Zero-free Regions and Bounds for the Number of Zeros of a Polynomial in a Region

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ABSTRACT

In this paper we find the zero-free regions and the number of zeros of a polynomial with restricted coefficients in a region. Our results generalize many already known results in this direction.

Mathematics Subject Classification (2010): 30C10, 30C15.

Key Words and Phrases: Coefficient, Open Disc, Polynomial, Zeros.

I. INTRODUCTION

Recently Gulzar et al [2] proved the following results regarding the upper bounds for the zeros of a polynomial with restricted coefficients.

Theorem A. Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some positive numbers k_1, k_2, ρ and some integer λ with $k_1 \ge 1, k_2 \ge 1, 0<\rho \le 1, 0<\lambda \le n-1$,

$$k_1 a_n \ge a_{n-1} \ge ...... \ge k_2 a_\lambda \ge a_{\lambda-1} \ge ...... \ge a_1 \ge \rho a_0 \ .$$

Then all the zeros of P(z) lie in the closed disk

$$|z+k_1-1| \le \frac{1}{|a_n|}\{k_1 a_n + 2(k_2-1)|a_\lambda| - \rho(a_0+|a_0|) + 2|a_0|\}.$$

Theorem B. Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $a_j=\alpha_j+i\beta_j$,

$j=0,1,...,n$, where α_j and β_j are real numbers, such that for some positive numbers k_1, k_2, ρ and some integer λ with $k_1 \ge 1, k_2 \ge 1, 0<\rho \le 1, 0<\lambda \le n-1$,

$$k_1\alpha_n \ge \alpha_{n-1} \ge ...... \ge k_2\alpha_\lambda \ge \alpha_{\lambda-1} \ge ...... \ge \alpha_1 \ge \rho\alpha_0 .$$

Then all the zeros of P(z) lie in the closed disk

$$\left|z+\frac{\alpha_n}{a_n}(k_1-1)\right| \le \frac{1}{|a_n|}\{k_1\alpha_n + 2(k_2-1)|\alpha_\lambda| - \rho(\alpha_0+|\alpha_0|) + 2|\alpha_0| + 2\sum_{j=0}^{n}|\beta_j|\}.$$

Theorem C. Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients such that for some positive numbers k_1, k_2, ρ and some integer λ with $k_1 \ge 1, k_2 \ge 1, 0 < \rho \le 1, 0 < \lambda \le n-1$,

$$k_1|a_n| \ge |a_{n-1}| \ge ...... \ge k_2|a_\lambda| \ge |a_{\lambda-1}| \ge ...... \ge |a_1| \ge \rho|a_0|$$

and for some real numbers α, β ,

$$|\arg a_j - \beta| \le \alpha \le \frac{\pi}{2},\ j = 0,1,2,.....,n .$$

Then all the zeros of P(z) lie in the closed disk

$$|z+k_1-1| \le \frac{1}{|a_n|}\{k_1|a_n|(\cos\alpha+\sin\alpha) + 2(k_2-1)|a_\lambda| + 2k_2|a_\lambda|\sin\alpha$$

$$+2|a_0| - \rho|a_0|(\cos\alpha - \sin\alpha + 1)\ \}.$$

II. MAIN RESULTS

The aim of this paper is to find zero-free regions of the polynomials in theorems A,B,C and also to find bounds for the number of their zeros in specified regions. In fact, we prove the following results.

Theorem 1. Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some positive numbers k_1, k_2, ρ and some integer λ with $k_1 \ge 1, k_2 \ge 1, 0 < \rho \le 1, 0 < \lambda \le n-1$,

$$k_1 a_n \ge a_{n-1} \ge ...... \ge k_2 a_\lambda \ge a_{\lambda-1} \ge ...... \ge a_1 \ge \rho a_0 .$$

Then for any R>0,P(z) does not vanish in $|z| < \frac{|a_0|}{M}$, where

$$M = |a_n|R^{n+1} + R^n\left[\ k_1(|\alpha_n| + \alpha_n) - |\alpha_n| + 2(k_2 - 1)|\alpha_\lambda| + |\alpha_0| - \rho(|\alpha_0| + \alpha_0) + 2\sum_{j=0}^{n}|\beta_j|\ \right]$$

for $R \geq 1$

and

$$M = |a_n|R^{n+1} + R\left[\ k_1(|\alpha_n| + \alpha_n) - |\alpha_n| + 2(k_2 - 1)|\alpha_\lambda| + |\alpha_0| - \rho(|\alpha_0| + \alpha_0) + 2\sum_{j=0}^{n}|\beta_j|\ \right].$$

for $R \leq 1$.

