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A NOTE ON FORA TYPE FIXED POINT THEOREMS

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ABSTRACT

Singal and Lal generalized Nadler's result to a larger class of spaces and a for larger class of mappings. Rhoades discussed a number of fixed point theorems dealing with rational type contractive conditions of some authors. Gupta generalized Nadler's result to a larger class of spaces and a for larger class of mappings according to rational type contractive conditions of Rhoades. In this paper we generalize Fora's result according to some contractive conditions of Rhoades which involves single mappings.

Keywords: Complete Metric Space, Topological Space, Locally contraction in the first variable, Continuous function, Fixed Point Property.

I. INTRODUCTION

Definition 1 [5] Let (X, d) be a metric space and Z be any space. Let f be a mapping from X×Z into X×Z. Then f is said to be locally contraction mapping in the first variable if and only if for any $z \in Z$ there exist an open set V(z) containing z and a real number $\lambda(z) \in [0,1)$ such that for each $x, x_* \in X$ and all $u \in V(z)$

$$d(\pi_1 f(x, u), \pi_1 f(x_*, u)) \leq \lambda(z)\, d(x, x_*),$$ where π_1 is the projection of X×Z on X along Z.

or $$d(f_u(x), f_u(x_*)) \leq \lambda\, d(x, x_*).$$

Theorem 2 [5] Let (X, d) be a complete metric space. Let Z be a topological space with the fixed point property (f.p.p) and let f be a continuous function from X×Z into X×Z. If f is locally contraction mapping in the first variable, then f has a fixed point.

In what follows X will denote a complete metric space, Z a topological space which has the f.p.p. and f is a mapping from X×Z into X×Z. $(m)'$ for $1 \leq m \leq 10$ will denote the condition $(m)'$ from Rhoades [3] and also mentioned in Gupta [4] with the modification that constants or functions that appear in $(m)'$ depend on z. Thus $f \in (4)'$ locally in the first variable means the following:

for each $z \in Z$, there exist an open set V(z) containing z and numbers $\alpha, \beta \geq 0$ with $\alpha + \beta < 1$ such that for each $x, x_* \in X$, $x_* \in 0(x)$ and all $u \in V(z)$,

$$d(f_u(x), f_u(x_*)) \leq \alpha \frac{d(x, f_u(x))\, d(x_*, f_u(x_*))}{d(x, x_*)} + \beta d(x, x_*),$$

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where 0(x) denotes the orbit of the function defined on X.

II. MAIN RESULTS

Now we prove the following results with the help of [4] and [5]:

Theorem 3 Let (X, d) be a complete metric space, Z a topological space which has the f.p.p. and let f: X×Z→X×Z be a continuous mapping. If f satisfies (3)' locally in the first variable, then f has a fixed point.

Proof: We define a sequence $\{t_n\}$ in X as follows:

For a fixed x_0 in X, and any z ∈Z,

$$f_z^0(x_0) = t_0 = x_0, t_n(z) = t_n = f_z^n(x_0) = \pi_1 f(f_z^{n-1}(x_0), z); n \geq 1.$$

Step I: $\{t_n\}$ is a Cauchy sequence in X

Since f∈(3)' locally in the first variable i.e. for each z ∈ Z, there exist an open set V(z) containing z and numbers a, b, c ≥ 0 with a + b + c < 1 such that for each x, x_* ∈ X, x_* ∈ 0(x) and all u ∈ V(z),

$$d(f_u(x), f_u(x_*)) \leq \frac{a[1+d(x, f_u(x))].d(x_*, f_u(x_*))}{1+d(x, x_*)} + \frac{b.d(x, f_u(x)).d(x_*, f_u(x_*))}{d(x, x_*)} + c.d(x, x_*)$$

(3.1)

Equation (3.1) implies that

$$d(f_z(x), f_z(x_*)) \leq \frac{a[1+d(x, f_z(x))].d(x_*, f_z(x_*))}{1+d(x, x_*)} + \frac{b.d(x, f_z(x)).d(x_*, f_z(x_*))}{d(x, x_*)} + c.d(x, x_*)$$

