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A BRIEF STUDY ON CARDINAL B- SPLINE WAVELETS

Rupali Gupta[1], Gagan Batra[2]

[1,2]*Mathematics Department, Khalsa College For Women, Ludhiana, Punjab (India)*

ABSTRACT

This paper focuses on the concepts of Cardinal B-Spline wavelets, their Properties and description of their relations. Spline wavelets are extremely regular and usually symmetric or anti-symmetric. They can be designed to have compact support and to achieve optimal time-frequency localization (B-spline wavelets). The underlying scaling functions are the B-splines, which are the shortest and most regular scaling functions of order L which will be further classified into linear, quadratic or biquadratic spline wavelets. Their graphical representations are enclosed herein.

***Keywords*: Compact Structures, Control points, Interpolation,Orthonormal, Spline wavelets.**

I. INTRODUCTION

In this paper, we try to visualize the basic concepts of B- spline cardinal wavelets and their properties[1].

A spline wavelet is a wavelet constructed using a spline functionwhich is basically orthogonal, but do not have any compact supports. The interpolatory spline wavelets were introduced by C.K. Chui and J.Z. Wang. There is a certain class of wavelets, unique in some sense, having compact supports with some special properties. These special wavelets are called B-spline wavelets or cardinal B-spline wavelets. A Basis spline, often called as B-spline is a spline function that has the minimum support with respect to some given degree, sleekness, and domain partition. All spline functions of a given degree can be expressed as Linear combination[6] of B-splines of that degree. Due to this reason, B-splines of order 'n' are basis functions for spline functions of the same order defined over the same knots. Cardinal B-splines have knots that are equidistant from each other.

1.1 Definition

B-spline is a combination of flexible bands that are continuous and passes through a number of points called control points. A B-spline of order n is a polynomialfunction of degree $n-1$ in a variable t that guarantees the continuity and derivability of the function uptoorder $n-1$. It is defined over $1+m$ locations t_j, called knots or breakpoints, which must be in non-descending order .The B-spline always contribute only in the range between the first and last of these knots and is zero elsewhere. If each knot is placed at the same distance $t_{j+1}-t_j$ from its predecessor and typically positioned at the integers, the knot vector and the corresponding B-splines are called 'uniform'.

Let a vector known as the knot vector be defined as $T = \{t_0, t_1, t_2, t_3, \ldots\ldots\ldots\ldots\ldots t_m\}$ where T is a non descending sequence with $t_i \in [0,1]$ and define control points $P_0, P_1, P_2 \ldots\ldots\ldots P_n$. Define the degree as $p \equiv m - n - 1$. The "knots" $t_{p+1}, t_{p+2}, \ldots\ldots t_{m-p-1}$ are called internal knots.

Define the basis function as

$$N_{i,0}(t) = \begin{cases} 1 & \text{if } t_i \le t \le t_{i+1} \text{ and } t_i < t_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

$$N_{i,j}(t) = \frac{t - t_i}{t_{i+j} - t_i} N_{i,j-1}(t) + \frac{t_{i+j+1} - t}{t_{i+j+1} - t_{i+1}} N_{i+1,j-1}(t) \tag{1}$$

where $j = 1,2,3,\ldots\ldots\ldots\ldots p$. Then the curve defined by $C(t) = \sum_{i=0}^{n} P_i N_{i,p}(t)$ is a B-spline. A cardinal B-spline is a special type of cardinal spline. For any positive integer m the cardinal B-spline of order m, denoted by $N_m(t)$ is defined recursively as follows which is basically Schoenberg's representation of splines in terms of B splines basis functions [2][3]

$$N_1(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$N_m(t) = \int_0^1 N_{m-1}(t - x)\,dt \quad \text{for } m > 1 \tag{2}$$

The two-scale relation[4] for B-spline scaling functions of general order m is

$$N_m(t) = \sum_{k=0}^{m} p_k N_m(2t - k) \tag{3}$$

where the two-scale sequence $\{p_k\}$ for B-spline scaling functions are given by :

$$p_k = 2^{-m+1}\left({}^{m}C_k\right), \text{ for } 0 \le k < m \tag{4}$$

The two-scale relation for B-spline wavelets for general order m is given by

$$\psi_m(x) = \sum_{k=0}^{3m-2} q_k N_m(2x - k) \tag{5}$$

where $q_k = (-1)^k 2^{1-m} \sum_{l=0}^{m} \left({}^{m}C_l\right) N_{2m}(k + 1 - l)$ (6)

