International Journal of Advance Research in Science and Engineering
Vol. No.6, Issue No. 08, August 2017
www.ijarse.com
IJARSE
ISSN (O) 2319 - 8354
ISSN (P) 2319 - 8346

EXTENSION OF CERTAIN BERNSTEIN-TYPE INEQUALITIES TO RATIONAL FUNCTIONS

Ajaz Wani

Department of Mathematics, University of Kashmir, Srinagar (India)

ABSTRACT

In this paper, we shall use a parameter β and obtain Bernstein-type inequality for rational functions with prescribed poles. The result shall generalize as well as refine some already proved results in this direction.

MATHEMATICS SUBJECT CLASSIFICATION: 30A10, 30C10, 30D15.

keywords and phrases:* *Rational function, Polynomial, Poles, Zeros.

I .INTRODUCTION

Let P_n denote the class of all complex polynomials of degree at most n. If $P \in P_n$ then concerning the estimate of $|P'(z)|$ on $|z|=1$, we have

$$|P'(z)| \leq \max_{|z|=1} |P(z)| \qquad \ldots\ldots\ldots\ldots\ldots (1)$$

Inequality (1) is a famous result due to Bernstein [2], who proved it in 1912.

It is worth to mention that equality holds in (1) if and only if $|P(z)|$ has all its zeros at the origin, so it is natural to seek improvements under appropriate assumption on the zeros of $|P(z)|$. If we restrict ourselves to the class of polynomials $|P(z)|$ having no zeros in $|z|<1$, then (1) can be replaced by

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)| \qquad \ldots\ldots\ldots\ldots\ldots (2)$$

Where as if $|P(z)|$ has no zeros in $|z|>1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)| \qquad \ldots\ldots\ldots\ldots\ldots (3)$$

Inequality (2) was conjectured by $Erd\ddot{o}s$ and later verified by Lax [3], whereas inequality (3) is due to $Tura'n$ [4] .

Li, Mohapatra and Rodriguez [5] gave a new perspective to the above inequalities (1), (2), (3) and extended them to rational functions with prescribed poles. Essentially, in the inequalities referred to, they replaced the polynomial $P(z)$ by a rational function $r(z)$ with prescribed poles $a_1, a_2, ..., a_n$ and z^n by a Blashke product $B(z)$. Before proceeding towards their results, let us introduce the set of rational functions involved.

For $a_j \in C$ with $j = 1,2,...,n$, let

$$W(z)=\prod_{j=1}^{n}(z-a_j)$$

And let

$$B(z)=\prod_{j=1}^{n}\frac{(1-\bar{a}_j z)}{(z-a_j)},\quad R_n=R_n(a_1,\,a_2,\,...,\,a_n)=\left\{\frac{P(z)}{W(z)},\ P\in P_n\right\}.$$

Then R_n is the set of rational functions with poles $a_1,\,a_2,\,...,\,a_n$ at most n and with finite limit at ∞. We shall always assume that these poles lie in $|z|>1$.

Note that $B(z)\in R_n$ and $B(z)=1$ for $|z|=1$. For $r(z)=\dfrac{P(z)}{W(z)}\in R_n$, the conjugate transpose $r*(z)$ of r is defined by $r*(z)=B(z)\overline{r(1/\bar{z})}$.

As an extension of (2) to rational functions, Li, Mohapatra and Rodriguez [5] showed that if $r\in R_n$, and $r(z)\neq 0$ in $|z|<1$, then

$$|r'(z)|\leq\frac{|B'(z)|}{2}\sup_{|z|=1}|r(z)|. \qquad ……………(4)$$

II MAIN RESULTS

In this paper, we establish Bernstein-type inequality for rational functions with prescribed poles which improves the result of Li, Mohapatra and Rodriguez [5]. More precisely, we prove

Theorem 1. Suppose $r\in R_n$ and all the n zeros of r lie in $|z|\geq 1$. If $r(z)=\dfrac{P(z)}{W(z)}$, where $P(z)=\sum_{j=0}^{n}c_j z^j$, then for $|z|=1$,

$$|r'(z)|\leq\frac{1}{2}\left\{|B'(z)|+\left(\frac{|c_n|-|c_0|}{|c_n|+|c_0|}\right)\left(\frac{|r(z)|^2}{\|r(z)\|^2}\right)\right\}\|r(z)\|, \qquad ……………(5)$$

Where $\|r(z)\|=\max_{|z|=1}|r(z)|$.

The result is best possible and equality in (5) holds for $r(z)=B(z)+\lambda,\ |\lambda|=1.$

Remark 1. Since all the zeros of $r(z)=\dfrac{P(z)}{W(z)}$ and hence of $P(z)=\sum_{j=0}^{n}c_j z^j$ lie in $|z|\geq 1$, therefore, $|c_0|\geq|c_n|$, which shows that Theorem 1 is an improvement of (4).

IJARSE
ISSN (O) 2319 - 8354
ISSN (P) 2319 - 8346

III.LEMMAS

For the proofs of these theorems, we shall make use of the following lemmas.

Lemma 1. If $r \in R_n$ and $r*(z) = B(z)\overline{r(1/\bar{z})}$, then for $|z|=1$, we have

$$|r'(z)| + |(r*(z))'| \leq |B'(z)| \max_{|z|=1} |r(z)|.$$

The above lemma is due to Li, Mohapatra and Rodrigues[5].

