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AN INVITATION TO ABSTRACT MATHEMATICS VIA CO-PRODUCTS

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ABSTRACT

In this article our aim is to acquaint the reader of varied disciplines with abstract thinking in mathematics. We achieve this by examining the origin of the concept of mappings and universal principles. The universl property of coproduct is taken as an illustrating example.

I INTRODUCTION

There is an old story which we probably heard in the school, of a shepherd desperately trying to find a way to figure out any loss of sheep that he took out to graze. For the poor fellow had no concept of counting and was dependent only on his intuition. He must have had nightmares of slowly but surely losing his livestock right under his very nose. Until one day, he hit upon an ingenious plan to find out if he gets all his sheep back every day. He collected a lot of stones and placed one at the entrance to his farm every time a sheep went out. He followed that with another stone for the next one and so on until all the sheep were out of his farm. He did not know how many sheep he had. Indeed the concept of *how many* was not yet born! But he knew that day that he had exactly as many sheep as the stones that he had put at the entrance to his farm. What a discovery! What an excitement he must have felt and how anxiously he must have waited to bring the sheep back to the farm and test his theory. He must have picked up one stone for each sheep that entered back to the farm, picked the last stone for the last sheep that entered and might have been amazed at the ingenuity of the idea that he had discovered. To be able to tell which collection contains more or less number of things without ever counting how many each had is indeed an ingenious idea. It is like telling time without the invention of clocks. But this must be the natural order of discovery. Just as the concept of time must precede the invention of a clock, so must the concept of comparing sizes precede the concept of a number.

The above story (surely it is just a story!) is mathematical. The idea of comparing two different collections of things by corresponding to every member of a collection a unique member of the other collection is the beginning of abstract mathematical thought. This correspondence is given the technical name of a *function* or a *mapping* between the two collections. Notice that it does not answer the question as to how many members each collection has, rather it only tells us if the two collections have the same number of members or not. We will call any collection of things

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with which we can work unambiguously a *Set*. It is important to observe that it is not clear a priori as to which collections fall into this category of Sets. For only experience can tell us which collections let us work on them unambiguously! Mathematicians have given a set of rules called Axioms of Set Theory, to make sure that the collections that we work with are sets and lead to no contradictions, so far as we know. It is not the purpose of this work to enumerate all those Axioms of Set Theory. We mention them here only as a word of caution: Beware! All collections are not sets. Let us give a famous example to illustrate the point. The philosopher Bertrand Russell gave the following paradox:

Let R denote the collection of all those sets which are not their own members. So that A is a member of R precisely when A is not a member of A. We ask the simple question whether R is a member of R or not. If R is a member of R, then R is not a member of R since no member of R is a member of itself. On the other hand if R is not a member of itself, then it must indeed be a member of R since R includes all those sets which are not their own members.

It is clear then from the above example that it is not at all easy to decide as to which collections constitute a set. The resolution of the above paradox is simply that R is not a set. We will have more examples later of collections of well defined objects which cannot constitute a set. If we notice carefully, the major problem in the Russell paradox above and indeed in many such contradictions is the free use of the phrase "*is a member of*." So since such a simple thing is to be applied rather carefully, the mathematicians have agreed upon to use the symbolic form for "x is a member of A" and write it technically as "$x \in A$", and "x is not a member of A", is written as "$x \notin A$". This keeps us aware that words in mathematics often have different meanings than the common use of those words in everyday life. However we will not be very pedantic with the notation and freely move from the English to the symbolic or vice versa as per convenience. Also we use the braces notation for a set. In this symbolic form the Russell paradox says that if $R = \{A \mid A \notin A\}$, Then $R \in R$ if and only if $R \notin R$. Indeed R is not a set and is in fact called the Russell Class. For more details on the axiomatic foundations of Set Theory the reader is referred to the first two chapters of Dugundji [3]. We are now ready to make our first formal definition below.