II. PROPERTIES OF CARDINAL B SPLINE WAVELETS

a) By the support of a continuous function which vanishes outside some bounded interval, we mean the smallest closed set outside which the function vanishes identically. The standard notation is $SuppN_m$ Thus, $SuppN_m = [0,1]$ which implies that support of a cardinal B spline wavelet is always a closed interval.

b) The function $N_m(t)$ is non-negative, which implies $N_m(t) > 0 \ for \ 0 < m < 1$

c) $\sum_{r=-\infty}^{\infty} N_m(t-r) = 1 \ \forall t$.

d) $\int_{-\infty}^{\infty} N_m(t) \ dt = 1$

e) Few Efficient algorithms for computing $N_m(t)$ and all its derivatives are available[1]. The derivative of $N_m(t)$ is given by $N'_m(t) = N_{m-1}(t) - N_{m-1}(x-1)$.

f) The B-splines are symmetric. $N_m(t)$ is symmetric for even m and anti symmetric for odd m about the center $x = \frac{m}{2}$

g) The cardinal B splines of order $m \ and \ m-1, that \ is, \ N_m \ and \ N_{m-1}$ are related as

$$N_m(t) = \frac{t}{m-1} N_{m-1}(t) + \frac{m-t}{m-1} N_{m-1}(t-1) \qquad (7)$$

This relation is useful to compute $N_m(t)$ at some integer values. Non-zero values of $N_m(t)$ for some small m are summarized in Table 1.

Table 1: Non-zero $N_m(k)$ values for $m = 2, \ldots, 6$

$N_m(t)$		t					
		0	1	2	3	4	5
m	2	0	1	0	………		
	3	0	½	½	0	……..	
	4	0	1/6	2/3	1/6	0	…….
	5	0	1/24	11/24	11/24	1/24	0
	6	0	1/120	26/120	66/120	26/120	1/120

h) Cardinal B splines are transformation invariant.

2.1 Reisz Basis

For any pair of integers m and j with $m \geq 2$, the family $S_j = \left\{ 2^{j/2} N_m \left(2^t t - k : k \in Z \right) \right\}$ is a Reisz basis of subspace $V_j^m \in L^2(R)$ with Reisz bounds A and B given as

$$A = \frac{1}{2m-1} \prod_{k=1}^{m} \frac{(1+\lambda_k)^2}{|\lambda_k|} > 0 \ and \ B = 1 \text{ and also } \sum_{k=-\infty}^{\infty} \left| N_m(\omega + 2\pi k) \right|^2 \geq \sum_{k=-\infty}^{\infty} \left| N_m(\pi + 2\pi k) \right|^2 \qquad (8)$$

The detailed proof and explanation are being referred [1]

2.2 Multi Resolution Analysis

A function $\varphi \in L^2(R)$ is said to generate a Multi Resolution Analysis (MRA)[5] if it generates a nested sequence of closed subspaces $W_j; j \in Z$ that satisfy the following properties:

a) $W_j \subseteq W_{j+1} \ \forall j$

b) $closure \ in \ L^2(R) \ of \left(\bigcup_{j \in Z} W_j \right) is \, equal \, to \, the \ whole \ space \ that \ is \ L^2(R)$

c) $\bigcap_{j \in Z} W_j = \{0\}$

d) $W_{j+1} = W_j \oplus V_j$

e) $f(x) \in W_j \ iff \ f(2x) \in W_{j+1} \ \forall j$

The property d) describes the orthonormality of the subspaces where V_j is the orthogonal complement of W_{j+1} in W_j

2.3 Spline Interpolation

The goal of cubic spline interpolation is to get an interpolation formula that is continuous in both the first and second derivatives, both within the intervals and at the interpolating nodes. This will give us a smoother interpolating function. In general, cubic splines always play an important role than linear interpolation.

The wavelet interpolation function $\overline{\psi}(x)$, is the wavelet function whose scaling function is the interpolation spline $\overline{\psi}(x) = \sum d_i \overline{\varphi}(2x - i)$. The Fourier transform of spline interpolation wavelet is

$$\overline{\psi}(\omega) = \frac{c(-z)}{c(z^2)} Q(z) \, \overline{\varphi}\left(\frac{\omega}{2}\right) \text{ where } \sum d_i z^i = \frac{c(-z)}{c(z^2)} Q(z) \tag{9}$$

III. TYPES OF CARDINAL B SPLINE WAVELETS

The different cardinal B splines of various orders are defined under the convention which starts from particularly B spline of order 1, namely, $N_1(t)$ which takes the value 1 in interval [0,1) and 0 elsewhere. This is called as CONSTANT B spline wavelet.

