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Fermionic and Majorana Zero Modes in Nanowires

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ABSTRACT

One dimensional fermionic chain is reviewed in the light of Su-Schrieffer-Heeger model and Kitaev model and idea of topological phase in such system is illustrated. Fermionic zero modes with four fold ground state degeneracy exist in Su-Schrieffer-Heeger model which considers liner carbon chain having alternate single and double bonds. The zero modes are found to exist near the edge of the chain and protected by sub-lattice symmetry. However, the Kitaev model predicts the existence of majorana zero modes with two- fold ground state degeneracy. The Kitaev model assumes identical spin less sites and therefore sub-lattice symmetry does not come in to play. The majorana zero modes are robust and not susceptible to external perturbations. Experimental detection of majorana zero modes will have great impact in the field of quantum computing using topological qubits.

Keywords: Majorana Fermion, Topology, Superconducting pairing, Sub-lattice symmetry

I. INTRODUCTION

In particle physics, majorana fermions [1] are spin 1/2 particles which are their own anti-particles. In 1937, Ettore Majorana [2] suggested a modification of Dirac equation [3], which would require only real numbers to describe spin 1/2 particles. Till date, no such elementary particle has been found to fit this description. In the Standard model each lepton is paired with an anti-lepton. However the concept of excitations that are their own antiparticle emerges in an entirely different setting with which we are already familiar. A few of the quasi-particle excitations emerging in solid state systems may share this property. An example of such an excitation is that of an exciton which is a bound state of an electron and a hole. But they are not fermions anymore. They behave as bosons. They are no longer half integral excitations but are integral excitations. In this sense they resemble photons. In context of condensed matter systems, we ask the following question. Can there ever be a situation in which half-integer spin particles are their own antiparticles? At first sight it seems hopeless to realize such a situation simply because electrons are charged, and therefore definitely different from their antimatter counterparts, the (oppositely charged) holes. Thus we can enlist at least one ingredient that we need to realize a majorana fermion: electron hole symmetry. We can find such symmetry in superconducting systems. Thus one can guess superconductivity to play a central role in realizing a majorana fermion. A deeper insight could be gained by studying free fermion models in second quantized formalism.

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In this article we will look at two different one dimensional free fermion chains. The first system to be discussed in section II is the Su-Schrieffer-Heeger (SSH) model of poly-acetylene [4,5]. The purpose of studying this model is to realize how different coupling between neighboring sites could give rise to different phases distinguished by topology. We will contrast this with Kitaev chain in section III. A discussion on the recipe to implement this model in a realistic setting in presented in section IV. An understanding of Bolgoliubov de Gennes theory [6] of superconductivity is assumed.

II. SSH MODEL

We consider a one dimensional open chain of carbon atoms with alternating single and double bonds. A toy model could be constructed by considering any two adjacent carbon atoms as two distinct sites labeled A and B within a unit cell as shown in Fig. 1. The two different types of bonds may be thought as two different

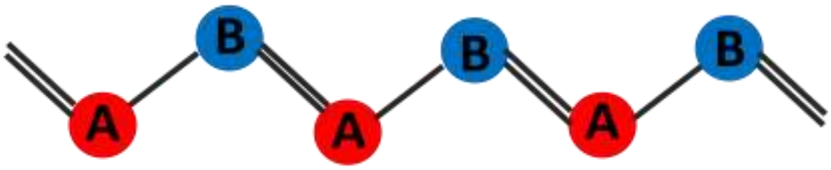

Figure 1. Fermionic chain (SSH Model)

magnitudes of coupling. Within a unit cell coupling will be labeled as J and coupling between unit cells will be labeled as t. A lattice model for the 3s orbital of the carbon atom is given by the following hamiltonian [4,5],

$$\mathcal{H} = -t\sum_x\left(c^\dagger_{x+1,A}\, c_{x,B} + \hbar.c.\right) - J\sum_x\left(c^\dagger_{x,A}\, c_{x,B} + \hbar.c.\right) - \mu\sum_{x,s} c^\dagger_{x,s}\, c_{x,s} \quad (1)$$