Theorem 2. Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some positive numbers

k_1, k_2, ρ and some integer λ with $k_1 \geq 1, k_2 \geq 1, 0 < \rho \leq 1, 0 < \lambda \leq n-1$,

$$k_1 a_n \geq a_{n-1} \geq \ldots\ldots \geq k_2 a_\lambda \geq a_{\lambda-1} \geq \ldots\ldots \geq a_1 \geq \rho a_0 .$$

Then for any R>0, the number of zeros of P(z) in $|z| \leq \frac{R}{c}, c > 1$ does not exceed

$$\frac{1}{\log c}\log\frac{K}{|P(0)|},$$

where

$$K = |a_n|R^{n+1} + |a_0| + R^n\left[\ k_1(|\alpha_n| + \alpha_n) - |\alpha_n| + 2(k_2 - 1)|\alpha_\lambda| + |\alpha_0| - \rho(|\alpha_0| + \alpha_0) + 2\sum_{j=0}^{n}|\beta_j|\ \right]$$

for $R \geq 1$ and

$$K = |a_n|R^{n+1} + |a_0| + R\left[\ k_1(|\alpha_n| + \alpha_n) - |\alpha_n| + 2(k_2 - 1)|\alpha_\lambda| + |\alpha_0| - \rho(|\alpha_0| + \alpha_0) + 2\sum_{j=0}^{n}|\beta_j|\ \right]$$

for $R \leq 1$.

Combining Theorem 1 and Theorem 2, we get the following result.

Theorem 3. Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some positive numbers k_1, k_2, ρ and some integer λ with $k_1 \geq 1, k_2 \geq 1, 0<\rho \leq 1, 0<\lambda \leq n-1$,

$$k_1 a_n \geq a_{n-1} \geq \ldots\ldots \geq k_2 a_\lambda \geq a_{\lambda-1} \geq \ldots\ldots \geq a_1 \geq \rho a_0 .$$

Then for any R>0,the number of zeros of P(z) in $\frac{|a_0|}{M} \leq |z| \leq \frac{R}{c}, c>1$ does not exceed

$$\frac{1}{\log c} \log \frac{K}{|P(0)|},$$

where

$$K=|a_n| R^{n+1}+|a_0|+R^n\left[\ k_1(|\alpha_n|+\alpha_n)-|\alpha_n|+2(k_2-1)|\alpha_\lambda|+|\alpha_0|-\rho(|\alpha_0|+\alpha_0)+2\sum_{j=0}^{n}|\beta_j|\ \right],$$

$$M=|a_n| R^{n+1}+R^n\left[\ k_1(|\alpha_n|+\alpha_n)-|\alpha_n|+2(k_2-1)|\alpha_\lambda|+|\alpha_0|-\rho(|\alpha_0|+\alpha_0)+2\sum_{j=0}^{n}|\beta_j|\ \right]$$

for $R \geq 1$ and

$$K=|a_n| R^{n+1}+|a_0|+R\left[\ k_1(|\alpha_n|+\alpha_n)-|\alpha_n|+2(k_2-1)|\alpha_\lambda|+|\alpha_0|-\rho(|\alpha_0|+\alpha_0)+2\sum_{j=0}^{n}|\beta_j|\ \right],$$

$$M=|a_n| R^{n+1}+R\left[\ k_1(|\alpha_n|+\alpha_n)-|\alpha_n|+2(k_2-1)|\alpha_\lambda|+|\alpha_0|-\rho(|\alpha_0|+\alpha_0)+2\sum_{j=0}^{n}|\beta_j|\ \right]$$

for $R \leq 1$.

For different values of the parameters, we get different results generalizing many known results already available in the literature. For example, for $\rho=1$, we get the following result from Theorem 3.

