International Journal of Advance Research in Science and Engineering
Vol. No.6, Issue No. 08, August 2017
www.ijarse.com
IJARSE
ISSN (O) 2319 - 8354
ISSN (P) 2319 - 8346

Algorithms for some geometric properties of non-transversasl intersection curve of hypersurfaces in $\mathbb{R}^5$

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Abstract

In this paper, we present the algorithms for calculating the differential geometric properties of the non-transversal intersection curve of four parametric hypersurfaces in $\mathbb{R}^5$. In transversal intersection the normals of the hypersurfaces at the intersection point are linearly independent and the tangential direction can be easily obtained as their cross product. While as in non-transversal intersection the normals of the hypersurfaces at the intersection point are linearly dependent, thus we need to devise whole new algorithms to derive tangential direction and other geometric quantities.

Keywords: Hypersurfaces, transversal intersection, non-transversal intersection.
MSC 2010:53A05, 53A04.

1. Introduction

The surface-surface intersection problem is a fundamental process needed in modeling shapes in CAD/CAM system. It is useful in the representation of the design of complex objects and animations. The two types of surfaces mostly used in geometric designing are parametric and implicit surfaces. For that reason, different methods have been given for either parametric-parametric or implicit-implicit surface intersection curves in $\mathbb{R}^3$. The numerical marching method is the most widely used method for computing the intersection curves in $\mathbb{R}^3$ and $\mathbb{R}^4$. The marching method involves generation of sequences of points of an intersection curve in the direction prescribed by the local geometry [2, 12]. To compute the intersection curve with precision and efficiency, approaches of superior order are necessary, that is, they are needed to obtain the geometric properties of the intersection curves. Differential geometry of a parametric curve in $\mathbb{R}^3$ can be found in textbooks such as Struik [23], Willmore [24], Stoker [22], do Carmo [18]. whereas differential geometry of parametric curves in $\mathbb{R}^n$ can be found in

International Journal of Advance Research in Science and Engineering
Vol. No.6, Issue No. 08, August 2017
www.ijarse.com
IJARSE
ISSN (O) 2319 - 8354
ISSN (P) 2319 - 8346

the textbook such as in klingenberg [26] and in the contemporary literature on Geometric Modeling [3, 7]. On the other hand, for the differential geometry of intersection curves, there exists a little literature. Willmore [24] and Aléssio [16] presented algorithms to obtain the unit tangent, unit principal normal, unit binormal, curvature and torsion of the transversal intersection curve of two implicit surfaces. Hartmann [4] provided formulas for computing the curvature of the intersection curves for all types of intersection problems in $\mathbb{R}^3$. Ye and Maekawa [27] presented algorithms for computing the differential geometric properties of both transversal and tangential intersection curves of two surfaces. Aléssio [14] formulated the algorithms for obtaining the geometric properties of intersection curves of three implicit hypersurfaces in $\mathbb{R}^4$. Based on the work of Aléssio [14], Mustufa Düldül [10] worked with three parametric hypersurfaces in $\mathbb{R}^4$ to derive the algorithms for differential geometric properties of transversal intersection. Abdel-All et al. [1] formulated algorithms for geometric properties of implicit-implict-parametric and implicit-parametric-parametric hypersurfaces in $\mathbb{R}^4$. Aléssio et al. [17] obtained algorithms for differential geometric properties of non-transversal intersection curves of three parametric hypersurfaces in $\mathbb{R}^4$. Recently Naeim Badr et al. [20] derived the algorithms for non-transversal intersection curves of implicit-parametric-parametric and implicit-implicit-parametric hypersurfacres in $\mathbb{R}^4$.

