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Bounds for the Zeros of Lacunary-Type Polynomials

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ABSTRACT

In this paper we give a bound for the zeros of a polynomial. The results so obtained generalizes as well as refines many known results proved in this direction.

***MATHEMATICS SUBJECT CLASSIFICATION:** 30C10, 30C15.*

KEY WORDS AND PHRASES: Coefficients, Polynomial, Zeros.

I. INTRODUCTION

The following result known as the Cauchy's Theorem [2] (see also [7, page 123]), is well-known on the location of zeros of a polynomial:

Theorem A. All the zeros of the polynomial $P(z)=\sum_{j=0}^{n} a_j z^j$ of degree n lie in the circle $|z|<1+M$, where $M=\max_{0\le j\le n-1}\left|\frac{a_j}{a_n}\right|$.

In the literature [6,7,9] , various bounds for all or some of the zeros of a polynomial

$$P(z)=a_0+a_1z+......+a_nz^n$$

are available. In either case the bounds are expressed as the functions of all the coefficients $a_0, a_1, ...,a_n$ of P(z).

An important class of polynomials is that of the lacunary type i.e. of the type

$$P(z)=a_0+a_1z+......+a_pz^p+a_{n_1}z^{n_1}+a_{n_2}z^{n_2}+........+a_{n_k}z^{n_k},$$

where $0<p=n_0<n_1<n_2<......<n_k; a_0a_pa_{n_1}a_{n_2}.......a_{n_k}\neq 0$, the coefficients $a_j, 0\le j\le p$, are fixed, $a_{n_j}, j=1,2,......,k$ are arbitrary and the remaining coefficients are zero. Landau[4,5] initiated the study of such polynomials in 1906-7 in connection with his study of the Picard's theorem and proved that every trinomial

$$a_0+a_1z+a_nz^n, a_1a_n\neq 0, n\ge 2$$

has at least one zero in $|z|\le 2\left|\frac{a_0}{a_1}\right|$ and every quadrinomial

$$a_0+a_1z+a_mz^m+a_nz^n, a_1a_ma_n\neq 0, 2\le m<n$$

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has at least one zero in $|z| \le \frac{17}{3}\left|\frac{a_0}{a_1}\right|$.

Q.G.Mohammad [8] in 1967 proved the following theorem:

Theorem B. All the zeros of the polynomial $P(z)=\sum_{j=0}^{n} a_j z^j$ of degree n lie in the circle

$$|z| \le \max(L_p, L_p^{\frac{1}{n}})$$

where

$$L_p = n^{\frac{1}{q}}\left\{\sum_{j=0}^{n}\left|\frac{a_j}{a_n}\right|^p\right\}^{\frac{1}{p}},$$

p>1,q>1with $\frac{1}{p}+\frac{1}{q}=1.$

Aziz and Rather [1] in 2013 proved the following result:

Theorem C. For every positive number t, all the zeros of the polynomial $P(z)=\sum_{j=0}^{n} a_j z^j$ of degree n lie in the circle

$$|z| \le (n+1)^{\frac{1}{q}}\left\{\sum_{j=0}^{n}\left|\frac{ta_j - a_{j-1}}{a_n t^{n-j}}\right|^p\right\}^{\frac{1}{p}},$$

where p>1, q>1with $\frac{1}{p}+\frac{1}{q}=1.$

However Gulzar and wani [3] proved the following result.

Theorem D. All the zeros of the polynomial

$$P(z)=a_0+a_1 z+......+a_\lambda z^\lambda + a_{n_1} z^n, a_\lambda \ne 0,\ 0 \le \lambda \le n-1$$

of degree n lie in the circle

$$|z| \le \max(L_p, L_p^{\frac{1}{n}})$$

where

$$L_p = (\lambda+2)^{\frac{1}{q}}\left\{\sum_{j=0}^{\lambda+1}\left|\frac{a_j - a_{j-1}}{a_n}\right|^p\right\}^{\frac{1}{p}},\ a_{\lambda+1}=0=a_{-1},$$

p>1,q>1with $\frac{1}{p}+\frac{1}{q}=1.$

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II. MAIN RESULTS

In this paper we consider the following generalization of Theorem C and Theorem D, more precisely we prove,

Theorem 1. For every positive number t, all the zeros of the polynomial

$$P(z)=a_0+a_1z+......+a_\lambda z^\lambda+a_{n_1}z^n, a_\lambda\neq 0,\ 0\leq\lambda\leq n-1$$

of degree n lie in the circle

$$|z|\leq t\ \max(L_p, L_p^{\frac{1}{n}})$$

where

$$L_p=(\lambda+2)^{\frac{1}{q}}\left\{\sum_{j=0}^{\lambda+1}\left|\frac{ta_j-a_{j-1}}{a_nt^{n-j+1}}\right|^p\right\}^{\frac{1}{p}},\ a_{\lambda+1}=0=a_{-1},$$

p>1,q>1with $\frac{1}{p}+\frac{1}{q}=1.$

Remark 1. For t =1, Theorem 1 reduces to Theorem D.

