International Journal of Advance Research in Science and Engineering
Vol. No.6, Issue No. 05, May 2017
www.ijarse.com
IJARSE
ISSN (O) 2319 - 8354
ISSN (P) 2319 - 8346

# SOME CASES OF REGULARITY OF BIRKHOFF INTERPOLATION PROBLEM

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**ABSTRACT**

*In this paper regularity of Birkhoff interpolation problem is investigated. Some nodes for evaluating the regularity of Birkhoff interpolation problem are studied. The domain of nodes is the complex plane and the regularity condition proves the existence of polynomial interpolation at the chosen complex nodes. The Birkhoff interpolation problem is one of the most general and important interpolation problems in multidimensional polynomial interpolation theory. Because of the great complexity of the problem, one has to concentrate on more particular aspects of the problem.*

***KeywordS- Birkhoff interpolation problem, Polynomial interpolation, Nodes, Regularity.***

## 1. INTRODUCTION

Birkhoff interpolation appears whenever observation gives scattered, irregular information about a function and itderivatives. Birkhoff interpolation differs radically from the more familiar Lagrange and Hermite interpolation in both its problems and its methods. It is an extension of Hermite interpolation in that it involves matching of values and derivatives at certain points but does not insist that these be consecutive. With this additional freedom, it is no longer automatic that such an interpolation problem is always regular, i.e., has exactly one solution for every choice of nodes.

The problem of Hermite-Birkhoff interpolation on real nodes has a well-developed theory which is embodied in the well-known book by Lorentz et al. [1].Chen et al. [2] considered the problems of $(o, m)$ and $(0,1, \dots, r-2, r)$interpolationon the zeros of $(z^{2n}+1)(z^2-1)$and$(z^{2n}+1)(z^n-1)$ respectively.De Bruin et al. [3] considered the same problems on the zeros

Of $(z^{3n}+1)(z-1)$, $(z^{3n}+1)(z^2+z+1)$and$(z^{3n}+1)(z^3-1)$. Study of Birkhoff interpolation problems on non-uniformly distributed nodes are of recent origin [4]-[7].

We examine the cases of $(o, m)$ interpolation problem on the zeros of $(z^{2n}+1)(z^2-\xi^2)$for any integer $m \geq 1, m < 2n+2$where $\xi^3=1, \xi \neq 0$and $(0,1, \dots, r-2, r)$ interpolation on the zeros of $(z^{2n}+1)(z^2-\xi^2)$where $\xi \neq 0$.

## II.. OBSERVATIONS

**Theorem 2.1: The problem of (0,m) interpolation on the zeros of $(z^{2n}+1)(z^2-\xi^2)$ is regular for any integer m $\geq$ 1, m< 2n+2, where $\xi^3=1, \xi\neq 0$.**

**Proof:** For $m=1$, it is Hermite interpolation so we take $m\geq 2$.

Here the total number of conditions is $2n+2+2n+2=4n+4$

As

$P(z_k)=a_k z_k^{2n}+1=0 \quad k=1,2,\dots,2n$ (1.1a)

$P^m(z_k)=b_k z_k^{2n}+1=0 \quad k=1,2,\dots,2n$ (1.1b)

$P(\pm\xi)=c$ (1.1c)

$P^m(\pm\xi)=d$ (1.1d)

It is enough to show that if $a_k=b_k=c=d=0$ then the polynomial $P(z)\in\wedge_{4n+3}$ is identically zero.