2. Preliminaries

Definition 2.1. Let $\{e_1, e_2, e_3, e_4, e_5\}$ be the standard basis of five dimensional Euclidean space E^5. Then the vector product of the vectors $x = \sum_{i=1}^{5} x_i e_i$, $y = \sum_{i=1}^{5} y_i e_i$, $z = \sum_{i=1}^{5} z_i e_i$ and $w = \sum_{i=1}^{5} w_i e_i$ is defined by

$$x \otimes y \otimes z \otimes w = \begin{vmatrix} e_1 & e_2 & e_3 & e_4 & e_5 \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ y_1 & y_2 & y_3 & y_4 & y_5 \\ z_1 & z_2 & z_3 & z_4 & z_5 \\ w_1 & w_2 & w_3 & w_4 & w_5 \end{vmatrix}. \tag{1}$$

The vector product $x \otimes y \otimes z \otimes w$ yields a vector that is orthogonal to x, y, z, w.

let $R \subset E^5$ be a regular hypersurface given by $\Phi = \Phi(u_1, u_2, u_3, u_4)$ and $\gamma : I \subset \mathbb{R} \to \Phi$ be an arbitrary curve with arc length parametrisation. If $\{t, n, b_1, b_2, b_3\}$ is the Frenet Frame along γ, then we have

$$\begin{cases} t' = \kappa_1 n, \\ n' = -\kappa_1 t + \kappa_2 b_1, \\ b_1' = -\kappa_2 n + \kappa_3 b_2, \\ b_2' = -\kappa_3 b_1 + \kappa_4 b_3, \\ b_3' = -\kappa_4 b_2, \end{cases} \tag{2}$$

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Vol. No.6, Issue No. 08, August 2017
www.ijarse.com
IJARSE
ISSN (O) 2319 - 8354
ISSN (P) 2319 - 8346

where t, n, b_1, b_2 and b_3 denote the tangent, the principal normal, the first binormal, the second binormal and third binormal vector fields. The normal vector n is the normalised acceleration vector γ''. The unit vector b_1 is determined such that n' can be decomposed into two components, a tangent one in the direction of t and a normal one in the direction of b_1. The unit vector b_2 is determined such that b_1' can be decomposed into two components - a normal and another in the direction of b_2. The unit vector b_3 is the unique unit vector field perpendicular to four dimensional subspace $\{t, n, b_1, b_2\}$. The functions κ_1, κ_2, κ_3 and κ_4 are the first, second, third and fourth curvatures of $\gamma(s)$. The first, second, third and fourth curvatures measure how rapidly the curve pulls away in a neighbourhood of s, from the tangent line, from planar curve, from three dimensional curve and from the four dimensional curve at s, respectively.
Now, using the Frenet Frame we have the derivatives of γ as

$$\begin{aligned}
\gamma' &= t, \quad \gamma'' = t' = \kappa_1 n, \quad \gamma''' = -\kappa_1^2 t + \kappa_1' n + \kappa_1\kappa_2 b_1, && (3)\\
\gamma^{(4)} &= -3\kappa_1\kappa_1' t + (-\kappa_1^3 + \kappa_1'' - \kappa_1\kappa_2^2)n + (2\kappa_1'\kappa_2 + \kappa_1\kappa_2')b_1 \\
&\quad + \kappa_1\kappa_2\kappa_3 b_2, && (4)\\
\gamma^{(5)} &= (-3(\kappa_1')^2 - 4\kappa_1 k_1'' + \kappa_1^4 + \kappa_1^2\kappa_2^2)t + (-6\kappa_1^2\kappa_1' + \kappa_1''' && (5)\\
&\quad -\kappa_1'\kappa_2^2 - 3\kappa_1\kappa_2\kappa_2' - 2\kappa_1'\kappa_{22}^2)n + (\kappa_1^3\kappa_2 - \kappa_1\kappa_2^3 && (6)\\
&\quad +3\kappa_1''\kappa_2 + 3\kappa_1'\kappa_2' + \kappa_1\kappa_2'' - \kappa_1\kappa_2\kappa_3^2)b_1 + (3\kappa_1'\kappa_2\kappa_3 \\
&\quad +2\kappa_1\kappa_2'\kappa_3 + 2\kappa_1\kappa_2'\kappa_3 + \kappa_1\kappa_2\kappa_3')b_2 \\
&\quad +\kappa_1\kappa_2\kappa_3\kappa_4 b_3. && (7)
\end{aligned}$$