Remark 2. For $\lambda=n-1$ in Theorem 1, we get Theorem C.

Remark 3. For $\lambda=n-1$ and t =1 in Theorem 1, we get a result due to Gulzar and Wani ([3], Corollary 1).

III. PROOF OF THE THEOREM

Proof of Theorem 1. Consider the polynomial

$$\begin{aligned}F(z)&=(t-z)P(z)\\&=(t-z)(a_nz^n+a_\lambda z^\lambda+a_{\lambda-1}z^{\lambda-1}+......+a_1z+a_0)\\&=-a_nz^{n+1}-a_\lambda z^{\lambda+1}+(ta_\lambda-a_{\lambda-1})z^\lambda+(ta_{\lambda-1}-a_{\lambda-2})z^{\lambda-1}+......+(ta_1-a_0)z+ta_0\\&=-a_nz^{n+1}+\sum_{j=0}^{\lambda+1}(ta_j-a_{j-1})z^j\end{aligned}$$

Therefore

$$\begin{aligned}|F(z)|&\geq|a_n||z|^{n+1}-\sum_{j=0}^{\lambda+1}|ta_j-a_{j-1}||z|^j\\&=|a_n||z|^{n+1}\left[1-\sum_{j=0}^{\lambda+1}\left|\frac{ta_j-a_{j-1}}{a_nt^{n-j+1}}\right|.\frac{t^{n-j+1}}{|z|^{n-j+1}}\right]\\&\geq|a_n||z|^{n+1}\left[1-\left\{\left(\sum_{j=0}^{\lambda+1}\left|\frac{ta_j-a_{j-1}}{a_nt^{n-j+1}}\right|^p\right)^{\frac{1}{p}}\left(\sum_{j=0}^{\lambda+1}\left(\frac{t^{n-j+1}}{|z|^{(n-j+1)}}\right)^q\right)^{\frac{1}{q}}\right\}\right]\end{aligned}$$

by applying Holder's inequality.

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Now , if $L_p \geq 1$ then $\max(L_p, L_p^{\frac{1}{n}}) = L_p$. Therefore, for $|z| \geq t$ so that $\left(\frac{t}{|z|}\right)^{n-j+1} \leq \frac{t}{|z|}$. Hence , for

$|z| > t\ L_p$,

$$|F(z)| \geq |a_n||z|^{n+1}\left[1-\left\{\left(\sum_{j=0}^{\lambda+1}\left|\frac{ta_j - a_{j-1}}{a_n t^{n-j+1}}\right|^p\right)^{\frac{1}{p}}\left(\sum_{j=0}^{\lambda+1}\left(\frac{t}{|z|}\right)^q\right)^{\frac{1}{q}}\right\}\right]$$

$$= |a_n||z|^{n+1}\left[1-\frac{t(\lambda+2)^{\frac{1}{q}}}{|z|}\left(\sum_{j=0}^{\lambda+1}\left|\frac{ta_j - a_{j-1}}{a_n t^{n-j+1}}\right|^p\right)^{\frac{1}{p}}\right]$$

$$= |a_n||z|^{n+1}\left[1-\frac{t\,L_p}{|z|}\right]$$

$$> 0.$$

Again , if , if $L_p \leq 1$ then $\max(L_p, L_p^{\frac{1}{n}}) = L_p^{\frac{1}{n}}$. Therefore, for $|z| \leq t$ so that $\left(\frac{t}{|z|}\right)^{n-j+1} \leq \left(\frac{t}{|z|}\right)^n$. Hence

,for $|z| > t\,L_p^{1/n}$,

$$|F(z)| \geq |a_n||z|^{n+1}\left[1-\left\{\left(\sum_{j=0}^{\lambda+1}\left|\frac{ta_j - a_{j-1}}{a_n t^{n-j+1}}\right|^p\right)^{\frac{1}{p}}\left(\sum_{j=0}^{\lambda+1}\left(\frac{t}{|z|}\right)^{nq}\right)^{\frac{1}{q}}\right\}\right]$$

$$= |a_n||z|^{n+1}\left[1-L_p\left(\frac{t}{|z|}\right)^n\right]$$

$$> 0.$$

From the above development it follows that F(z) does not vanish for

$$|z| > t\max(L_p, L_p^{\frac{1}{n}}).$$

Consequently all the zeros of F(z) and hence P(z) lie in

$$|z| \leq t\max(L_p, L_p^{\frac{1}{n}}).$$

That completes the proof of Theorem 1.

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