$$N_1(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & elsewhere \end{cases} \tag{10}$$

The 1st order B-Spline $N_1(t)$ is the Haar scaling function where Haar Scaling Function is defined as sequence of rescaled square shaped functions which together forms a wavelet family or basis. The Haar wavelet is till now the simplest known form of the wavelets studied over the years. There is a speciality in these wavelets that its disadvantage, not being continuous and hence not being differentiable even, is still an advantage for the analysis of signals with sudden transitions, such as monitoring of tool failure in machines. Mathematically, Haar Scaling function is representated as

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$$\phi(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & elsewhere \end{cases} \qquad (11)$$

The two-scale relation for Haar scaling function is

$$\phi(t) = \phi(2t) + \phi(2t-1) \qquad (12)$$

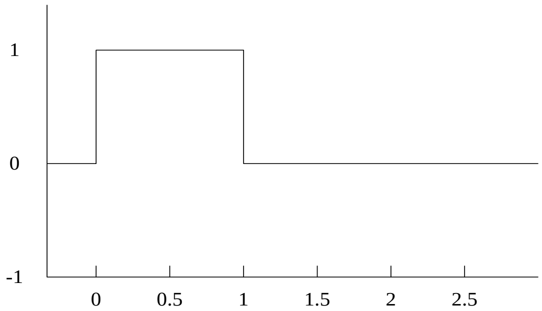

Fig.1: Haar scaling function ϕ(t).

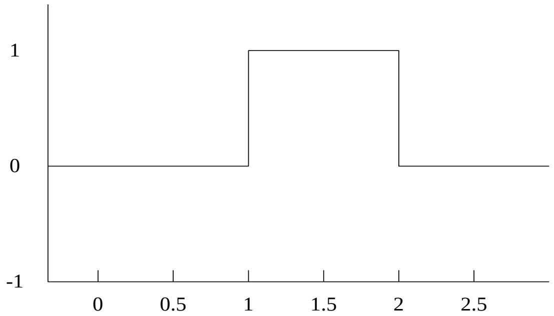

Fig.2: Haar scaling function ϕ(t-1).

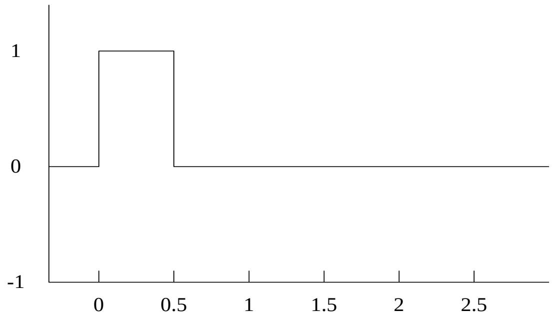

Fig.3: Haar scaling function ϕ(2t).

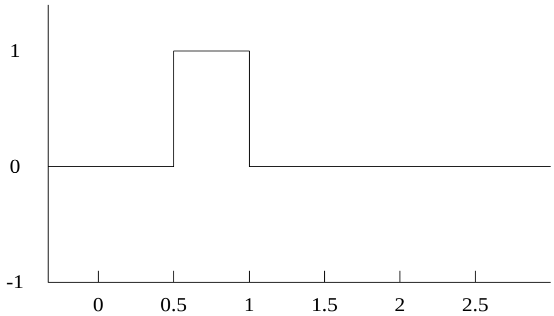

Fig.4: Haar scaling function ϕ(2t-1).

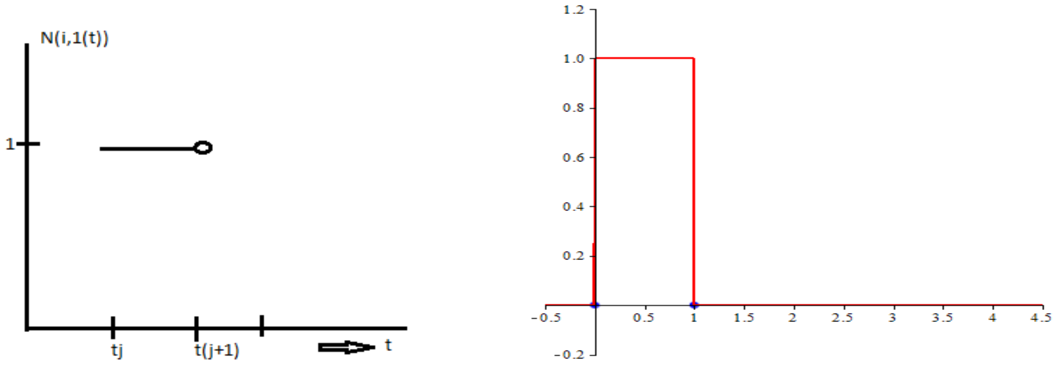

Fig. 5 Constant B Spline, $N_1(t)$

Further, B spline of order 2, namely, $N_2(t)$ is called as linear spline, derived from (2) by replacing $m = 2$ and defined as

$$N_2(t) = \begin{cases} t & 0 \le t < 1 \\ -t+2 & 1 \le t < 2 \\ 0 & elsewhere \end{cases} \tag{13}$$