Also since Φ is regular, the partial derivatives Φ_1, Φ_2, Φ_3, Φ_4, where ($\Phi_i = \frac{\partial\Phi}{\partial u_i}$) are linearly independent at every point of Φ , i.e., $\Phi_1 \otimes \Phi_2 \otimes \Phi_3 \otimes \Phi_4 \neq 0$. Thus, the unit normal vector of Φ is given by

$$N = \frac{\Phi_1 \otimes \Phi_2 \otimes \Phi_3 \otimes \Phi_4}{\|\Phi_1 \otimes \Phi_2 \otimes \Phi_3 \otimes \Phi_4\|}.$$

Furthermore, the first, second, and the third binormal vectors of the curve are given by

$$b_3 = \frac{\gamma' \otimes \gamma'' \otimes \gamma''' \otimes \gamma^{(4)}}{\|\gamma' \otimes \gamma'' \otimes \gamma''' \otimes \gamma^{(4)}\|}, \quad b_2 = \frac{b_3 \otimes \gamma' \otimes \gamma'' \otimes \gamma'''}{\|b_3 \otimes \gamma' \otimes \gamma'' \otimes \gamma'''\|},$$
$$b_1 = \frac{b_2 \otimes b_3 \otimes \gamma' \otimes \gamma''}{\|b_2 \otimes b_3 \otimes \gamma' \otimes \gamma''\|} \qquad (8)$$

and the curvatures are obtained by

$$\kappa_1 = \|\gamma''\|, \ \ \kappa_2 = \frac{\langle\gamma''', b_1\rangle}{\kappa_1}, \ \ \kappa_3 = \frac{\langle\gamma^{(4)}, b_2\rangle}{\kappa_1\kappa_2}, \ \ \kappa_4 = \frac{\langle\gamma^{(5)}, b_3\rangle}{\kappa_1\kappa_2\kappa_3}. \qquad (9)$$

On the other hand, since the curve $\gamma(s)$ lies on Φ, we may write

$$\gamma(s) = \Phi(u_1(s), u_2(s), u_3(s), u_4(s)).$$

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ISSN (O) 2319 - 8354
ISSN (P) 2319 - 8346

Then, we have

$$\gamma'(s) = \sum_{i=1}^{4} \Phi_i u_i', \tag{10}$$

$$\gamma''(s) = \sum_{i=1}^{4} \Phi_i u_i'' + \sum_{i,j=1}^{4} \Phi_{ij} u_i' u_j', \tag{11}$$

$$\gamma'''(s) = \sum_{i=1}^{4} \Phi_i u_i''' + 3\sum_{i,j=1}^{4} \Phi_{ij} u_i'' u_j' + \sum_{i,j,k=1}^{4} \Phi_{ijk} u_i' u_j' u_k', \tag{12}$$

$$\begin{aligned}\gamma^{(4)}(s) &= \sum_{i=1}^{4} \Phi_i u_i^{(4)} + 4\sum_{i,j=1}^{4} \Phi_{ij} u_i''' u_j' + 3\sum_{i,j=1}^{4} \Phi_{ij} u_i'' u_j'' \\ &+6\sum_{i,j,k=1}^{4} \Phi_{ijk} u_i'' u_j' u_k' + \sum_{i,j,k,l=1}^{4} \Phi_{ijkl} u_i' u_j' u_k' u_l'\end{aligned} \tag{13}$$

$$\begin{aligned}\gamma^{(5)} &= \sum_{i=1}^{4} \Phi_i u_i^{(5)} + 5\sum_{i,j=1}^{4} \Phi_{ij} u_i^{(4)} u_j' + 10\sum_{i,j=1}^{4} \Phi_{ij} u_i''' u_j'' \\ &+10\sum_{i,j,k=1}^{4} \Phi_{ijk} u_i''' u_j' u_k' + 15\sum_{i,j,k=1}^{4} \Phi_{ijk} u_i'' u_j'' u_k' \\ &+10\sum_{i,j,k,l=1}^{4} \Phi_{ijkj} u_i'' u_j' u_k' u_l' + \sum_{i,j,k,l,m=1}^{4} \Phi_{ijklm} u_i' u_j' u_k' u_l' u_m'.\end{aligned} \tag{14}$$