(3.2)

Set $x_* = f_z(x)$ in the above inequality to obtain

$$d(f_z(x), f_z^2(x)) \leq (a+b)d(f_z(x), f_z^2(x)) + c.d(x, f_z(x))$$

which implies that

$$d(f_z(x), f_z^2(x)) \leq \left(\frac{c}{1-a-b}\right).d(x, f_z(x))$$

(3.3)

Now, set x = x_*, then we have

$$d(f_z(x_*), f_z^2(x_*)) \leq \left(\frac{c}{1-a-b}\right).d(x_*, f_z(x_*))$$

(3.4)

Repeating above substitute we obtain

$$d(f_z^2(x), f_z^3(x)) \leq \left(\frac{c}{1-a-b}\right)^2.d(x, f_z(x))$$

Using induction we get

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$$d(f_z^n(x), f_z^{n+1}(x)) \le \left(\frac{c}{1-a-b}\right)^n .d(x, f_z(x))$$

Finally set x = x_0, we get

$$d(t_n, t_{n+1}) \le h^n .d(t_0, t_1), \quad where\ h = \left(\frac{c}{1-a-b}\right) < 1$$

(3.5)

Using triangle inequality, we find for m>n

$$d(t_n,t_m) \le d(t_n,t_{n+1}) + d(t_{n+1},t_{n+2}) + \ldots. + d(t_{m-1},t_m)$$

$$\le (h^n + h^{n+1} + \ldots. + h^{m-1}).d(t_0,t_1)$$

$$= \frac{h^n(1-h^{m-n}).d(t_0,t_1)}{1-h} < \frac{h^n.d(t_0,t_1)}{1-h}$$

Since $h^n \to 0$ as $n\to\infty$, this inequality shows that $\{t_n\}$ is a Cauchy sequence. Since X is a complete metric space, there exists a point t_z . in X such that $t_n \to t_z$.

Step II: $\pi_1 f(t_z, z) = t_z$

If possible let $t_z \ne \pi_1 f(t_z, z) = u_*$(say). Then $d(u_*, t_z) = \in > 0$. Since f is continuous, there exists an open set U×V in X×Z such that

$(t_z, z) \in U\times V$, $U \subset S_{\in/4}(t_z)$ and $f(U\times V) \subset S_{\in/4}(u_*) \times Z$

Since $\lim_{n\to\infty} t_n = t_z$, there is a natural number $k \ge 1$ such that $t_n \in U$ for all $n \ge k$. But $\pi_1 f(t_k, z) = t_{k+1} \in U$.

Therefore $f(t_k, z) \notin S_{\in/4}(u_*)\times Z$, which contradicts the fact that $f(U\times V) \subset S_{\in/4}(u_*)\times Z$ and $d(u_*, t_z) = \in > o$, so our assumption is false, hence the required conclusion.

Step III : Let g: Z → Z defined by $g(z) = \pi_2 f(t_z, z)$. Then g is a continuous mapping

Let $z \in Z$ and U be an open set containing g(z). Then $f(t_Z, z) \in X\times U$. Since f is continuous at (t_z, z), there exists an open set G in Z and a positive real number $\in > 0$ such that

$(t_z, z) \in S_\in(t_z) \times G$ and $f(S_\in(t_z) \times G) \subset X\times U$.

Let W be an open set in Z containing z and $h \in [0,1)$ be a real number. Then from above step I making decreasing sequence procedure we can write:

$d(\pi_1 f(x, v), \pi_1 f(x_*, v)) \le h.\ d(x, x_*)$, for all $x, x_* \in X$, $x_* \in 0(x)$ and all $v \in W$.

