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A WELL BEHAVED COMPACT STAR MODEL IN ISOTROPIC COORDINATES

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ABSTRACT

We present a static and spherically symmetric solution of the general relativistic field equations in isotropic coordinates for perfect fluid. The solution is regular and well behaved having positive finite central pressure and positive finite central density. The solution is well behaved for all the values of u lying in the range 0 < u < .16. The central red shift and surface red shift are positive and monotonically decreasing. we have constructed a model for superdense star by assuming the surface density $\rho_b = 2\times10^{14}\ g/cm^3$. *The mass of the compact star comes out to be 1.165* M_Θ *with linear dimension 10.796 Km and moment of inertia* $1.59\times10^{45} gm\,cm^2$. *The solution is not only well behaved but also having one of the simplest expressions so far known well behaved solutions.*

Keywords: ***Isotropic coordinates, Einstein's field equations, Superdense star, General relativity, Exact solutions.***

I. INTRODUCTION

Ever since the first exact solution of field equations for a perfect fluid sphere of constant density was obtained by Schwarzschild[1], the relativists have been proposing different models of immensely gravitating astrophysical objects by solving field equations both in curvature and isotropic coordinates. Exact solutions with well behaved nature of Einstein's field equations are of vital importance in relativistic astrophysics because the distribution of matter in the interior of stellar object can be easily understood in terms of simple algebraic relations. Due to the nonlinearity of Einstein's field Equations it becomes difficult to obtain the new exact solutions. Various exact solutions of Einstein's field equations are known till date but not all of them are physically relevant. Hence a new solution is always welcome which should be regular and well behaved .

A considerable number of known solutions of Einstein's field equations are of finite central density and finite central pressure which describe the internal structure of stellar objects[2]. These solutions are important for describing the interior of compact objects like neutron star and Quark star. Delgaty and Lake [2] also pointed out that only nine solutions so far are regular and well behaved; out of

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which seven in curvature coordinates(Tolman [3], Patvardhav and Vaidya[4], Mehra [5], Kuchowich [6], Matese and Whiteman[7], Durgapa[8] and only two solutions in isotropic coordinates [9-10]. Many solutions in isotropic coordinates have been explored so far[11-12]. In this paper we have tried to study in detail one of the solutions earlier obtained by Tewari[13] .

From the physical point of view, the mathematical solutions must satisfy certain physical requirements to render them physically meaningful. For well behaved nature of the solution in isotropic coordinates, the following conditions should be satisfied[14].

(i) The solution should be free from physical and geometrical singularities i.e. finite and positive values of central pressure , central density and non zero positive values of e^{α} and e^{β} .

(ii) The solution should have positive and monotonically decreasing expressions for pressure and density($p\, and \rho$)with the increase of *r*. The solution should have positive value of ratio of pressure-density and less than 1(weak energy condition) and less than 1/3(strong energy condition) throughout within the star, monotonically decreasing as well [15].

(iii) The solution should have positive and monotonically decreasing expression for fluid parameter $\frac{p}{\rho\, c^2}$ with the increase of *r*.

(iv) The solution should have positive and monotonically decreasing expression for velocity of sound $\left(\frac{dp}{d\rho}\right)$ with the increase of *r* and causality condition should be obeyed at the centre i.e. $\frac{dp}{c^2 d\rho} < 1$

(v) The red shift Z should be positive, finite and monotonically decreasing in nature with the increase of *r*.

(vi) $\frac{p}{\rho} \le \frac{dp}{d\rho}$, everywhere within the ball.

$$\gamma = \frac{d \log_e p}{d \log_e \rho} = \frac{\rho\, dp}{p\, d\rho} \Rightarrow \frac{dp}{d\rho} = \gamma \frac{p}{\rho}$$

For realistic matter $\gamma \ge 1$ (Pant and Maurya [16])

Keeping in view the above mentioned conditions we are trying to study one of the solutions obtained by Tewari [13] in isotropic coordinates. The solution is well behaved and have been used to construct the model of superdense objects.The mass of superdense objects based on this solution is maximized by assuming surface density $\rho_b = 2 \times 10^{14}$ g/cm^3.

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II. FIELD EQUATIONS IN ISOTROPIC COORDINATES:

Let us consider a spherical symmetric metric in curvature coordinates

$$ds^2 = e^{\alpha} c^2 dt^2 - e^{\beta}\left(dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right) \quad (1)$$

Where α and β are functions of r. Einstein's field equations of gravitation for a nonempty space-time are

$$-\frac{8\pi G}{c^4} T_{ij} = R_{ij} - \frac{1}{2} R g_{ij} \quad (2)$$

Where R_{ij} is a Ricci tensor. T_{ij} is energy- momentum tensor and R is the scalar curvature. The energy - momentum tensor T_{ij} is defined as

$$T_{ij} = (p + \rho) v_i v_j - p g_{ij} \quad (3)$$

Where p denotes the pressure distribution , ρ the density distribution and v_i the velocity vector, satisfying the relation

$$g_{ij} v^i v^j = 1 \quad (4)$$

Since we are dealing with static field, therefore,

$$v^1 = v^2 = v^3 = 0 \quad \text{and} \quad v^4 = \frac{1}{\sqrt{g_{44}}} \quad (5)$$

For the metric (1) the field equation (2) reduces to the following equations

$$\frac{8\pi G}{c^4} p = e^{-\beta}\left(\frac{(\beta')^2}{4} + \frac{\beta'}{r} + \frac{\beta'\alpha'}{2} + \frac{\alpha'}{r}\right) \quad (6)$$

$$\frac{8\pi G}{c^4} p = \left[\frac{\beta''}{2} + \frac{\alpha''}{2} + \frac{(\alpha')^2}{4} + \frac{\beta'}{2r} + \frac{\alpha'}{2r}\right] \quad (7)$$

$$\frac{8\pi G}{c^2} \rho = -e^{-\beta}\left[\beta'' + \frac{(\beta')^2}{4} + \frac{2\beta'}{r}\right] \quad (8)$$

where prime (') denotes the differentiation with respect to r .