$$\begin{aligned}\gamma^{(6)} &= \sum_{i=1}^{4} \Phi_i u_i^{(6)} + 6\sum_{i,j=1}^{4} \Phi_{ij} u_i^{(5)} u_j' + 15\sum_{i,j=1}^{4} \Phi_{ij} u_i^{(4)} u_j'' \\ &+15\sum_{i,j,k=1}^{4} \Phi_{ijk} u_i^{(4)} u_j' u_k' + 10\sum_{i,j=1}^{4} \Phi_{ij} u_i''' u_j''' \\ &+60\sum_{i,j,k=1}^{4} \Phi_{ijk} u_i''' u_j'' u_k' + 20\sum_{i,j,k,l=1}^{4} \Phi_{ijkl} u_i''' u_j' u_k' u_l' \\ &+15\sum_{i,j,k=1}^{4} \Phi_{ijk} u_i'' u_j'' u_k'' + 45\sum_{i,j,k,l=1}^{4} \Phi_{ijkl} u_i'' u_j'' u_k' u_l' \\ &+15\sum_{i,j,k,l,m=1}^{4} \Phi_{ijklm} u_i'' u_j' u_k' u_l' u_m' \\ &+\sum_{i,j,k,l,m,n=1}^{4} \Phi_{ijklmn} u_i' u_j' u_k' u_l' u_m' u_n'.\end{aligned} \tag{15}$$

Definition 2.2. Let Φ^1, Φ^2, Φ^3 and Φ^4 be the regular hypersurfaces, respectively. Then the unit normal of these hypersurfaces is obtained by

$$N_i = \frac{\Phi_1^i \otimes \Phi_2^i \otimes \Phi_3^i \otimes \Phi_4^i}{\|\Phi_1^i \otimes \Phi_2^i \otimes \Phi_3^i \otimes \Phi_4^i\|}, \qquad i = 1,2,3,4.$$

Assuming that the intersection of these hypersurfaces is a smooth curve $\gamma(s)$ with arc length parametrisation s. Let $\gamma(s_0) = p$. Now, if N_1, N_2, N_3, N_4 are linearly dependent we non-transversal intersection at p with the following subcases:

(1) Almost tangential intersection

$$N_4 = aN_1 + bN_2 + cN_3, \; a,b,c \in \mathbb{R}. \tag{16}$$

(2) Tangential intersection

$$N_1 = N_2 = N_3 = N_4. \tag{17}$$

In this paper, we shall be discussing only tangential intersection case.

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ISSN (O) 2319 - 8354
ISSN (P) 2319 - 8346

3. Tangential intersection of four parametric hypersurfaces

Here we assume that $N_1 = N_2 = N_3 = N_4 = N$. For the unit tangent vector, the algorithm is provided by the following theorem:

Theorem 3.1. *Let $\Phi_1, \Phi_2, \Phi_3, \Phi_4$ be the four parametric hypersurfaces in $\mathbb{R}^5$, then we have*

$$t = \frac{(\Phi_1^1 + \lambda_1 \Phi_4^1) + (\Phi_2^1 + \lambda_2 \Phi_4^1)\varepsilon + (\Phi_3^1 + \lambda_3 \Phi_4^1)\varpi}{\|(\Phi_1^1 + \lambda_1 \Phi_4^1) + (\Phi_2^1 + \lambda_2 \Phi_4^1)\varepsilon + (\Phi_3^1 + \lambda_3 \Phi_4^1)\varpi\|},$$