Corollary 1. Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some positive numbers k_1, k_2 and some integer λ with $k_1 \geq 1, k_2 \geq 1, 0<\lambda \leq n-1$,

International Journal of Advance Research in Science and Engineering
Volume No.07, Issue No.03, March 2018
www.ijarse.com
IJARSE
ISSN: 2319-8354

$$k_1 a_n \geq a_{n-1} \geq ...... \geq k_2 a_\lambda \geq a_{\lambda-1} \geq ...... \geq a_1 \geq a_0 .$$

Then for any R>0,the number of zeros of P(z) in $\frac{|a_0|}{M} \leq |z| \leq \frac{R}{c}, c > 1$ does not exceed

$$\frac{1}{\log c} \log \frac{K}{|P(0)|},$$

where

$$K = |a_n| R^{n+1} + |a_0| + R^n [\ k_1(|\alpha_n| + \alpha_n) - |\alpha_n| + 2(k_2 - 1)|\alpha_\lambda| - \alpha_0 + 2\sum_{j=0}^{n} |\beta_j|\ ],$$

$$M = |a_n| R^{n+1} + R^n [\ k_1(|\alpha_n| + \alpha_n) - |\alpha_n| + 2(k_2 - 1)|\alpha_\lambda| - \alpha_0 + 2\sum_{j=0}^{n} |\beta_j|\ ]$$

for $R \geq 1$ and

$$K = |a_n| R^{n+1} + |a_0| + R[\ k_1(|\alpha_n| + \alpha_n) - |\alpha_n| + 2(k_2 - 1)|\alpha_\lambda| - \alpha_0 + 2\sum_{j=0}^{n} |\beta_j|\ ],$$

$$M = |a_n| R^{n+1} + R[\ k_1(|\alpha_n| + \alpha_n) - |\alpha_n| + 2(k_2 - 1)|\alpha_\lambda| - \alpha_0 + 2\sum_{j=0}^{n} |\beta_j|\ ]$$

for $R \leq 1$.

For $k_2 = 1$, we get the following result due to Gulzar [4] from Theorem 3.

Corollary 2. Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some positive numbers k_1, ρ and some integer λ with $k_1 \geq 1, 0 < \rho \leq 1, 0 < \lambda \leq n-1$,

$$k_1 a_n \geq a_{n-1} \geq ...... \geq a_\lambda \geq a_{\lambda-1} \geq ...... \geq a_1 \geq \rho a_0 .$$

Then for any R>0,the number of zeros of P(z) in $\frac{|a_0|}{M} \leq |z| \leq \frac{R}{c}, c > 1$ does not exceed

IJARSE
ISSN: 2319-8354

$$\frac{1}{\log c}\log\frac{K}{|P(0)|},$$

where $K = |a_n|R^{n+1} + |a_0| + R^n\left[\ k_1(|\alpha_n| + \alpha_n) - |\alpha_n| + |\alpha_0| - \rho(|\alpha_0| + \alpha_0) + 2\sum_{j=0}^{n}|\beta_j|\ \right]$,

$$M = |a_n|R^{n+1} + R^n\left[\ k_1(|\alpha_n| + \alpha_n) - |\alpha_n| + |\alpha_0| - \rho(|\alpha_0| + \alpha_0) + 2\sum_{j=0}^{n}|\beta_j|\ \right]$$

for $R \geq 1$ and

$$K = |a_n|R^{n+1} + |a_0| + R\left[\ k_1(|\alpha_n| + \alpha_n) - |\alpha_n| + |\alpha_0| - \rho(|\alpha_0| + \alpha_0) + 2\sum_{j=0}^{n}|\beta_j|\ \right],$$

$$M = |a_n|R^{n+1} + R\left[\ k_1(|\alpha_n| + \alpha_n) - |\alpha_n| + |\alpha_0| - \rho(|\alpha_0| + \alpha_0) + 2\sum_{j=0}^{n}|\beta_j|\ \right]$$

for $R \leq 1$.

For R=1 in Theorem 3, we get the following result.

Corollary 3. Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some positive numbers k_1, k_2, ρ and some integer λ with $k_1 \geq 1, k_2 \geq 1, 0 < \rho \leq 1, 0 < \lambda \leq n-1$,

$$k_1 a_n \geq a_{n-1} \geq ...... \geq k_2 a_\lambda \geq a_{\lambda-1} \geq ...... \geq a_1 \geq \rho a_0 .$$

Then the number of zeros of P(z) in $\frac{|a_0|}{M} \leq |z| \leq \frac{1}{2}$ does not exceed

$$\frac{1}{\log c}\log\frac{K}{|P(0)|},$$

where

$$K = |a_n| + |a_0| + k_1(|\alpha_n| + \alpha_n) - |\alpha_n| + 2(k_2 - 1)|\alpha_\lambda| + |\alpha_0| - \rho(|\alpha_0| + \alpha_0) + 2\sum_{j=0}^{n}|\beta_j|\ \right],$$

$$M = |a_n| + k_1(|\alpha_n| + \alpha_n) - |\alpha_n| + 2(k_2 - 1)|\alpha_\lambda| + |\alpha_0| - \rho(|\alpha_0| + \alpha_0) + 2\sum_{j=0}^{n}|\beta_j| \ ].$$

If a_j is real i.e. $\beta_j = 0, \forall j = 0,1,....,n$, we get the following result from Theorem 3.