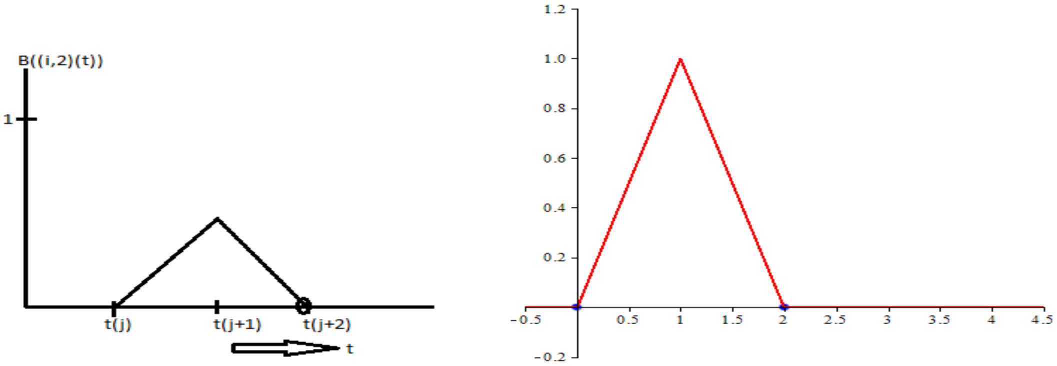

Fig.6 LinearB Spline, $N_2(t)$

And, the two-scale relation for Linear B-Spline wavelets is derived from (5) by substituting*m=2*

$$\psi_2(x) = \sum_{k=0}^{4} q_k N_2(2x-k) \tag{14}$$

where
$$\begin{aligned} q_k &= (-1)^k 2^{-1} \sum_{l=0}^{2} \binom{2}{l} N_4(k+1-l) \\ &= \left(-\tfrac{1}{2}\right)^k \{N_4(k+1) + 2N_4(k) + N_4(k-1)\} \end{aligned} \tag{15}$$

The two scale sequence q_k is obtained by substituting the values of *k=0,1,2,3,4* and thus, obtaining the Linear B spline wavelet of order 2.

$$\psi_2(x) = \frac{1}{12} N_2(2x) - \frac{1}{2} N_2(2x-1) + \frac{5}{6} N_2(2x-2) - \frac{1}{2} N_2(2x-3) + \frac{1}{12} N_2(2x-4) \tag{16}$$

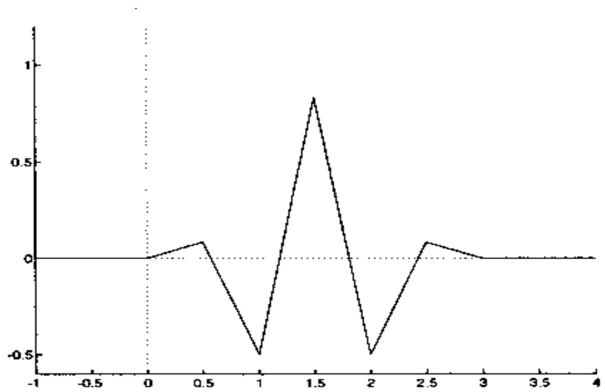

Fig.7 LinearB Spline wavelet, $\psi_2(x)$

B splines of order 3 and order 4, namely, $N_3(t)$ *and* $N_4(t)$ are called as Quadratic B spline and Cubic B spline scaling functions respectively and defined as:

$$N_3(t)=\begin{cases}\dfrac{t^2}{2} & 0\le t<1\\ -t^2+3t-\dfrac{3}{2} & 1\le t<2\\ \dfrac{t^2}{2}-3t+\dfrac{9}{2} & 2\le t<3\\ 0 & elsewhere\end{cases} \qquad (17)$$