or

$$t = \frac{(\Phi_1^1 + \lambda_1 \Phi_4^1)\varepsilon + (\Phi_2^1 + \lambda_2 \Phi_4^1) + (\Phi_3^1 + \lambda_3 \Phi_4^1)\varpi}{\|(\Phi_1^1 + \lambda_1 \Phi_4^1)\varepsilon + (\Phi_2^1 + \lambda_2 \Phi_4^1) + (\Phi_3^1 + \lambda_3 \Phi_4^1)\varpi\|},$$

or

$$t = \frac{(\Phi_1^1 + \lambda_1 \Phi_4^1)\varepsilon + (\Phi_2^1 + \lambda_2 \Phi_4^1)\varpi + (\Phi_3^1 + \lambda_3 \Phi_4^1)}{\|(\Phi_1^1 + \lambda_1 \Phi_4^1)\varepsilon + (\Phi_2^1 + \lambda_2 \Phi_4^1)\varpi + (\Phi_3^1 + \lambda_3 \Phi_4^1)\|}.$$

Proof. Projecting γ'' onto the common normal vector, we obtain

$$\begin{cases} \langle \gamma'', N_1 \rangle = \langle \gamma'', N_2 \rangle \Rightarrow \sum_{i=1}^4 h_{ij}^1 u_i' u_j' = \sum_{i=1}^4 h_{ij}^2 v_i' v_j', \\ \langle \gamma'', N_1 \rangle = \langle \gamma'', N_3 \rangle \Rightarrow \sum_{i=1}^4 h_{ij}^1 u_i' u_j' = \sum_{i=1}^4 h_{ij}^3 w_i' w_j', \\ \langle \gamma'', N_1 \rangle = \langle \gamma'', N_4 \rangle \Rightarrow \sum_{i=1}^4 h_{ij}^1 u_i' u_j' = \sum_{i=1}^4 h_{ij}^4 r_i' r_j'. \end{cases} \tag{18}$$

Replacing v_i', w_i', r_i' by u_i', we denote the first two equations of (18) as system

$$\begin{aligned} &e_{11}(u_1')^2 + e_{12}u_1'u_2' + e_{13}u_1'u_3' + e_{14}u_1'u_4' + e_{22}(u_2')^2 + e_{23}u_2'u_3' \\ &+ e_{24}u_2'u_4' + e_{33}(u_3')^2 + e_{34}u_3'u_4' + e_{44}(u_4')^2 = 0 \end{aligned} \tag{19}$$

$$\begin{aligned} &f_{11}(u_1')^2 + f_{12}u_1'u_2' + f_{13}u_1'u_3' + f_{14}u_1'u_4' + f_{22}(u_2')^2 + f_{23}u_2'u_3' \\ &+ f_{24}u_2'u_4' + f_{33}(u_3')^2 + f_{34}u_3'u_4' + f_{44}(u_4')^2 = 0 \end{aligned} \tag{20}$$

respectively, where e_{ij}, f_{ij}, $i, j = 1,2,3,4$ are scalars.
Since, $\langle t, N \rangle = \sum_{i=1}^4 \langle \Phi_i^1, N \rangle u_i' = 0$, this means that at least one of the products is non-vanishing.
Suppose $\langle \Phi_4^1, N \rangle \neq 0$, this implies that $u_4' = \lambda_1 u_1' + \lambda_1 u_2' + \lambda_1 u_3'$, where $\lambda = -\frac{\langle \Phi_i^1, N \rangle}{\langle \Phi_4^1, N \rangle}, i = 1,2,3$.
Thus (19) and (20) reduces to

$$m_{11}(u_1')^2 + m_{12}u_1'u_2' + m_{13}u_1'u_3' + m_{22}(u_2')^2 + m_{23}u_2'u_3' + m_{33}(u_3')^2 = 0, \tag{21}$$

$$n_{11}(u_1')^2 + n_{12}u_1'u_2' + n_{13}u_1'u_3' + n_{22}(u_2')^2 + n_{23}u_2'u_3' + n_{33}(u_3')^2 = 0 \tag{22}$$

where m_{ij}, n_{ij}, $i, j = 1,2,3$ are scalars. Now, if we denote

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ISSN (O) 2319 - 8354
ISSN (P) 2319 - 8346