x labels the unit cell. This runs from 1 to L. $s \in \{A, B\}$, sub-lattice sites. c and $c^\dagger$ are the fermionic operators and μ is the chemical potential. We will assume t and J to be positive and real. Diagonalizing this Hamiltonian in the momentum space, the energy spectrum of excitations of the model can be written as,

$$\epsilon_k = -\mu \pm \sqrt{J^2 + t^2 + 2Jt\ \cos k} \quad (2)$$

For the rest of the discussion we will set chemical potential at zero. The ground state of this system corresponds to a completely filled lower band. There are two types of excitations in this system. The first is a neutral excitation costing at least $2|J - t|$. This is an excitation from the lower band to the upper band. The second is a charged excitation costing at least $|J - t|$. This is an excitation from lower band to the level of chemical potential. Notice that the above expression is symmetric with respect to interchanging J with t. Does this meanwe have same ground state for $J > t$ and $J <$ t cases?

To understand this qualitatively, we first set $t = 0$ and keep J finite. Then looking at the hamiltonian, $\mathcal{H}$ we find an absence of inter-unit cell coupling. The energy of this system depends on all the fermionic sites. The ground state corresponds to all the coupled fermionic sites (A, B) within each individual unit cells. This is in contrast to the case when we set $J = 0$ and keep t at some finite value. In this case again inspecting the Hamiltonian, $\mathcal{H}$ we find that there are two dangling fermionic sites $(1, A)$ and (L, B) of which the system's

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energy is independent. This gives rise to a 4-fold degeneracy in this ground state. (It does not matter whether $(1, A)$ or (L, B) site is filled up or empty). These fermionic modes are called zero modes because they commute with the Hamiltonian and it costs no extra energy to occupy these modes. They are also known as edge modes because they lie at the end points of the chain. This observation could be extended when $t > J > 0$. However, now the zero modes are not sharply localized at the edges of the chain but extend over few lattice sites. The zero modes decay exponentially with the distance from the edge.

Further one can verify that adding coupling between $A - A$ sites and $B - B$ sites make these zero modes to vanish. This means that the zero modes are not robust to any local perturbations of the hamiltonian. However they are protected, in this case, by sub-lattice symmetry because there are only $A - B$ coupling terms in the hamiltonian. One can distinguish the two regimes of the SSH hamiltonian discussed above, based on topology. A topological phase [7,8] (in the SSH model this corresponds to $t > J > 0$ regime, where there are edge modes) is characterized by a non vanishing topological invariant. If the topological invariant is zero, then we call the phase to be topologically trivial. A sketch for finding the topological invariant in context of SSH model is as follows: The topological invariant for the SSH model is the winding number of the eigen values defined in a complex plane. The hamiltonian, $\mathcal{H}$ in (1) is first written in momentum space using a sub-lattice basis and then the eigen values are found of this 2×2 matrix. In this model topological phase transition occurs at $t = J$. At this point the spectrum is gapless (easily verified from (2)). The eigen values could be represented in the complex plane to lie on the perimeter of a circle centered at $(J, 0)$ with radius t. Winding number is non zero if the interior of the circle encloses the origin and is zero if not. Adding any arbitrary term to the hamiltonian which respects the sub-lattice symmetry, the circle could be deformed into any closed loop. However, the results discussed above still hold. Now we generalize the definition of winding number as: the number of times the loop runs around origin when the hamiltonian is gapped. Winding number is not defined for the $(t = J)$ case since the loop intersects the origin (and also the hamiltonian has a gapless spectra). Table 1 summarizes the topological phase classification of SSH model.