Let us take

$$P(z)=P_0(z)+z^{2n}P_1(z)+z^{4n}P_2(z)$$

Where $P_0(z), P_1(z)\in\wedge_{2n-1}$ and $P_2(z)\in\wedge_3$ then $P(z)\in\wedge_{4n+3}$

We suppose that $P_i(z)=\sum_{j=0}^{2n-1}a_{i,j}z^j \qquad i=0,1,2$ (1.2)

From $P_2(z)\in\wedge_3$, we have $a_{2,j}=0$ for $j\geq 4$

Now $P(z_k)=0$

$P_0(z_k)+z_k^{2n}p_1(z_k)+z_k^{4n}P_2(z_k)=0$

$$\sum_{j=0}^{2n-1}[a_{0,j}-a_{1,j}+a_{2,j}]z_k^j=0$$

Thus we have the identity

$\sum_{j=0}^{2n-1}[a_{0,j}-a_{1,j}+a_{2,j}]z^j=0$ (1.3)

This gives $a_{0,j}-a_{1,j}+a_{2,j}=0$ when $j=0,1,2,3$ (1.4a)

$a_{0,j}-a_{1,j}=0$ when $j=4,5,\dots,2n-1$ (1.4b)

On setting

$(j)_m=j(j-1),\dots,(j-m+1)$ for $j\geq m>0$

$(j)_m=1$ for $m\leq 0$ and $(j)_m=0$ for $m>j$

Similarly from $P^m(z_k)=0$

$(j)_m a_{0,j}-(2n+j)_m a_{1,j}+(4n+j)_m a_{2,j}=0, \quad j=0,1,2,3$ (1.5a)

And $(j)_m a_{0,j}-(2n+j)_m a_{1,j}=0 \qquad j=4,5,\dots,2n-1$ (1.5b)

Since $(j)_m=0$ for $m\geq 4 \quad j=0,1,2,3$

IJARSE
ISSN (O) 2319 - 8354
ISSN (P) 2319 - 8346

Therefore $a_{0,j} = a_{1,j} = a_{2,j} = 0$ for $j = 4, 5, \ldots, 2n-1$

From $P(\pm\xi) = 0$ for $\xi^3 = 1$, we get

$\sum_{j=0}^{3}[a_{0,j} - a_{i,j} + a_{2,j}]\xi^j = 0$ (1.6)

And $\sum_{j=0}^{3}[a_{0,j} - a_{i,j} + a_{2,j}](-\xi)^j = 0$ (1.7)

On adding equations (1.6) and (1.7), we get

$$\sum_{j=0}^{1}[a_{0,2j} - a_{1,2j} + a_{2,2j}]\xi^{2j} = 0$$

$\sum_{j=0}^{1}[a_{0,2j} - a_{1,2j} + a_{2,2j}] = 0$ (1.8)

From

$P^m(\pm\xi) = 0$ for $\xi^3 = 1$, we get $\sum_{j=0}^{3}[(j)_m a_{0,j} - (2n+j)_m a_{1,j} + (4n+j)_m a_{2,j}]\xi^j = 0$ (1.9)

And $\sum_{j=0}^{3}[(j)_m a_{0,j} - (2n+j)_m a_{1,j} + (4n+j)_m a_{2,j}](-\xi)^j = 0$ (1.10)

On adding equations (1.9) and (1.10), we get

$\sum_{j=0}^{1}[(2j)_m a_{0,2j} + (2n+2j)_m a_{1,2j} + (4n+2j)_m a_{2,2j}]\xi^{2j} = 0$ (1.11)

On subtracting equations (1.6) and (1.7), we get

$$\sum_{j=0}^{1}[a_{0,2j+1} + a_{1,2j+1} + a_{2,2j+1}]\xi^{2j+1} = 0$$

$\sum_{j=0}^{1}[a_{0,2j+1} + a_{1,2j+1} + a_{2,2j+1}] = 0$ (1.12)

On subtracting equations (1.9) and (1.10), we get

$$\sum_{j=0}^{1}[(2j+1)_m a_{0,2j+1} + (2n+2j+1)_m a_{1,2j+1} + (4n+2j+1)_m a_{2,2j+1}]\xi^{2j+1} = 0$$

From equation (1.4a), we get

$a_{0,0} + a_{2,0} = a_{1,0}$

And $a_{0,2} + a_{2,2} = a_{1,2}$ (1.13)

From equation (1.8), we get

$a_{1,0} + a_{1,2} = 0$ (1.14)