$$\varepsilon = \frac{u_2'}{u_1'}, \quad \varpi = \frac{u_3'}{u_1'} \quad \text{when} \quad \chi_1 = \begin{vmatrix} m_{22} & m_{33} \\ n_{22} & n_{33} \end{vmatrix} \neq 0,$$

or

$$\varepsilon = \frac{u_1'}{u_2'}, \quad \varpi = \frac{u_3'}{u_2'} \quad \text{when} \quad \chi_2 = \begin{vmatrix} m_{11} & m_{33} \\ n_{11} & n_{33} \end{vmatrix} \neq 0,$$

or

$$\varepsilon = \frac{u_1'}{u_3'}, \quad \varpi = \frac{u_2'}{u_3'} \quad \text{when} \quad \chi_3 = \begin{vmatrix} m_{11} & m_{22} \\ n_{11} & n_{22} \end{vmatrix} \neq 0$$

we obtain

$$\left.\begin{array}{l} m_{22}\varepsilon^2 + m_{33}\varpi^2 + m_{12}\varepsilon + m_{13}\varpi + m_{23}\varepsilon\varpi + m_{11} = 0, \\ n_{22}\varepsilon^2 + n_{33}\varpi^2 + n_{12}\varepsilon + n_{13}\varpi + n_{23}\varepsilon\varpi + n_{11} = 0, \end{array}\right\} \tag{23}$$

or

$$\left.\begin{array}{l} m_{11}\varepsilon^2 + m_{33}\varpi^2 + m_{12}\varepsilon + m_{23}\varpi + m_{13}\varepsilon\varpi + m_{22} = 0, \\ n_{11}\varepsilon^2 + m_{33}\varpi^2 + n_{12}\varepsilon + n_{23}\varpi + n_{13}\varepsilon\varpi + n_{22} = 0, \end{array}\right\} \tag{24}$$

or

$$\left.\begin{array}{l} m_{11}\varepsilon^2 + m_{22}\varpi^2 + m_{13}\varepsilon + m_{23}\varpi + m_{12}\varepsilon\varpi + m_{33} = 0, \\ n_{11}\varepsilon^2 + n_{22}\varpi^2 + n_{13}\varepsilon + n_{23}\varpi + n_{12}\varepsilon\varpi + n_{33} = 0 \end{array}\right\} \tag{25}$$

respectively. We, see that (23),(24) and (25) are pairs of conics with respect to ε and ϖ. The intersection point (ε, ϖ) can be found by any known methods of conic solutions, thus the unit tangent vector is obtained by

$$t = \frac{(\Phi_1^1 + \lambda_1\Phi_4^1) + (\Phi_2^1 + \lambda_2\Phi_4^1)\varepsilon + (\Phi_3^1 + \lambda_3\Phi_4^1)\varpi}{\|(\Phi_1^1 + \lambda_1\Phi_4^1) + (\Phi_2^1 + \lambda_2\Phi_4^1)\varepsilon + (\Phi_3^1 + \lambda_3\Phi_4^1)\varpi\|}, \tag{26}$$

or

$$t = \frac{(\Phi_1^1 + \lambda_1\Phi_4^1)\varepsilon + (\Phi_2^1 + \lambda_2\Phi_4^1) + (\Phi_3^1 + \lambda_3\Phi_4^1)\varpi}{\|(\Phi_1^1 + \lambda_1\Phi_4^1)\varepsilon + (\Phi_2^1 + \lambda_2\Phi_4^1) + (\Phi_3^1 + \lambda_3\Phi_4^1)\varpi\|}, \tag{27}$$

or

$$t = \frac{(\Phi_1^1 + \lambda_1\Phi_4^1)\varepsilon + (\Phi_2^1 + \lambda_2\Phi_4^1)\varpi + (\Phi_3^1 + \lambda_3\Phi_4^1)}{\|(\Phi_1^1 + \lambda_1\Phi_4^1)\varepsilon + (\Phi_2^1 + \lambda_2\Phi_4^1)\varpi + (\Phi_3^1 + \lambda_3\Phi_4^1)\|}, \tag{28}$$

respectively. □

Note that, we can use this method if $\chi_i \neq 0$, $i = 1,2,3$, or if $\sum_{i,j=1}^{3}(m_{ij}^2 + n_{ij}^2) \neq 0$ and $\chi_1 = \chi_2 = \chi_3 = 0$, then t may not exist.