Lemma 2. Suppose $r \in R_n$ be such that $r(z) = \dfrac{P(z)}{W(z)}$ where $P(z) = \sum_{j=0}^{n} c_j z^j$ and all the zeros of r lie in $|z|>1$. Then for $|z|=1$, we have

$$\operatorname{Re}\left(\frac{zr'(z)}{r(z)}\right) \leq \frac{1}{2}\{|B'(z)|\}.$$

The above lemma is due to Aziz and Shah [1].

IV PROOF OF THEOREM 1

Since $r(z) = \dfrac{P(z)}{W(z)}$ where $P(z) = \sum_{j=0}^{n} c_j z^j$ and $r(z)$ has all its zeros in $|z| \geq 1$. Since

$r*(z) = B(z)\overline{r(1/\bar{z})}$, we have

$$z(r*(z))' = zB'(z)\overline{r(1/\bar{z})} - \frac{B(z)}{2}\overline{r'(1/\bar{z})},$$

and therefore for $|z|=1$ (so that $z = \dfrac{1}{\bar{z}}$), we get

$$|(r*(z))'| = \left|zB'(z)\overline{r(z)} - B(z)\overline{zr'(z)}\right|,$$

$$= |B(z)|\left|\frac{zB'(z)}{B(z)}\overline{r(z)} - \overline{zr'(z)}\right|. \qquad \text{……………} (6)$$

Also

$$\frac{zB'(z)}{B(z)} = |B'(z)| > 0$$

we get from (6) for $|z|=1$ with $r(z) \neq 0$,

International Journal of Advance Research in Science and Engineering
Vol. No.6, Issue No. 08, August 2017
www.ijarse.com
IJARSE
ISSN (O) 2319 - 8354
ISSN (P) 2319 - 8346

$$\left|\frac{z(r*(z))'}{r(z)}\right|^2 = \left| |B'(z)| - \frac{zr'(z)}{r(z)}\right|^2$$

$$=|B'(z)|^2 + \left|\frac{zr'(z)}{r(z)}\right|^2 - 2|B'(z)|\operatorname{Re}\left(\frac{zr'(z)}{r(z)}\right),$$

which gives by using Lemma 2 for $|z|=1$ with $r(z)\neq 0$, that

$$\left|\frac{z(r*(z))'}{r(z)}\right|^2 = |B'(z)|^2 + \left|\frac{zr'(z)}{r(z)}\right|^2 - 2|B'(z)|\left\{|B'(z)| - \frac{|c_0|-|c_n|}{|c_0|+|c_n|}\right\}$$

$$= \left|\frac{zr'(z)}{r(z)}\right|^2 + \left(\frac{|c_0|-|c_n|}{|c_0|+|c_n|}\right)|B'(z)|.$$

Which implies for $|z|=1$, that

$$|r'(z)|^2 + \left(\frac{|c_0|-|c_n|}{|c_0|+|c_n|}\right)|B'(z)||r(z)|^2 \leq |(r*(z))'|^2.$$

Combining this with Lemma 1, we get for $|z|=1$, that

$$|r'(z)| + \left\{|r'(z)|^2 + \left(\frac{|c_0|-|c_n|}{|c_0|+|c_n|}\right)|B'(z)||r(z)|^2 \leq |(r*(z))'|^2\right\}^{\frac{1}{2}}$$

$$\leq |r'(z)| + |(r*(z))'|$$

$$\leq |B'(z)|\,\|r(z)\|,$$

or equivalently

$$|r'(z)|^2 + \left(\frac{|c_0|-|c_n|}{|c_0|+|c_n|}\right)|B'(z)||r(z)|^2 \leq |(r*(z))'|^2 \leq |B'(z)|^2\,\|r(z)\|^2 - 2|B'(z)||r'(z)|\,\|r(z)\| + |r'(z)|^2,$$

which on using the fact that $|B'(z)|\neq 0$ and after a simplification gives for $|z|=1$, that

$$|r'(z)| \leq \frac{1}{2}\left\{|B'(z)| + \left(\frac{|c_n|-|c_0|}{|c_n|+|c_0|}\right)\left(\frac{|r(z)|^2}{\|r(z)\|^2}\right)\right\}\|r(z)\|,$$

This completes the proof of Theorem 1.

REFERENCES

[1] A. Aziz and W. M. Shah, Some refinements of Bernstein-type inequalities for rational functions, Glas. Mate., 32(1997), 29-37.

International Journal of Advance Research in Science and Engineering
Vol. No.6, Issue No. 08, August 2017
www.ijarse.com
IJARSE
ISSN (O) 2319 - 8354
ISSN (P) 2319 - 8346

[2] S. Bernstein, Sur e' ordre de la meilleure approximation des functions continues par des polynomes de $\deg re'$ $donne'$, Mem. Acad. R. Belg., 4(1912), 1-103.

[3] P. D. Lax, Proof of a conjecture of P. Erd□os on the derivative of a polynomial, Bull. Amer. Math. Soc., 50 (1944), 509-513.

[4] P. $Tura'n\ \ddot{U}ber$ die Ableitung von Polynomen, Compos. Math., 7(1939), 89-95.

[5] Xin Li, R. N. Mohapatra and R. S. Rodriguez, Bernstein-type inequalities for rational functions with prescribed poles, J. London Math. Soc., 51(1995), 523-531.