Corollary 4. Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some positive numbers k_1, k_2, ρ and some integer λ with $k_1 \geq 1, k_2 \geq 1, 0 < \rho \leq 1, 0 < \lambda \leq n-1$,

$$k_1 a_n \geq a_{n-1} \geq ...... \geq k_2 a_\lambda \geq a_{\lambda-1} \geq ...... \geq a_1 \geq \rho a_0 .$$

Then for any R>0,the number of zeros of P(z) in $\frac{|a_0|}{M} \leq |z| \leq \frac{R}{c}, c > 1$ does not exceed

$$\frac{1}{\log c} \log \frac{K}{|P(0)|},$$

where

$$K = |a_n|R^{n+1} + |a_0| + R^n[\ k_1(|a_n| + a_n) - |a_n| + 2(k_2 - 1)|a_\lambda| + |a_0| - \rho(|a_0| + a_0)\ ],$$

$$M = |a_n|R^{n+1} + R^n[\ k_1(|a_n| + a_n) - |a_n| + 2(k_2 - 1)|a_\lambda| + |a_0| - \rho(|a_0| + a_0)\ ]$$

for $R \geq 1$ and

$$K = |a_n|R^{n+1} + |a_0| + R[\ k_1(|a_n| + a_n) - |a_n| + 2(k_2 - 1)|a_\lambda| + |a_0| - \rho(|a_0| + a_0)\ ],$$

$$M = |a_n|R^{n+1} + R[\ k_1(|a_n| + a_n) - |a_n| + 2(k_2 - 1)|a_\lambda| + |a_0| - \rho(|a_0| + a_0)\ ]$$

for $R \leq 1$.

III.LEMMA

Lemma. If f(z) is analytic , $f(0) \neq 0$ and $|f(z)| \leq M$ for $|z| \leq R$, then the number of zeros of f(z) in $|z| \leq \frac{R}{c}, c > 1$ does not exceed $\frac{1}{\log c} \log \frac{M}{|f(0)|}$.

(For reference see [1]).

IV.PROOFS OF THEOREMS

Proof of Theorem 1. Consider the polynomial

$$F(z)=(1-z)P(z)$$

$$=-a_n z^{n+1}+(a_n-a_{n-1})z^n+......+(a_{\lambda+1}-a_\lambda)z^{\lambda+1}+(a_\lambda-a_{\lambda-1})z^\lambda+.....+(a_1-a_0)z+a_0$$
$$=-a_n z^{n+1}+(k_1\alpha_n-\alpha_{n-1})z^n-(k_1-1)\alpha_n z^n+(\alpha_{n-1}-\alpha_{n-2})z^{n-1}+....+(\alpha_{\lambda+1}-k_2\alpha_\lambda)z^{\lambda+1}$$

$$+(k_2-1)\alpha_\lambda z^{\lambda+1}+(k_2\alpha_\lambda-\alpha_{\lambda-1})z^\lambda-(k_2-1)\alpha_\lambda z^\lambda+(\alpha_{\lambda-1}-\alpha_{\lambda-2})z^{\lambda-1}+.....$$
$$+(\alpha_2-\alpha_1)z^2+(\alpha_1-\rho\alpha_0)z+(\rho-1)\alpha_0 z+i\{\sum_{j=1}^{n}(\beta_j-\beta_{j-1})z^j\}+a_0$$

$$=a_0+G(z)$$

where

$$G(z)=-a_n z^{n+1}+(k_1\alpha_n-\alpha_{n-1})z^n-(k_1-1)\alpha_n z^n+(\alpha_{n-1}-\alpha_{n-2})z^{n-1}+....+(\alpha_{\lambda+1}-k_2\alpha_\lambda)z^{\lambda+1}$$

$$+(k_2-1)\alpha_\lambda z^{\lambda+1}+(k_2\alpha_\lambda-\alpha_{\lambda-1})z^\lambda-(k_2-1)\alpha_\lambda z^\lambda+(\alpha_{\lambda-1}-\alpha_{\lambda-2})z^{\lambda-1}+.....$$
$$+(\alpha_2-\alpha_1)z^2+(\alpha_1-\rho\alpha_0)z+(\rho-1)\alpha_0 z+i\{\sum_{j=1}^{n}(\beta_j-\beta_{j-1})z^j\}.$$