Table 1. Topological phases in SSH model

	Spectra	Winding No.	Zero Modes	Phase
$J > t > 0$	Gapped	Zero	No	Trivial
$J = t$	Gapless	Undefined	-	Transition point
$t > J > 0$	Gapped	Non zero	Yes	Topological

III. KITAEV CHAIN

Relevant Kitaev proposed a simple, one dimensional model containing a tight-binding chain of spin-less electrons and a p-wave pairing superconducting term [9-11] as shown in Fig. 2. Conventional s-wave pairing is

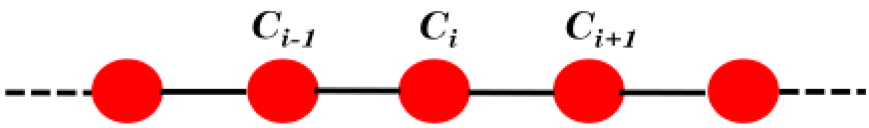

Figure 2. Free fermioinc chain (Kitaev Model)

not allowed because the electrons considered in this model are spin-less and hence Pauli exclusion prevents putting two spin-less electrons at the same site. Considering an open chain, the hamiltonian reads,

$$\mathcal{H} = -w\sum_j \left(c_j^\dagger\, c_{j+1} + h.c.\right) + \Delta\sum_j \left(c_j\; c_{j+1} + h.c.\right) - \mu\sum_j c_j^\dagger\, c_j \tag{3}$$

Δ is the superconducting pairing term. The effect of this term could be realized by putting the system in close proximity to a superconductor. The strength of the nearest neighbour site coupling is quantified by w. We have assumed Δ and w to be real. The index j runs from 1 to L (last fermionic site of the chain). Now one introduces majorana operators, where two majorana fermions (MF) describe one fermionic state,

$$\gamma_{2j-1} = c_j + c_j^\dagger \quad and \quad \gamma_{2j} = i\left(c_j^\dagger - c_j\right) \tag{4}$$

With the properties

$$\gamma_{2j-1}^\dagger = \gamma_{2j-1} \quad and \quad \gamma_{2j-1}^\dagger = \gamma_{2j-1} \tag{5}$$

$$\gamma_m^2 = 1 \tag{6}$$

$$\{\gamma_m\,, \gamma_n\,\} = 2\delta_{m,n} \tag{7}$$

$\{.\}$ is the anti-commutator. Inverting (4) one gets

$$c_j^\dagger = \frac{\gamma_{2j-1} - i\gamma_{2j}}{2} \quad \text{and} \quad c_j^\dagger = \frac{\gamma_{2j-1} - i\gamma_{2j}}{2} \tag{8}$$

Using (8) and the properties listed (5-7) one can write the hamiltonian (3) of the Kitaev chain after dropping the constant terms as,

$$\mathcal{H} = \frac{i}{4}\sum_j \left[-2\mu\gamma_{2j-1}\gamma_{2j} + 2(\Delta + w)\gamma_{2j}\gamma_{2j+1} + 2(\Delta - w)\gamma_{2j-1}\gamma_{2j+2}\right] \tag{9}$$

There is no extra information content in the Hamiltonian written in majorana operators as opposed to fermionic operators. However this form makes it apparent how the majoranas at each site are coupled with respect to each other.

Just like the qualitative analysis which we did for the SSH model (by tuning the coupling parameters), we can repeat the same analysis here. Let usconsider the case when $\Delta = w$. The last term in (9) drops out. One can see direct analogies with the SSH hamiltonian. Recall, the role of relative intra unit cell coupling and inter unit cell couplings J and t respectively are now played by μ and Δ which controls the coupling between odd to even majorana sites and even to odd majorana sites respectively. If we make even to odd coupling stronger than the odd to even coupling, then we find two dangling majorana fermions at the end of the chain when we take the limiting case $\mu = 0$. $\gamma_0$and γ_{2L}does not appear in the hamiltonian (9). Therefore these two correspond to the zero energy modes of the Hamiltonian. Hence they are referred as majorana zero modes in literature. These are the edge modes because they appear only at the ends of a chain. The ground state of the Kitaev chain has a twofold degeneracy. In general we will findnd zero modes whenever$|2w| > |\mu|$.