Remark 3.1. Depending upon the real intersection points of the conic pairs, we have following

1. When the conics do not have any point in common p is an isolated contact point.
2. When the conics have one point in common, t is unique.
3. When the conics have two or more points in common, then p is a branch point.
4. When $\sum_{i,j=1}^{3}(m_{ij}^2 + n_{ij}^2) = 0$, then (23), (24), (25) vanishes for any values of u_i', i.e., p is a higher order contact point. If all the second fundamental coefficients of the hypersurfaces vanishes at p, then p is a flat point of the hypersurfaces.

International Journal of Advance Research in Science and Engineering
Vol. No.6, Issue No. 08, August 2017
www.ijarse.com
IJARSE
ISSN (O) 2319 - 8354
ISSN (P) 2319 - 8346

3.1. Curvature vector of tangential intersection

To find the curvature vector, we need to find u_i'', which needs a system of four linear equations in $u_1'', u_2'', u_3'', u_4''$. Representing v_i'', w_i'', r_i'' in terms of linear combination of u_i''. Also denoting γ^i be the curve associated with $\Phi^i, i = 1,2,3,4$ hypersurface. Thus the first three equations of the required system of equations is given by the following projections

$$\begin{cases} \langle (\gamma^1)''', N_1 \rangle = \langle (\gamma^2)''', N_2 \rangle, \\ \langle (\gamma^1)''', N_1 \rangle = \langle (\gamma^3)''', N_3 \rangle, \\ \langle (\gamma^1)''', N_1 \rangle = \langle (\gamma^4)''', N_4 \rangle, \end{cases} \tag{29}$$

and the last equation is given by $\langle (\gamma^1)', (\gamma^1)'' \rangle = 0$. If the coefficient determinant of the system (29) is non-zero, then we can find $u_i'', i = 1,2,3,4$. substituting the results into (11) gives the curvature vector. Consequently, the first curvature can be found from (9) and n, κ_1' can be found from (3). Otherwise, if the coefficient determinant of (29) is zero, then among the first three equations of (29) one or more equations vanishes, or one equation is proportional to some other. In this case that equation(vanishing) is replaced by $\langle (\gamma^1)'', N_1' \rangle = \langle (\gamma^i)'', N_i' \rangle$, or $\langle (\gamma^1)'', N_1'' \rangle = \langle (\gamma^i)'', N_i'' \rangle, i = 2,3,4$.

and the last equation is given by $\langle (\gamma^1)', (\gamma^1)'' \rangle = 0$. If the coefficient determinant of the system (29) is non-zero, then we can find $u_i'', i = 1,2,3,4$. substituting the results into (11) gives the curvature vector. Consequently, the first curvature can be found from (9) and n, κ_1' can be found from (3). Otherwise, if the coefficient determinant of (29) is zero, then among the first three equations of (29) one or more equations vanishes, or one equation is proportional to some other. In this case that equation(vanishing) is replaced by $\langle (\gamma^1)'', N_1' \rangle = \langle (\gamma^i)'', N_i' \rangle$, or $\langle (\gamma^1)'', N_1'' \rangle = \langle (\gamma^i)'', N_i'' \rangle, i = 2,3,4$.