Since $h^m \to 0$ as $m \to \infty$, we can choose a natural number $n \ge 1$ such that

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$h^n < \frac{\in}{8}\left(\frac{1-h}{d(t_0,t_1)+(\in/8)}\right)$ and $d(t_z, t_m) < \frac{\in}{8}$ for all $m \geq n$

Since $f(t_n, z) \in X \times U$ and f is continuous at (t_n, z), there exists a basic open set $U_n \times V_n$ in $X \times Z$ such that

$(t_n, z) \in U_n \times V_n$, $U_n \subset S_{\in/8}(t_z)$, $V_n \subset G \cap W$ and $f(U_n \times V_n) \subset X \times U$.

Inductively, suppose that sets $U_n, U_{n-1}, \ldots U_i, V_n, V_{n-1}, \ldots, V_i$ $(1 \leq i \leq (n-1))$ are choosen such that

$(t_i, z) \in U_i \times V_i$, $U_i \subset S_{\in/8}(t_i)$, $V_i \subset V_{i+1}$ and $f(U_i \times V_i) \subset U_{i+1} \times Z$.

Since f is continuous at (t_{i-1}, z) and $f(t_{i-1}, z) \in U_i \times Z$, there exists a basic open set $U_{i-1} \times V_{i-1}$ in $X \times Z$ such that,

$(t_{i-1}, z) \in U_{i-1} \times V_{i-1}$, $U_{i-1} \subset S_{\in/8}(t_{i-1})$, $V_{i-1} \subset V_i$ and $f(U_{i-1} \times V_{i-1}) \subset U_i \times Z$.

In this way, we collect the open sets $U_n, U_{n-1}, \ldots, U_0, V_n, V_{n-1}, \ldots, V_0$ which are defined with the above mentioned properties.

Now, we claim that $g(V_0) \subset U$:

Let $y \in V_0$. Then from the above mention properties we have $(t'_0, y) \in U_0 \times V_0$, where $t'_0 = xo$.

Thus $f(t'_0, y) \in U_1 \times Z$ ie. , $t'_1 = \pi_1 f(t'_0, y) \in U_1$, and consequently $d(t'_1, t_1) < \frac{\in}{8}$.

Using the triangular inequality, we have

$$d(t'_0, t'_1) = d(t_0, t_1') \leq d(t_0, t_1) + d(t_1, t'_1) < d(t_0, t_1) + \frac{\in}{8}.$$

Since $f(U_1 \times V_1) \subset U_2 \times Z$ and $(t'_1, y) \in U_1 \times V_1$, therefore $f(t'_1, y) \in U_2 \times Z$ ie $t'_2 = \pi_1 f(t'_1, y) \in U_2$.

In this way we find the sequence $t'_n(y) = t'_n$, for which $t'_i = \pi_1 f(t'_{i-1}, y) \in Ui$; i =1,2,..., n.

Moreover, $t'_n \in U_n$ and $U_n \subset S_{\in/8}(t_z)$, therefore $d(t'_n, t_z) < \frac{\in}{8}$.

Using the triangular inequality, we find, for $m \geq n$.

$$d(t'_n, t_z) \leq d(t_z, t'_n) + d(t'_n, t'_{n+1}) + \ldots\ldots\ldots + d(t'_{m-1}, t'_m)$$

$$< \frac{\in}{8} + h^n \cdot d(t'_0, t'_1) + \text{-----} + h^{m-1} d(t'_0, t'_1)$$

$$= \frac{h^n}{1-h}(1-h^{m-n}).\ d(t'_0, t'_1) + \frac{\in}{8}$$

$$< \left(\frac{h^n}{1-h}\right).[d(t'_0, t'_1) + \frac{\in}{8}] + \frac{\in}{8}$$

$$< \frac{\in}{8} + \frac{\in}{8} = \frac{\in}{4}$$

If $t_y = \lim t'_n$, then the above inequality shows that $d(t_y, t_z) \leq \in/4$.

Clearly $(t_y, y) \in S_\in(t_z) \times G$ and consequently $f(t_y, y) \in X \times U$, ie. , $g(y) = \pi_2 f(t_y, y) \in U$.