We can express the general form of hamiltonian (9) as,

$$\mathcal{H} = \frac{1}{4}\sum_{n,m} \gamma_m M_{m,n} \gamma_n \quad (10)$$

where $M_{m,n}$is purely imaginary and anti-symmetric. This follows from taking a hermitian conjugate on both sides of equation (9) and recalling that $\mathcal{H}$ is always hermitian followed by using anti-commutative property of majorana modes. In matrix notation, M could be expresses as,

$$M = iA \quad (11)$$

where A is real, anti-symmetric matrix. Now consider the time independent Schrodinger equation for Kitaev chain,

$$\mathcal{H}\psi = iA\psi = E\psi \quad (12)$$

Taking complex conjugate we find ,

$$\mathcal{H}\psi^* = -iA\psi^* = E\psi^* \quad (13)$$

Combining (12) and (13) implies that the eigen values of hamiltonian of the generic form (10) come in pairs $(E, -E)$. Thus we have a pair of robust zero modes at zero energy. Now we will analyze the robustness of these zero modes by perturbing the Hamiltonian. The terms in the Hamiltonian which are likely to shift the majorana zero modes from zero energy must have some combinations of γ_1 and γ_{2L}. Let us consider here a few such combinations: (i) γ_1, (ii) γ_1, (iii) $i\gamma_1\gamma_{2L}$, (iv) γ_1^2 and (v) γ_{2L}^2.

Terms proportional to (i) and (ii) are not allowed in the hamiltonian because it breaks the fermion parity, whereas (iv) and (v) give no contribution because it is unity using (6). One can argue that terms proportional to (iii) are highly un-likely because of its extreme nonlocal character. Thus we conclude that the zero modes are indeed quite robust. Differences with the SSH modelare summed up in the table below:

Table 2. Differences between SSH and Kitaev Model

SSH Model	Kitaev Model
Need sub-lattice symmetry to protect the zero modes	No symmetry constraints

SSH Model	Kitaev Model
4 fold ground state degeneracy	2 fold ground state degeneracy
Fermionic zero modes	Mazorana zero modes

IV. EXPERIMENTAL REALIZATION OF KITAEV MODEL

One requires three ingredients to realize the Kitaev model discussed above [12,13]. A typical system consists of a semiconductor nanowire with strong spin-orbit interaction like InAs or InSb placed over a s-wave superconductor. On top of this one requires to break time reversal symmetry by introducing an in-plane magnetic field along the length of the nanowire. The action of each ingredient could be visualized in energy momentum space as follows: We start with two fold spin degenerate electron bands (kinetic term; two overlapping parabolas). We use the Rashba spin-orbit effect to shift the spin up and spin down bands horizontally along k-axis. Turning on a magnetic field will open up a gap where there is a crossing. This will result in two spin-less bands. Actually half of the band left of the crossing is of one spin and the other half is of the other spin. After splitting each band has an equal character of spin up and spin down electrons. Now introducing superconductivity through proximity effect which pairs these two spin species now belonging to the same band and if the chemical potential is properly tuned such that we have an odd number of band crossings, we will end up in a topological phase with majorana zero modes at the end of the chain. A relative qualitative strength of these effects for observing a topological phase are as follows: spin-orbit interaction $>$ magnetic field strength $>$ superconducting pairing strength.

V. CONCLUSSION

Free fermionic chain is revisited comparing SSH and Kitaev Model to explore the existence of fermionic and majorana zero modes. In SSH model, the chain of carbon atoms having alternate single and double bond, encounters both site to site and inter unit cell coupling. Ferminonic zero modes with topologic phase exist for $t > J > 0$ at the edge of the chain however is not precisely localized but extends over few lattice sites. Sub-lattice symmetry become important to protect the zero modes and this phase is called symmetry protected phase. In contrary, the Kitaev model considers all the sites to be identical and spin less. Therefore neither sub-lattice symmetry nor the inter unit cell coupling exists. Rather in Kitaev model superconducting pairing terms comes in to play exhibiting proximity effect. For the limiting case of $\mu = 0$, majorana zero modes are formed, localized at the edges of the chain and are shown to have a two-fold ground state degeneracy. Since these zero modes are not symmetry protected, are robust against external perturbations. Existence of majorana zero modes can help forming topological qubits and revolutionize the area of quantum computing. Extensive experimental investigation is required to conclusively detect and manipulate majorana zero modes.

VI. ACKNOWLEDGEMENT

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