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Skew Laplacian Energy of Complete Bipartite Digraphs

Bilal Chat[1] S. Pirzada[2]

[1,2]Department of Mathematics University of Kashmir-Srinagar (India)

ABSTRACT

Let D be a digraph with skew-adjacency matrix S(D). The skew energy of D is defined as the sum of the norms of all eigen values of S(D). Two digraphs are said to be skew equienergetic if their energies are equal. In this paper we obtain the skew laplacian energy $\widetilde{SL}(D)$ of complete directed bipartite graph $K_{n,2}$ and $K_{n,m}$

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I. INTRODUCTION

Let D be a digraph with n vertices $v_1, v_2, ...v_n$ and m arcs. Let $d_i^+ = d^+(v_i), d^i = d^-(v_i)$ and $d_i = d_i^+ + d_i^-$, $i = 1, 2, \ldots, n$ be the outdegree, indegree and degree of vertices of D, respectively. The out-adjacency matrix $A^+(D) = (a_{ij})$ of a digraph D is the $n \times n$ matrix, where $a_{ij} = 1,$ if (v_i, v_j) is an arc and $a_{ij} = 0$, otherwise. The in adjacency matrix $A^-(D) = (a_{ij})$ of a digraph D is the $n \times n$ matrix, where $a_{ij} = 1,$ if (v_j, v_i) is an arc and $a_{ij} = 0$. It is clear that $A^-(D) = (A^+(D))^t$.

The skew adjacency matrix S(D) = (s_{ij}) of a digraph D is the $n \times n$ matrix, where (s_{ij}) = 1, if there is an arc from v_i to v_j, (s_{ij}) = -1, if there is an arc from v_j to v_i and (s_{ij}) = 0, otherwise.

It is clear that S(D) is a skew symmetric matrix, so all its eigenvalues are zero or purely imaginary. The energy of the matrix S(D) was considered in [1], and is defined as

$$E_s(D) = \sum_{i=1}^{n} |\zeta_i|,$$

where $\zeta_1, \zeta_2, \ldots, \zeta_n$ are the eigen values of S(D). This energy of a digraph D is called the skew energy by Adiga et al. [1]. For recent developments in the theory of skew energy, see the survey [10].

Let $D^+(G) =$ diag($d_1^+, d_2^+, \ldots, d_n^+$), $D^-(G) =$ diag($d_1^-, d_2^-, \ldots, d_n^-$) and D(G) = diag($d_1, d_2, \ldots, d_n$) be the diagonal of vertex out degrees, vertex in degrees and vertex degrees of D respectively.

Many results have been obtained on the skew spectra and skew spectral radii of oriented graphs [2, 5, 4, 6, 7, 9].

Recently (in 2013) Cai et al. [3] defined a new type of skew laplacian matrix $\widetilde{SL}(D)$ of a digraph D as follows.

Let $D^{+}(D)$ and $D^{-}(G)$ respectively be the diagonal matrices of vertex out degree and vertex in degree and let $A^{+}(D)=(a_{ij})$ and $A^{-}(D)=(a_{ij})$ respectively be the out-adjacency and in-adjacency matrix of a digraph D. If A(G) is the adjacency matrix of the underlying graph G of the digraph D, then it is clear that A(G) = $A^{+}(D)$ + $A^{-}(D)$ and S(D) = $A^{+}(D)$ - $A^{-}(D)$ where S(D) is the skew adjacency matrix of D. Therefore, following the definition of Laplacian matrix of a graph, Cai et al. called the matrix

$\widetilde{SL}(D) = (D^{+}(D)- D^{-}(D)) - (A^{+}(D)- A^{-}(D))$

$= \widetilde{D}(D)-S(D)$

where $\widetilde{D}(D)=D^{+}(D)-D^{-}(D)$ as the skew laplacian matrix of the digraph D. It is clear that the matrix $\widetilde{SL}(D)$ is not symmetric, so its eigen values need not be real. However, we have the following observation.

Theorem 1.1.

(i) $v_1, v_2, \ldots, v_n$ are the eigen values of $\widetilde{SL}(D)$, then $\sum_{i=1}^{n} v_i = 0$

(ii) 0 is an eigen value of $\widetilde{SL}(D)$ with multiplicity p, where p is the number of components of D with all ones vector (1, 1, 1,…, 1) as the corresponding eigen vector.

Following the definition of matrix energy given by Nikifrov and Cai et al. [3] defined the skew laplacian energy of a digraph D, as the sum of the absolute values of the eigen values of the matrix $\widetilde{SL}(D)$ and obtained various bounds.

In this paper, we will confinene ourselves to the definition of laplacian energy of a digraph given by Cai et al. [3].

