore $\exists\ a \in A$ and $b \in B$, $s.t.$
$u = T(a)$ and $v = T(b)$. Using in (24), we get

$$Tx_n \to T(a) \ \ and \ \ Ty_n \to T(b). \tag{26}$$

Also from (25), we get

$$T(a) = T(b). \tag{27}$$

Now using triangular inequality, (1), (2), (26), (27) and Lemma 1.8, we get

$$\begin{aligned} d(T(a), F(a,b)) &\leq d(T(a), Tx_{n+1}) + d(Tx_{n+1}, F(a,b)) \\ &= d(T(a), Tx_{n+1}) + d(F(y_n, x_n), F(a,b)) \\ &\leq d(T(a), Tx_{n+1}) + \phi(max\{d(Ty_n.T(a)), d(Tx_n, T(b))\}) \\ &\to 0 \quad as \quad n \to \infty. \end{aligned}$$

Thus from above, we have

$$F(a,b) = T(a). \tag{28}$$

Again by using triangular inequality, (1), (2), (26), (27) and Lemma 1.8, we get

$$\begin{aligned} d(T(b), F(b,a)) &\leq d(T(b), Ty_{n+1}) + d(Ty_{n+1}, F(b,a)) \\ &= d(T(b), Ty_{n+1}) + d(F(x_n, y_n), F(b,a)) \\ &\leq d(T(b), Ty_{n+1}) + \phi(max\{d(Tx_n.T(b)), d(Ty_n, T(a))\}) \\ &\to 0 \quad as \quad n \to \infty. \end{aligned}$$

From above, we have

$$F(b,a) = T(b). \tag{29}$$

Hence (28) and (29) shows that $(a,b) \in A \times B$ is the coupled coincidence point of F and T.

Corollary 2.1.6. It should be noted that the above condition also gives a symmetric point of F in $A \times B$, i.e. there exists a point $(a,b) \in A \times B$ s.t. $F(a,b) = F(b,a)$. This can be easily see by using (27) in (28) and (29) of Theorem 2.1.5, we get $F(a,b) = F(b,a)$.

Corollary 2.1.7. If we take $T = I$ (the identity map) and A and B the closed subsets, then Theorem 2.1.5 will reduce to Theorem 1.10 by H.Aydi [3] for partial metric spaces not necessary complete.
Proof : The proof can be easily verified by using (27) and (28) of Theorem 2.1.5 and the fact that I is one-one map, so $a = b$ and hence $A \cap B \neq \emptyset$ and $F(a,a) = a$.

Note 2.1.8. It should be noted that if T is one-one, then by the asssumption $T(A) \subseteq A$ and $T(B) \subseteq B$, we have T is identity map on A and B and uniqueness can be proved by Corollary 2.1.7.

Example 2.1.9. Let $X = (-5,5)$ be the metric space with respect usual metric d on X, i.e. $d(x,y) =\mid x - y \mid$. Let $A = [0,2]$ and $B = [0,4]$ be the complete subspaces of X. Let us define $F : X \times X \to X$ by

$$F(x,y)=\begin{cases}2, & 0\le x,y\le 2\\ \frac{x+y}{24}, & elsewhere.\end{cases} \tag{30}$$

Also we define $T: X\to X$ by

$$T(x)=\begin{cases}2, & 0\le x\le 2\\ 4, & x>2.\end{cases} \tag{31}$$

We define $\phi:[0,\infty)\to[0,\infty)$ by

$$\phi(t)=\begin{cases}\frac{2}{3}t, & 0\le t\le \frac{47}{24}\\ \frac{47}{24}, & t>\frac{47}{24}.\end{cases} \tag{32}$$

Clearly $\phi\in\Phi$.
Now from (31), we have

$$T(A)=\{2\}\subseteq A \ \ and \ \ T(B)=\{2,4\}\subseteq B.$$

Also $T(A)$ and $T(B)$ are closed in A and B respectively. Thus $T: X\to X$ is SCC-Map (w.r.t. A and B).
Now we will show $F: X\times X\to X$ is coupling (w.r.t. A and B).
Let $x\in A$ and $y\in B$. Here two cases will arise for y,
case(i): $0\le y\le 2$, i.e. $y\in A$,
case(ii): $2<y\le 4$.
For case (i) i.e. $x,y\in A$, by using (30), we have
$F(x,y)=2\in B$ and $F(y,x)=2\in A$. Thus F is coupling (w.r.t. A and B) and we are done in this case.
For case (ii) i.e. $x\in A$ and $2<y\le 4$, by using (30), we have

$$F(x,y)=\frac{x+y}{24}$$

i.e.