From equations (1.5a) and (1.11), we get

$(2n)_m a_{1,0} + (2n+2)_m a_{1,2} = 0$ (1.15)

Equations (1.14) and (1.15) give $a_{1,0} = a_{1,2} = 0$

From equation (1.13), we get

$a_{0,0} + a_{2,0} = 0 \quad a_{0,2} + a_{2,2} = 0$

From equation (1.5a), we get

$(0)_m a_{0,0} + (4n)_m a_{2,0} = 0$

IJARSE
ISSN (O) 2319 - 8354
ISSN (P) 2319 - 8346

And $\quad (2)_m\, a_{0,2} + (4n)_m a_{2,2} = 0$

Thus $\quad a_{0,0} = a_{2,0} = a_{0,2} = a_{2,2} = 0$

On combining these equations, we have

$a_{0,2j} = 0$ and $a_{2,2j} = 0, \quad$ for $\quad j = 0,1$

Similarly we get $a_{0,2j+1} = a_{1,2j+1} = a_{2,2j+1} = 0, \quad$ for $\quad j = 0,1$

Hence $\quad P(z) \equiv 0$

This completes our proof.

**Theorem 2.2:The problem of$(0,1,\dots,r-2,r)$interpolation on the zeros of$(z^{2n}+1)(z^2-\xi^2)$is regular for $\xi \neq 0$.**

**Proof:**Here the number of conditions is$r(2n+2)$

We set$Q(z) = (z^{2n}+1)^{r-1}(z^2-\xi^2)^{r-1}P(z), \qquad P(z) \in \pi_{2n+1}$

It is enough to show that if $Q^{(j)}(z_k) = 0, j = 0,1,\dots,r-2$ for$z_k{}^{2n} = -1,$

$Q^{(j)}(\xi) = 0$ and$Q^{(j)}(-\xi) = 0$ , then $Q(z) \equiv 0.$

From $Q^{(j)}(\xi) = 0$ and$Q^{(j)}(-\xi) = 0,$ we get

$$2\xi P'(\xi) + (j-1)(2n+1)P(\xi) = 0 \tag{2.1}$$

$$-2\xi P'(-\xi) - (j-1)(2n+1)P(-\xi) = 0 \tag{2.2}$$

If we set $P(z) = \sum_{i=0}^{2n+1} a_i z^i$ then equations (2.1) and (2.2) give

$$\sum_{i=0}^{2n+1}\{2i + (j-1)(2n+1)\}\, a_i = 0 \tag{2.3}$$

And $\quad \sum_{i=0}^{2n+1}\{2i - (j-1)(2n+1)\}\, a_i(-\xi^i) = 0 \quad (2.4)$

From equations (2.3) and (2.4), we at once obtain

$$\sum_{i=0}^{n}\{4i + (j-1)(2n+1)\}\, a_{2i} = 0 \tag{2.5}$$

$$\sum_{i=0}^{n}\{2(2i+1) + (j-1)(2n+1)\}\, a_{2i+1} = 0 \tag{2.6}$$

From $Q^{(j)}(z_k) = 0, j = 0,1,\dots,r-2,$ we get

$z_k{}^2\{2z_k P'(z_k) + (j-1)(2n+3)P(z_k)\} = 2z_k P'(z_k) + (j-1)(2n-1)P(z_k)$

On substituting the values of$P(z_k), P'(z_k)$ and using the fact $z_k{}^{2n} = -1,$ we have

$$\sum_{i=2}^{2n-1}\{2(i-2) + (j-1)(2n+3)\}a_{i-2}z_k{}^i - \sum_{i=0}^{3}\{2(i+2n-2) + (j-1)(2n+3)\}a_{2n+i}z_k{}^i$$
$$= \sum_{i=0}^{2n-1}\{2i + (j-1)(2n-1)\}\, a_i z_k{}^i - \sum_{i=0}^{1}\{2(i+2n) + (j-1)(2n-1)\}\, a_i z_k{}^i$$