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Therefore our claim is proved and hence g is continuous mapping.

Now as in step III of the theorem 3, g: Z$\to$Z is continuous mapping. Since Z has the f.p.p., there is a point $z_0 \in Z$ such that $g(z_0) = z_0$ and by step II above, we have, $\pi_1 f(t_{z0}, z_0) = t_{z0}$. But $z_0 = g(z_0) = \pi_2 f(t_{z0}, z_0)$. Hence $f(t_{zo}, z_0)=(t_{zo}, z_o)$ i.e. (t_{z0},z_0) is a fixed point of f. This completes the proof.

Theorem 4: Let (X, d) be a complete metric space, Z a topological space with the f.p.p and f: X×Z$\to$ X×Z be a continuous mapping. If f satisfies any one of the conditions $(2)'$, $(5)'$, $(6)'$, $(7)'$, $(8)'$, $(9)'$, and $(10)'$ locally in the first variable then f has a fixed point.

Proof: We define a sequence $t_n(z)=t_n$ in X as follows:

For a fixed x_0 in X and any $z \in Z$,

$$t_o= x_0,\ t_n= \pi_1 f(t_{n-1}, z);\ n \geq 1$$

Step-I: $\{t_n\}$ is a Cauchy sequence in X

If f satisfies any one of the conditions $(2)'$, $(5)'$, $(6)'$, $(7)'$, $(8)'$, $(9)'$, and $(10)'$ locally in the first variable then we have find following equations respectively,

$$d(t_n,t_{n+1}) \leq \left(\frac{\beta}{1-\alpha}\right)^n d(t_0,t_1) \tag{4.1}$$

$$d(t_n,t_{n+1}) \leq k^n d(t_0,t_1) \tag{4.2}$$

$$d(t_n,t_{n+1}) \leq \left(\frac{\beta_1+\beta_2+\beta_4}{1-\alpha_1-\beta_3-\beta_4}\right)^n d(t_0,t_1) \tag{4.3}$$

$$d(t_n,t_{n+1}) \leq \beta^n d(t_0,t_1) \tag{4.4}$$

$$d(t_n,t_{n+1}) \leq q_1^n d(t_0,t_1) \tag{4.5}$$

$$d(t_n,t_{n+1}) \leq q_2^n d(t_0,t_1) \tag{4.6}$$

and $$d(t_n,t_{n+1}) \leq \left(\frac{\gamma_2+\gamma_3+\gamma_4+\gamma_5}{1-\gamma_1-\gamma_2-\gamma_4-\gamma_5}\right)^n d(t_0,t_1) \tag{4.7}$$

In each of these cases we see that $\{t_n\}$ is a Cauchy sequence in X. However, by the completeness of X, there is a point t_z in X such that t_n converges to t_z.

Step II: $\pi_1 f(t_z, z) = t_z$

Step III: Let g : Z$\to$Z defined by $g(z) = \pi_2 f(t_z, z)$. Then g is a continuous mapping:

(As by theorem 3)

Finally, since Z has the f.p.p., there is a point $z_0 \in Z$ such that $g(z_0) =z_0$ and by step II above, we have, $\pi_1 f(t_{z0}, z_0) = t_{z0}$. But $z_0 = g(z_0) = \pi_2 f(t_{z0}, z_0)$.Hence $f(t_{zo}, z_0) = (t_{zo}, z_o)$ i.e. (t_{z0},z_0) is a fixed point of f.

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III. CONCLUSION

We observe from [4] that conditions (1)', (4)' are stronger than (3)'.Therefore the above theorem 3 has two corollaries corresponding to each of these two conditions. Also we observe that condition (4)' is stronger than conditions (6)'and (8)'. Hence the above theorem 4 have one corollary corresponding to condition (4)'. Clearly Nadler's result [2] is also corollary of above theorem 3 and theorem 4.

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