Definition 1.2. Skew laplacian energy of a digraph. Let D be a digraph of order n with m arcs and having skew laplacian eigen values $(\mu_1, \mu_2, \ldots, \mu_n)$. The skew laplacian energy of D is denoted by $\widetilde{SLE}(D)$ and is defined as

$$\widetilde{SLE}(D) = \sum_{i=1}^{n} |\mu_i|$$

This concept was introduced in 2013 by Cai et al. [3]. The idea of Cai et al. was to conceive a graph energy like quantity for a digraph, that instead of skew adjacency eigen values is defined in terms of skew laplacian eigen values and that hopefully would preserve the main features of the original graph energy. The definition of $\widetilde{SLE}(D)$ was therefore so chosen that all the properties possessed by graph energy should be preserved.

In [8], we show that every even positive integer is indeed the skew Laplacian energy of some digraph.

Theorem 1.3. Every even positive integer 2(n - 1) is the skew laplacian energy of a directed star.

We prove the following main result.

Theorem 1.4. Every positive integer 4(n-1) is the skew laplacian energy of a complete oriented bipartite graph $K_{n,2}$.

Proof. Let $V(K_{n,2}) = \{v_1, v_2, \ldots, v_n, v_{n+1}, v_{n+2}\}$ be the vertex set of $K_{n,2}$. If $V_1 = \{v_1, v_2, \ldots, v_n\}$ is the partite set and $V_2 = \{v_{n+1}, v_{n+2}\}$ is the another partite set of the complete directed bipartite graph orient all edges towards $V_1 = \{v_1, v_2, \ldots, v_n\}$ from $V_2 = \{v_{n+1}, v_{n+2}\}$. Then

$$S(K_{n,2}) = \begin{bmatrix} 0 & 0 & 0 & \cdots & -1 & -1 \\ 0 & 0 & 0 & \cdots & -1 & -1 \\ 0 & 0 & 0 & & -1 & -1 \\ \vdots & \vdots & & & \vdots & \vdots \\ & & & & & \\ 1 & 1 & 1 & \cdots & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 \end{bmatrix}$$

$$D(K_{n,2}) = \begin{bmatrix} -2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -2 & 0 & \ldots & 0 & 0 \\ 0 & 0 & -2 & & 0 & 0 \\ \vdots & \vdots & & & \vdots & \vdots \\ & & & & & \\ 0 & 0 & 0 & \ldots & n & 0 \\ 0 & 0 & 0. & \ldots. & 0 & n \end{bmatrix}$$

Therefore,

$$\widetilde{SL}(D) = \begin{bmatrix} -2 & 0 & 0 & \cdots & 1 & 1 \\ 0 & -2 & 0 & \cdots & 1 & 1 \\ 0 & 0 & -2 & & 1 & 1 \\ \vdots & \vdots & & & \vdots & \vdots \\ & & & & & \\ -1 & -1 & -1 & \cdots & -n & 0 \\ -1 & -1 & -1 & \cdots & 0 & n \end{bmatrix}$$

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By direct calculation it can easily be seen that skew laplacian characteristic polynomial of this matrix is $x[x-(n-2](x-2)^{n-1}(x-n)$. Therefore it is easy to see that the eigen values of this matrix are ($n-2, 0, (-2)^{n-1}, n$) and so $S\tilde{L}(E(K_{n,2})) = 4(n-1)$.

On the other hand, we orient all the edges from V_2 to V_1 then it can be seen that

$$S\tilde{L}(K_{n,2}) = \begin{bmatrix} 2 & 0 & 0 & \cdots & -1 & -1 \\ 0 & 2 & 0 & \cdots & -1 & -1 \\ 0 & 0 & 2 & & -1 & -1 \\ \vdots & \vdots & & & \vdots & \vdots \\ & & & & & \\ 1 & 1 & 1 & \cdots & -n & 0 \\ 1 & 1 & 1 & \cdots & 0 & -n \end{bmatrix}$$

having the skew laplacian characteristic polynomial of this matrix is $x[x+(n-2](x+2)^{n-1}(x+n)$ and so eigen values (-($n-2$), 0, $(2)^{n-1}, -n)$, so $S\tilde{L}E(K_{n,2}) = 4(n-1)$. Thus for a complete oriented bipartite graph $K_{n,2}$, we have $S\tilde{L}E(K_{n,2}) = 4(n-1)$.

Example 1.5. Let D = $K_{3,2}$ be a digraph as shown below with partite sets $V_1 = \{v_1, v_2, v_3\}$ and $V_2 = \{u_1, u_2\}$ oriented all edges from V_2 to

Clearly,

$$D(K_{3,2}) = \begin{bmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \quad \text{and} \quad S(K_{3,2}) = \begin{bmatrix} 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Therefore,

$$S\tilde{L}(K_{3,2}) = \begin{bmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

Hence skew laplacian spectrum of $\widetilde{S}L(K_{3,2}) = (1,0,-2,-2,3)$ so, $\widetilde{S}LE(K_{3,2}) = 4(n-1) = 4(3-1).$ =8 = 1+ 0+ 2+ 2+3. Similarly the skew laplacian spectrum of $\widetilde{S}LE(K_{3,2}) = (-1,0,2,2,-3)$ if all edges are oriented from V_1 to V_2. Hence, $\widetilde{S}LE(K_{3,2}) = 4(n-1) = 4(3-1).$ = 8 = 1+ 0+ 2+ 2+3.

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