3.2. Second curvature of the tangential intersection

For the the second curvature, we need to find γ'''. To find this, we need to evaluate $u_i''', i = 1,2,3,4$. Hence a system of four linear equations in $u_i''', i = 1,2,3,4$ is needed. On representing v_i''', w_i''', r_i''' in terms of linear combination of u_i''', the first three equations are given by the following system

$$\begin{cases} \langle (\gamma^1)^{(4)}, N_1 \rangle = \langle (\gamma^2)^{(4)}, N_2 \rangle, \\ \langle (\gamma^1)^{(4)}, N_1 \rangle = \langle (\gamma^3)^{(4)}, N_3 \rangle, \\ \langle (\gamma^1)^{(4)}, N_1 \rangle = \langle (\gamma^4)^{(4)}, N_4 \rangle, \end{cases} \tag{30}$$

and the last equation is obtained from (3) as $\langle (\gamma^1)', (\gamma^1)''' \rangle = -\kappa_1^2$. If the coefficient matrix is non-singular, then u_i''' are easily found, while as if any of the equations in (30) vanishes, then that equation is replaced by $\langle (\gamma^1)''', N_1' \rangle = \langle (\gamma^i)''', N_i' \rangle$, or $\langle (\gamma^1)''', N_1'' \rangle = \langle (\gamma^i)''', N_i'' \rangle$, or $\langle (\gamma^1)''', N_1''' \rangle = \langle (\gamma^i)''', N_i''' \rangle i = 2,3,4$.

3.3. Third curvature of the tangential intersection

To find the fourth curvature, we calculate $u_i^{(4)}, i = 1,2,3,4$. Representing $v_i^{(4)}$, $w_i^{(4)}$, $r_i^{(4)}$ in terms of linear combination of $u_i^{(4)}$, then the first three equations are

$$\begin{cases} \langle (\gamma^1)^{(5)}, N_1 \rangle = \langle (\gamma^2)^{(5)}, N_2 \rangle, \\ \langle (\gamma^1)^{(5)}, N_1 \rangle = \langle (\gamma^3)^{(5)}, N_3 \rangle, \\ \langle (\gamma^1)^{(5)}, N_1 \rangle = \langle (\gamma^5)^{(5)}, N_5 \rangle. \end{cases} \tag{31}$$

Also the last equation depending on $u_i^{(4)}$ is $\langle(\gamma^1)',(\gamma^1)^{(4)}\rangle = -3\kappa_1\kappa_1'$. If the determinant of coefficient matrix is non-vanishing, then $u_i^{(4)}$ are easily found, while as if any of the equations in (31) vanishes, then that equation is replaced by $\langle(\gamma^1)^{(4)},N_1^{(j)}\rangle = \langle(\gamma^i)^{(4)},N_i^{(j)}\rangle$, $i=2,3,4$. and $j=1\cdots4$.

3.4. *Fourth curvature of the tangential intersection*

Finally to find the fourth curvature, we find $u_i^{(5)}, i=1,2,3,4$. Writting $v_i^{(5)}$, $w_i^{(5)}$, $r_i^{(5)}$ in terms of linear combination of $u_i^{(5)}$, we have

$$\begin{cases} \langle(\gamma^1)^{(6)},N_1\rangle = \langle(\gamma^2)^{(6)},N_2\rangle, \\ \langle(\gamma^1)^{(6)},N_1\rangle = \langle(\gamma^3)^{(6)},N_3\rangle, \\ \langle(\gamma^1)^{(6)},N_1\rangle = \langle(\gamma^6)^{(6)},N_6\rangle. \end{cases} \tag{32}$$

The fourth equation depending on $u_i^{(5)}$ is $\langle(\gamma^1)',(\gamma^1)^{(5)}\rangle = -3(\kappa_1)^2 - 4\kappa_1'k_1'' + k_1^{(4)} + k_1^2k_2^2$. If the determinant of coefficient matrix is non-zero, then $u_i^{(5)}$ are easily found, while as if any of the equations in (32) vanishes, then that equation is replaced by $\langle(\gamma^1)^{(5)},N_1^{(j)}\rangle = \langle(\gamma^i)^{(5)},N_i^{(j)}\rangle$, $i=2,3,4$. and $j=1\cdots5$.

Acknowledgement: The authors are very thankful to second author Manzoor Ahmad Lone for his help particularly in handling the softwares, mathematica and matlab.

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