1.1 Definition. A **mapping** f, also called a **function**, from a set A to a set B is a rule that associates a unique member of B to each member of A. We write it as $f : A \to B$. A is called the **domain** of f and B is called the **co-domain** of f. If each member of B is associated with some member of A, we say that the mapping is **onto** or **surjective**. If distinct members of A correspond to distinct members of B, we say that the map f is **injective** or **one to one**. If the mapping f is both injective as well as surjective, we say that f is a **bijection** or a **one to one correspondence**.

To write more succinctly, for a map $f : A \to B$, if x is any member of A, we use the symbol f(x) to denote the unique member of B which f associates to x. f(x) is called the **image of x under f**. In this notation,

(i) f is injective, if $f(x_1) = f(x_2)$ *implies* that $x_1 = x_2$. Equivalently f is injective if whenever $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$;

(ii) f is surjective, if for each member y of B, there exists a member x of A, such that f(x) = y.

If our shepherd corresponds different stones to different sheep, then his function of associating stones with sheep is injective from the set of his sheep to the set of stones that he has. If all his stones have been so used to correspond to sheep, then his function is onto. Notice very importantly that he must correspond exactly one stone with each sheep for otherwise we do not even have a function.

1.2 Remark on the word "implies" : The word *implies* is used in precise mathematical sense. The statement "S *implies* T" is *not* to be read as "there is a proof of statement T from the statement S". If "S *implies* T", it means that if S is true, then T is true. Equivalently, if T is not true then S also cannot be true. Further if S is false, then the statement "S *implies* T" is necessarily true irrespective of whether T is true or not! For the only way to prove that the statement "S *implies* T" is false is to have the truth of S on one hand and the falsity of T on the other. This will not be the case if S is false! So for instance the statement "New York is not in USA *implies* India is hotter than the Sun" is a TRUE statement! The statement "S *implies* T" does not at all refers to the truth or falsity of S or T rather it only gives a logical dependence of the truth of T on the truth of S. It should then be very clear to the reader that the use of the word *implies* in mathematics is very different to its use in everyday life. In mathematics, a *false statement implies everything*! Finally we note that the statement of the form "S if and only if T" means that statement S implies statement T *and* statement T implies statement S. We say that S and T are logically equivalent.

The following definition gives another fundamental concept about which we will have a lot more to say in the future.

1.3 Definition Any set of members of a set X is called a **subset** of X and we write $A \subset X$, for "A is subset of X". Notice that $A \subset X$ precisely when each member of A is a member of X. In particular $X \subset X$. Also, two sets X and Y are equal if and only if each member of X is a member of Y and vice versa, so that X = Y if and only if $X \subset Y$ *and* $Y \subset X$. Further we call a collection that consists of no members at all as the **empty set** and agree to always denote it by the symbol ϕ. Notice that $\phi \subset X$ for any set X and that $\{\phi\}$ is not an empty set since it consists of a member, namely ϕ.

Further progress is almost impossible without the following:

1.4 Definition Given any mappings $X \to Y \to Z$ we define the **composition of mappings** as the mapping $h \circ p : X \to Z$ given by $(h \circ p)(x) = h(p(x))$. The h o p image of x is the h image of the p image of x.

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II CO-PRODUCT OF SETS

As our application of the idea of mappings, we acquaint ourselves with the concept of a *universal mapping property* through an important set operation. In many cases, given some sets we need to find a single set which includes the given sets in some best possible way. So, if for instance we have two subsets A and B of a set Γ, we seek another subset, say, A + B, which will somehow include A and B and will do so in the best possible way. For the sake of simplicity we assume, for the time being, that the two sets A and B have nothing in common. There is no common member of A and B. We say in such cases that A and B are **disjoint**. In this case we note the following two plausible requirements for A + B

(i) Since A + B should include A as well as B, it should be possible to go from either of A or B to the set A + B that we seek. We may therefore assume that there will be two mappings, one each from the set A and from the set B into A + B. This should give us a diagram of the form i_1:A → A + B ← B: i_2

(ii) For any other diagram of the form given in (i) above, say, A → X ← B, consisting of mappings from A and B to any set X, we must have some property ensuring that the diagram i_1:A → A + B ← B: i_2 turns out to be **the best possible one**.