The quadratic B spline wavelet is given by

$$\begin{aligned}\psi_3(x)=&\frac{1}{480}N_3(2x)-\frac{29}{480}N_3(2x-1)\\ &+\frac{147}{480}N_3(2x-2)-\frac{303}{480}N_3(2x-3)\\ &+\frac{303}{480}N_3(2x-4)-\frac{147}{480}N_3(2x-5)\\ &+\frac{29}{480}N_3(2x-6)-\frac{1}{480}N_3(2x-7)\end{aligned}$$

$$(18)\, The\ cubic\ slpine\quad N_4(t)=\begin{cases}\dfrac{t^3}{6} & 0\le t<1\\ -\dfrac{t^3}{2}+2t^2-2t+\dfrac{2}{3} & 1\le t<2\\ \dfrac{t^3}{2}-4t^2+10t-\dfrac{22}{3} & 2\le t<3\\ -\dfrac{t^3}{6}+2t^2-8t+\dfrac{32}{3} & 3\le t<4\\ 0 & elsewhere\end{cases}$$

(19)

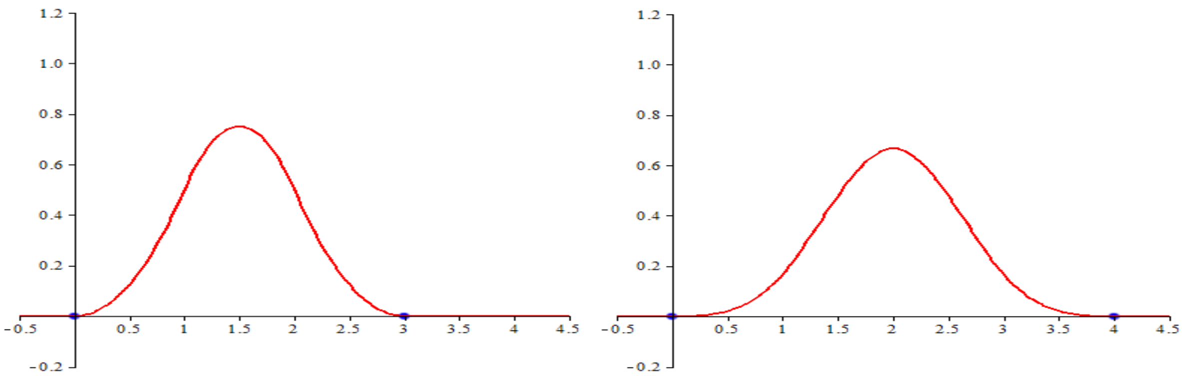

Fig. 8Quadratic B Spline $N_3(t)$ and Cubic B spline $N_4(t)$

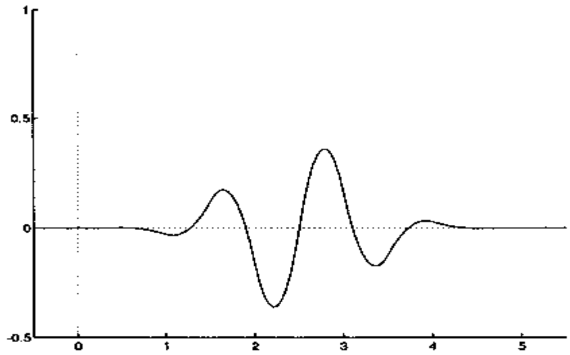

Fig. 9 Quadratic B Spline wavelet $\psi_3(x)$

IV. APPLICATIONS

Due to the orthonormal bases, wavelets provide fast algorithms in numerical aspects in taking approximations. Semi-orthogonal compactly supported B spline wavelets behave better and easier than other wavelets in a bounded interval or closed intervals. For these reasons, they are good alternatives for solving integral equations. Splines are smooth, regular and well behaved functions. Splines of degree n are (n-1) times continuously differentiable due to which splines have excellent approximation properties. Convergence properties of splines are relevant for coding applications. Bsplines and their wavelet counterparts have excellent localization properties so they are good templates for timefrequency signal analysis. The compactly supported Bspline wavelets have been found to be powerful tool in many scientific and practical application including mathematical approximation, the finite element method, image processing and compression and computer-aided geometric design.In computer aided design, computer aided manufacturing and computer graphics, a powerful extension of Bsplines is non-uniform rational Bsplines (NURBS). NURBS are essentially Bsplines in homogeneous coordinates. Like Bsplines, NURBS control points determine the shape of the curve.

V. CONCLUSION

Wavelets constructed via multiresolution analysis taking Bspline as scaling function generate orthonormal basis or semi-orthonormal basis depending on order of spline for the wavelet space. Constructed B-spline wavelets have a compact support and explicit formulae which reduces the calculation effort. These wavelets are easier to handle in bounded interval and due to polynomial function have excellent approximation property.

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