For $|z|\le R$, we have by using the hypothesis

$$|G(z)|\le|a_n|R^{n+1}+|(k_1-1)\alpha_n|R^n+|k_1\alpha_n-\alpha_{n-1}|R^n+|\alpha_{n-1}-\alpha_{n-2}|.R^{n-1}+....+|\alpha_{\lambda+1}-k_2\alpha_\lambda|.R^{\lambda+1}$$

$$+|(k_2-1)\alpha_\lambda|.R^{\lambda+1}+|k_2\alpha_\lambda-\alpha_{\lambda-1}|.R^\lambda+|(k_2-1)\alpha_\lambda|.R^\lambda$$
$$+|\alpha_{\lambda-1}-\alpha_{\lambda-2}|.R^{\lambda-1}+.....+|\alpha_2-\alpha_1|.R^2+|\alpha_1-\rho\alpha_0|.R+|(\rho-1)\alpha_0|.R$$
$$+\sum_{j=1}^{n}(|\beta_j|+|\beta_{j-1}|)R^j$$

$$\le|a_n|R^{n+1}+R^n[\ (k_1-1)|\alpha_n|+k_1\alpha_n-\alpha_{n-1}+\alpha_{n-1}-\alpha_{n-2}+......+\alpha_{\lambda+1}-k_2\alpha_\lambda$$

$$+(k_2-1)|\alpha_\lambda|+k_2\alpha_\lambda-\alpha_{\lambda-1}+(k_2-1)|\alpha_\lambda|+\alpha_{\lambda-1}-\alpha_{\lambda-2}+.....$$
$$+\alpha_2-\alpha_1+\alpha_1-\rho\alpha_0+(1-\rho)|\alpha_0|+2\sum_{j=0}^{n}|\beta_j|\ ]$$

$$=|a_n|R^{n+1}+R^n[\ k_1(|\alpha_n|+\alpha_n)-|\alpha_n|+2(k_2-1)|\alpha_\lambda|+|\alpha_0|-\rho(|\alpha_0|+\alpha_0)+2\sum_{j=0}^{n}|\beta_j|\ ]$$
$$=M$$

for $R\geq 1$.

For $R\leq 1$,

$$|G(z)|\leq|a_n|R^{n+1}+R[\ k_1(|\alpha_n|+\alpha_n)-|\alpha_n|+2(k_2-1)|\alpha_\lambda|+|\alpha_0|-\rho(|\alpha_0|+\alpha_0)+2\sum_{j=0}^{n}|\beta_j|\ ]$$

$$=M'.$$

Since G(z) is analytic for $|z|\leq R$ and G(0)=0, it follows by Schwarz lemma that

$$|G(z)|\leq M|z| \text{ for } R\geq 1 \text{ and } |G(z)|\leq M'|z| \text{ for } R\leq 1 \text{ in } |z|\leq R.$$

Hence for $R\geq 1$,

$$|F(z)|=|a_0+G(z)|$$
$$\geq|a_0|-|G(z)|$$
$$\geq|a_0|-M|z|$$
$$>0$$

if $|z|<\dfrac{|a_0|}{M}$.

And for $R\leq 1$, $|F(z)|>0$ if $|z|<\dfrac{|a_0|}{M'}$.

In other words, all the zeros of F(z) lie in $|z|\geq\dfrac{|a_0|}{M}$ for $R\geq 1$ and in $|z|\geq\dfrac{|a_0|}{M'}$ for $R\leq 1$.

Since the zeros of P(z) are also the zeros of F(z), it follows that all the zeros of P(z) lie in $|z| \geq \frac{|a_0|}{M}$ for $R \geq 1$

and in $|z| \geq \frac{|a_0|}{M'}$ for $R \leq 1$.

That completes the proof of Theorem 1.