After a little thought, we see that the first requirement above can be easily fulfilled if we take the set A + B = {x ∈ Γ | x ∈ A OR x ∈ B}. That is to say that the set A + B consists of all those members of Γ that are either the members of the set A or members of the set B. For this would then ensure that i_1 and i_2 can simply be taken as the maps i_1(a) = a for each a in A and i_2(b) = b for each b in B. For obvious reasons, these are called the inclusion maps and they give us the diagram of the form i_1: A → A + B ← B : i_2. For this diagram to satisfy the second requirement, consider the following situation:

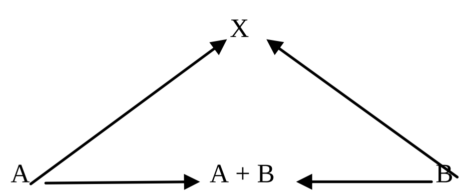

Fig 1.

Here X is any set and the arrows from A and B to X represent any mappings from A and B to X, say f : A → X and g : B → X . The horizontal arrows are our inclusion maps i_1 and i_2. Now, since A + B is defined in such a way that there is no extraneous member of A + B, that is the only members of A + B are either members of A or of B, therefore there is an obvious way of going from A + B to X by using the mappings from A and B into X. Namely, there is a mapping u : A + B → X, given by u(x) = f(x) if x is in A and u(x) = g(x) if x is in B. The map u is unambiguously defined precisely because the sets A and B are assumed to be disjoint so that we may safely apply either f(x) or g(x) when computing u(x). We have the following diagram:

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$$\begin{tikzcd} & X & \\ A \arrow[ur] \arrow[r] & A + B \arrow[u] & B \arrow[ul] \arrow[l] \end{tikzcd}$$

Fig 2.

The mapping u is such that $u(i_1(a) = f(a)$ and $u(i_2(b) = g(b)$. This means that the mappings f and g can be written in terms of the inclusion maps and the mapping u. It means that f and g do not contain any more information than that which is already contained in the inclusion maps i_1 and i_2. We extract the information about f and g from the inclusion maps with the help of the mapping u. The mapping u is usually denoted by the symbol [f , g]. In terms of composition of mappings we have in Figure 2, $u \circ i_1 = f$ and $u \circ i_2 = g$. We say that Figure 2 is **commutative**. We also note that the mapping u is the unique mapping that makes Figure 2 commutative.

The reader should at this point check the uniqueness of [f , g] in the commutative diagram in Figure 2.

We have now proved our first theorem:

2.1 Theorem (Universal Characterization for Co-Products). *For any disjoint sets A and B, there is a set A + B and a diagram of mappings $i_1 :\ A \to A + B \leftarrow B : i_2$ such that for any diagram of the form $f : A \to X \leftarrow B : g$, where X is any set, there is a unique mapping, denoted by $[f , g]: A + B \to X$, such that the following diagram is commutative*

$$\begin{tikzcd} & X & \\ A \arrow[ur] \arrow[r] & A + B \arrow[u] & B \arrow[ul] \arrow[l] \end{tikzcd}$$

Fig 3.

2.2 Remarks (i) The property of the triple $(A + B, i_1, i_2)$ proved in the above theorem is an example of a **universal mapping property.** The word "*universal*" signifying that among *all the diagrams of the form $f: A \to X \leftarrow B :g$, the diagram $A \to A + B \leftarrow B$ is the best in the sense that any such diagram can be decomposed or factorized in terms of the diagram $A \to A + B \leftarrow B$ in a unique manner in the sense of the Theorem 2.1 above*. So in a way i_1: *$A \to A + B \leftarrow B : i_2$ encodes in it the information about all diagrams of the form $A \to X \leftarrow B$ and the map $[f , g]$ helps us to decode that information.*

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(ii) Any diagram of the form A → Q ← B is called a **co-product diagram** and Q is called the **co-product** of A and B precisely when it has the universal mapping property of the Theorem 2.1. Formally, we make the following

2.3 Definition. For given sets A and B, a diagram of the form A → Q ← B is called a **co-product diagram** whenever for any other diagram of the same form, say, A → X ← B, there is a unique mapping Q → X such that the following diagram is commutative

Fig 4.

$$\begin{tikzcd} & X & \\ A \arrow[ur] \arrow[r] & Q \arrow[u] & B \arrow[ul] \arrow[l] \end{tikzcd}$$

The triple (Q, j, k) is called the **co-product** of A and B and is then said to satisfy the universal mapping property of co-products. Q is usually written as A + B.