On comparing like powers of $z_k$ on both sides, we have following system of equations

$\{2(i-2) + (j-1)(2n+3)\}a_{i-2} = \{2i + (j-1)(2n-1)\}a_i,$for $i = 4,5,\dots,2n-1$(2.7)

IJARSE
ISSN (O) 2319 - 8354
ISSN (P) 2319 - 8346

$$a_{i-2} = \frac{\{2i + (j-1)(2n-1)\}}{\{2(i-2) + (j-1)(2n+3)\}} a_i$$

$$a_{2q-2} = \frac{\{4q + (j-1)(2n-1)\}}{\{4(q-1) + (j-1)(2n+3)\}} a_{2q}$$

$$a_{2q-2} = \mathcal{A}_{q-1} a_{2q}, \quad (2.8)$$

Where

$$\mathcal{A}_{q-1} = \frac{4q + (j-1)(2n-1)}{4(q-1) + (j-1)(2n+3)}$$

And for $i = 0, 1, 2, 3$, we have

$\{2(2n-2) + (j-1)(2n+3)\} a_{2n-2} = 4n a_0$ (2.9)

$\{2(2n-1) + (j-1)(2n+3)\} a_{2n-1} = 4n a_1$ (2.10)

$-\{4n + (j-1)(2n+3) a_{2n} + (j-1)(2n+3)\} a_0 = \{4 + (j-1)(2n-1)\} a_2$ (2.11)

$-\{2(2n+1) + (j-1)(2n+3)\} a_{2n+1} + \{2 + (j-1)(2n+3)\} a_1 = \{6 + (j-1)(2n-1)\} a_3$

Fromequation (2.8), we have $\mathcal{A}_{q-1} < 1$ for $j \geq 2, (j = 2, \ldots, n-1)$

If we set

$$\mathcal{A}_{n-1} = \frac{4n}{4(n-1) + (j-1)(2n+3)} < 1$$

Then from equation (2.9), we have $a_{2n-2} = \mathcal{A}_{n-1} a_0$

Fromequations (2.5), (2.6) and (2.8), we get

$$(j-1)(2n+1) a_0 + \sum_{i=0}^{n-1} \{4i + (j-1)(2n+1)\} a_{2i} + \{4i + (j-1)(2n+1)\} a_{2n}$$

It can be written as

$\mathcal{A}_n a_0 + \{4n + (j-1)(2n-1)\} a_{2n} = 0,$ (2.12)

Where $0 < \mathcal{A}_n = (j-1)(2n+1) + \sum_{i=1}^{n-1} \{4i + (j-1)(2n+1)\} \prod_{\mu=j}^{n-1} \mathcal{A}_\mu$

From equation (2.11), we have

$(j-1)(2n+3) a_0 - \{4 + (j-1)(2n-1)\} a_2 - \{4n + (j-1)(2n+3)\} a_{2n} = 0$

$\mathcal{B}_n a_0 - \{4n + (j-1)(2n+3)\} a_{2n} = 0,$ (2.13)

Where $\mathcal{B}_n = (j-1)(2n+3) - \{4 + (j-1)(2n-1)\} \prod_{\mu=1}^{n-1} \mathcal{A}_\mu$

Since $0 < \mathcal{A}_n < 1$, we have

$\mathcal{B}_n > (j-1)(2n+3) - \{4 + (j-1)(2n-1)\}$

$> 4(j-2) \geq 0$ if $r \geq 2$

The determinant of two homogeneous equations (2.12) and (2.13) is clearly non-zero. Hence $a_0 = a_{2n} = 0$ and equation (2.8) shows that all $a_j$'s are zero.

This completes our proof.

IJARSE
ISSN (O) 2319 - 8354
ISSN (P) 2319 - 8346

## III. CONCLUSION

The posed problems of $(o, m)$ interpolation and $(0,1,\ldots,r-2,r)$ interpolation on the zeros of $(z^{2n}+1)(z^2-\xi^2)$ are found to be regular.

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