$$\frac{1}{12}<F(x,y)\le\frac{1}{4}, \quad \Rightarrow F(x,y)\in B.$$

and

$$\frac{1}{12}<F(y,x)\le\frac{1}{4}, \quad \Rightarrow F(y,x)\in A.$$

Thus in both the cases we get F is a coupling (w.r.t. A and B).
Now we show that F is ϕ-contraction type T-coupling (w.r.t. A and B).
Let $x,v\in A$ and $y,u\in B$, three cases will arise for y,u,
case(i): when both $y,u\in A$, i.e. $0\le y,u\le 2$.
case(ii): when one is in A and other outside A.
case(iii): when both y,u lie outside A, i.e. $2<y,u\le 4$.
For case(i),*i.e.* $x,y,u,v\in A$, we have from (31)

$$T(x)=T(y)=T(u)=T(v)=2.$$

So, $d(T(x), T(u)) = d(T(y), T(v)) = 0$,
thus

$$max\{d(T(x), T(u)), d(T(y), T(v))\} = 0.$$

Using (32) in above, we get

$$\phi(max\{d(T(x), T(u)), d(T(y), T(v))\}) = \phi(0) = 0. \tag{33}$$

Also for $x, y, u, v \in A$, we have from (30)

$$F(x, y) = F(u, v) = 2, \quad \Rightarrow \quad d(F(x, y), F(u, v)) = 0. \tag{34}$$

Thus from (33) and (34), we get

$$d(F(x, y), F(u, v)) = \phi(max\{d(T(x), T(u)), d(T(y), T(v))\}).$$

Hence we have proved in this case.
For case(ii), *i.e.* $x, v \in A$ and either y or $u \in$ A. Without loss of generality we assume $y \in A$ and u outside A i.e. $2 < u \leq 4$.
thus for $x, y, v \in A$ and $2 < u \leq 4$, we have from (31)

$$T(x) = T(y) = T(v) = 2 \;\; and \;\; T(u) = 4.$$

$$\text{so} \quad d(T(x), T(u)) = 2 \;\; and \;\; d(T(y), T(v)) = 0$$

thus

$$max\{d(T(x), T(u)), d(T(y), T(v))\} = 2.$$

Using (32) in above, we get

$$\phi(max\{d(T(x), T(u)), d(T(y), T(v))\}) = \phi(2) = \frac{47}{24}. \tag{35}$$

Also for $x, y, v \in A$ and $2 < u \leq 4$, we have from (30)

$$F(x, y) = 2 \quad and \quad \frac{1}{12} < F(u, v) \leq \frac{1}{4}. \tag{36}$$

Therefore from (36), we have

$$\begin{aligned} d(F(x, y), F(u, v)) &< (2 - \frac{1}{12}) \\ &= \frac{23}{12} \\ &< \frac{47}{24}. \end{aligned} \tag{37}$$

Thus from (35) and (37), we get

$$d(F(x, y), F(u, v)) < \phi(max\{d(T(x), T(u)), d(T(y), T(v))\}).$$

Which proves case(ii).
For case(iii), *i.e.* $x, v \in A$ and $2 < y, u \leq 4.$, we have from (31)

$$T(x) = T(v) = 2 \; and \; T(y) = T(u) = 4,$$

so,

$$d(T(x),T(u)) = d(T(y),T(v)) = 2.$$

and

$$max\{d(T(x),T(u)), d(T(y),T(v))\} = 2.$$

Using (32) in above, we get

$$\phi(max\{d(T(x),T(u)), d(T(y),T(v))\}) = \phi(2) = \frac{47}{24}. \quad (38)$$

Also for $x, v \in A$ and $2 < y, u \leq 4$, we have from (30)

$$\frac{1}{12} < F(x,y) \leq \frac{1}{4} \quad and \quad \frac{1}{12} < F(u,v) \leq \frac{1}{4} \quad (39)$$

from (39), we have

$$\begin{aligned} d(F(x,y),F(u,v)) < (\frac{1}{4} - \frac{1}{12}) &= \frac{1}{6} \\ &< \frac{47}{24}. \end{aligned} \quad (40)$$

From (38) and (40), we have

$$d(F(x,y),F(u,v)) < \phi(max\{d(T(x),T(u)), d(T(y),T(v))\}).$$

Thus in all the cases we have proved that F is ϕ-contraction type T-coupling (w.r.t. A and B).

Hence all the assumptions of Theorem 2.1.5 are satisfied, therefore F and T have coupled coincidence point in $A \times B$.
For $a \in A$ and $b \in B$ $s.t. 0 \leq b \leq 2$, then from (30) and (31)

$$F(a,b) = 2 = T(a) \quad and \quad F(b,a) = 2 = T(b). \quad (41)$$

This shows that (a,b) is coupled coincidence point of F and T.
The above example also shows that F and T have infinitely many coupled coincidence points.

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