The set operation involved in the construction of A + B is formalized in the following

2.4 Definition For any subsets A and B of a set Γ, the set $\{x \in \Gamma \mid x \in A \text{ OR } x \in B\}$ is called the **union** of the sets A and B and is denoted by $A \cup B$.

We have proved in Theorem 2.1 that *the disjoint union of A and B along with the inclusion maps i_1 and i_2 can be taken as the* ***co-product*** *of A.*

2.5 Remark It is pertinent to remark that in the definition $A \cup B = \{x \in \Gamma \mid x \in A \text{ OR } x \in B\}$, we have another instance where common words are used in a special mathematical sense and differ from their everyday use. The condition "$x \in A$ OR $x \in B$" in the definition of $A \cup B$ is satisfied for a member x if x is a member of at least one of A or B. So, in particular it may be a member of both. This is a little different from the use of "OR" in everyday life, where it is usually assumed that one of the two given condition is to be satisfied and not the both. In mathematics, "Statement S OR Statement T" means that *at least* one of S or T is true.

What happens when the sets A and B whose co-product we seek are not disjoint? Then is there no best possible diagram of the form A → Q ← B? We can still get the best possible diagram of this form, except that we cannot use the set $A \cup B$ for Q. However, something *essentially the same* as their unions will do the job for us. We have the following

2.6 Theorem *For any two subsets A and B of a set Γ, there is a set Q and a co-product diagram of the form* $A \to Q \leftarrow B$*. The set Q can therefore be denoted by* $A + B$*.*

Proof. Define the set A^1 as the set whose members are couples of the form (a , 1) where a is a member of the set A. That is, to each member of A we associate a symbol 1 and whenever we have a member a of A we have the member (a , 1) of A^1. Similarly, we define the set B^2, whose members consist of all the couples of the form (b , 2) where b is a member of the set B. Even if some x is a member of both A and B, it gives distinct members of A^1 and B^2, namely (x , 1) and (x , 2) respectively. A^1 and B^2 are therefore disjoint sets. We will see that the union of disjoint sets, A^1 and B^2 can be taken as the co-product of A and B. For this we define the mappings from A and B into $A^1 \cup B^2$, namely $q_1 : A \to A^1 \cup B^2$ given by $q_1(a) = (a , 1)$ and $q_2 : A \to A^1 \cup B^2$ given by $q_2(b) = (b , 2)$. Then for any diagram of the form $f : A \to X \leftarrow B : g$, where X is any set, define the mapping $u : A^1 \cup B^2 \to X$ by $u(a , 1) = f(a)$ and $u(b , 2) = g(b)$. Then $u \circ q_1 = f$ and $u \circ q_2 = g$. So the following diagram is commutative

$$\begin{tikzcd} & X & \\ A \arrow[ur] \arrow[r] & A^1 \cup B^2 \arrow[u] & B \arrow[ul] \arrow[l] \end{tikzcd}$$

Fig 5.

We leave it for the reader to verify that the map u is unique. Hence $A \to A^1 \cup B^2 \leftarrow B$ is a co-product diagram.

In view of Theorems above we can always say that the co-product set for given sets A and B is their disjoint union, even if A and B are not disjoint, in which case by their disjoint union we mean, the union of the disjoint sets A^1 and B^2 obtained in the above Theorem 2.6.

III CONCLUSION

1. The point of the above is not to proved new results. The concept of co-product is well known (see for instance [1]), rather it is to introduce to the reader a way of thinking about ordinary mathematical constructs (like disjoint unions in our case) in a way which allows us to see them as answers to a universal problem given in terms of universal mapping property. 2. Every concept in mathematics is associated with its universal property. The deep relationship between topological properties and universal mapping properties is well documented in Brown [2].
The author intends to discuss more universal properties in the forthcoming papers.

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