Proof of Theorem 2. For the polynomial

$$F(z) = (1-z)P(z)$$

$$\begin{aligned}
&= -a_n z^{n+1} + (a_n - a_{n-1})z^n + ... + (a_{\lambda+1} - a_\lambda)z^{\lambda+1} + (a_\lambda - a_{\lambda-1})z^\lambda + ... + (a_1 - a_0)z + a_0 \\
&= -a_n z^{n+1} + (k_1\alpha_n - \alpha_{n-1})z^n - (k_1 - 1)\alpha_n z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1} + ... + (\alpha_{\lambda+1} - k_2\alpha_\lambda)z^{\lambda+1} \\
&\quad + (k_2 - 1)\alpha_\lambda z^{\lambda+1} + (k_2\alpha_\lambda - \alpha_{\lambda-1})z^\lambda - (k_2 - 1)\alpha_\lambda z^\lambda + (\alpha_{\lambda-1} - \alpha_{\lambda-2})z^{\lambda-1} + ... \\
&\quad + (\alpha_2 - \alpha_1)z^2 + (\alpha_1 - \rho\alpha_0)z + (\rho - 1)\alpha_0 z + i\{\sum_{j=1}^{n}(\beta_j - \beta_{j-1})z^j\} + a_0,
\end{aligned}$$

We have , for $|z| \leq R$,

$$\begin{aligned}
|F(z)| &\leq |a_n|R^{n+1} + |a_0| + |(k_1 - 1)\alpha_n|R^n + |k_1\alpha_n - \alpha_{n-1}|R^n + |\alpha_{n-1} - \alpha_{n-2}|.R^{n-1} + ... + |\alpha_{\lambda+1} - k_2\alpha_\lambda|.R^{\lambda+1} \\
&\quad + |(k_2 - 1)\alpha_\lambda|.R^{\lambda+1} + |k_2\alpha_\lambda - \alpha_{\lambda-1}|.R^\lambda + |(k_2 - 1)\alpha_\lambda|.R^\lambda \\
&\quad + |\alpha_{\lambda-1} - \alpha_{\lambda-2}|.R^{\lambda-1} + ... + |\alpha_2 - \alpha_1|.R^2 + |\alpha_1 - \rho\alpha_0|.R + |(\rho - 1)\alpha_0|.R \\
&\quad + \sum_{j=1}^{n}(|\beta_j| + |\beta_{j-1}|)R^j
\end{aligned}$$

$$\begin{aligned}
&\leq |a_n|R^{n+1} + |a_0| + R^n[\ (k_1 - 1)|\alpha_n| + k_1\alpha_n - \alpha_{n-1} + \alpha_{n-1} - \alpha_{n-2} + ... + \alpha_{\lambda+1} - k_2\alpha_\lambda \\
&\quad + (k_2 - 1)|\alpha_\lambda| + k_2\alpha_\lambda - \alpha_{\lambda-1} + (k_2 - 1)|\alpha_\lambda| + \alpha_{\lambda-1} - \alpha_{\lambda-2} + ... \\
&\quad + \alpha_2 - \alpha_1 + \alpha_1 - \rho\alpha_0 + (1 - \rho)|\alpha_0| + 2\sum_{j=0}^{n}|\beta_j|\ ]
\end{aligned}$$

$$= |a_n|R^{n+1} + |a_0| + R^n[\ k_1(|\alpha_n| + \alpha_n) - |\alpha_n| + 2(k_2 - 1)|\alpha_\lambda| + |\alpha_0| - \rho(|\alpha_0| + \alpha_0) + 2\sum_{j=0}^{n}|\beta_j|\ ]$$

for $R \geq 1$.

And for $R \leq 1$,

$$F(z) \leq |a_n|R^{n+1} + |a_0| + R[\ k_1(|\alpha_n| + \alpha_n) - |\alpha_n| + 2(k_2 - 1)|\alpha_\lambda| + |\alpha_0| - \rho(|\alpha_0| + \alpha_0) + 2\sum_{j=0}^{n}|\beta_j|\ ].$$

Hence, by using the Lemma, it follows that the number of zeros of F(z) and therefore P(z) in $|z| \leq \frac{R}{c}, c > 1$ does not exceed

$$\frac{1}{\log c}\log\frac{K}{|P(0)|},$$

where

$$K = |a_n|R^{n+1} + |a_0| + R^n[\ k_1(|\alpha_n| + \alpha_n) - |\alpha_n| + 2(k_2 - 1)|\alpha_\lambda| + |\alpha_0| - \rho(|\alpha_0| + \alpha_0) + 2\sum_{j=0}^{n}|\beta_j|\ ]$$

for $R \geq 1$ and

$$K = |a_n|R^{n+1} + |a_0| + R[\ k_1(|\alpha_n| + \alpha_n) - |\alpha_n| + 2(k_2 - 1)|\alpha_\lambda| + |\alpha_0| - \rho(|\alpha_0| + \alpha_0) + 2\sum_{j=0}^{n}|\beta_j|\ ]$$

for $R \leq 1$.

That completes the proof of Theorem 2.

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