From (6) and (7) we obtain following differential equation in α and β

$$\beta'' + \alpha'' + \frac{(\alpha')^2}{2} - \frac{(\beta')^2}{2} - \beta'\alpha' - \left(\frac{\beta'}{r} + \frac{\alpha'}{r}\right) = 0 \quad (9)$$

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Our task is to explore the solution of eq. (9) and obtain the fluid parameters p and ρ from Eqs. (6) and (8).

III. SOLUTION TO BE STUDIED

Equation (9) is solved by assuming an ad hoc relation of gravitational potential g_{11} and r, analogus to Tewari [13]and Boutros[17] and considering the arbitrary constants in such a manner that the solution should be well behaved. We assume

$$e^{\beta/2} = \mathbf{B}\left(1+Cr^2\right)^{-\frac{12}{89}} \qquad x = Cr^2 \qquad \text{and} \qquad y = \frac{d\alpha}{dx} \tag{10}$$

We get the following Riccati differential equation in y

$$\frac{dy}{dx} + \frac{y^2}{2} + \frac{24}{89}\frac{y}{1+x} = -\frac{1848}{7921(1+x^2)} \tag{11}$$

Which yields the following solution,

$$e^{\frac{\alpha}{2}} = \frac{\left\{1+A\left(1+Cr^2\right)^{\frac{23}{89}}\right\}\left(1+Cr^2\right)^{\frac{21}{89}}}{\mathbf{B}^2} \tag{12}$$

Where A, B, and C are arbitrary constants.

The expressions for pressure and density are obtained as :

$$\frac{8\pi G\rho}{c^2} = \frac{4}{7921B^2\left(1+Cr^2\right)^{\frac{154}{89}}}\left(3204C + 924C^2 r^2\right) \tag{13}$$

$$\frac{8\pi G p}{c^4} = \frac{4}{7921B^2(1+Cr^2)^{\frac{154}{89}}}\left[(801C + 441C^2 r^2) + \frac{23A(1+Cr^2)^{\frac{23}{89}}(65C^2 r^2 + 89C)}{\left(1+A(1+Cr^2)^{\frac{23}{89}}\right)}\right] \tag{14}$$

I. Properties of the new solution:

The central values of pressure and density are given by

$$\left(\frac{8\pi G p}{c^4}\right)_0 = \frac{4}{89B^2}\left[9C + \frac{23AC}{(1+A)}\right] \tag{15}$$

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$$\left(\frac{8\pi G\rho}{c^2}\right)_0 = \frac{144C}{89B^2} \tag{16}$$

The central values of pressure and density will be non zero positive definite, if the following conditions will be satisfied.

$$A > -9/32\,, \quad c > 0. \tag{17}$$

Subjecting the condition that positive value of ratio of pressure-density and less than 1 at the centre i.e. $\frac{p_0}{\rho_0 c^2} \leq 1$ which leads to the following inequality,

$$\frac{1}{36}\left[9 + \frac{23A}{1+A}\right] \leq 1 \tag{18}$$

All the values of A which satisfy equation (17) will lead to the condition $\frac{p_0}{\rho_0 c^2} \leq 1$

Differentiating (13) with respect to r,

$$\frac{dp}{dr} = \frac{4}{704969B^2\left\{1 + A\left(1+Cr^2\right)^{\frac{23}{89}}\right\}^2 \left(1+Cr^2\right)^{\frac{243}{89}}} \left[\begin{array}{l} A^2(1+Cr^2)^{\frac{46}{89}}\left(-532576C^2r - 251680C^3r^3\right) + \\ A\left(1+Cr^2\right)^{\frac{23}{89}}\left(-606624\,C^2r - 240240C^3r^3\right) \\ -168210C^2r - 57330C^3r^3 \end{array}\right] \tag{19}$$

Thus extrema of p occur at the centre if

$$p' = 0 \quad \Rightarrow \quad r = 0 \tag{20}$$

$$\frac{8\pi G}{c^4}\left(p''\right)_{r=0} = \frac{4}{7921B^2}\frac{1}{(1+A)^2}\left[-5984A^2C^2 - 6816AC^2 - 1890C^2\right] \tag{21}$$

$$= \text{-ve} \quad \text{if} \quad 2992A^2 + 3408A + 945 > 0 \tag{22}$$

Thus the expression of right hand side of equation (20) is negative showing thereby that the pressure p is maximum at the centre and monotonically decreasing.

Now differentiating equation (12) with respect to r.

$$\frac{d\rho}{dr} = \frac{4C}{704969\,B^2\left(1+Cr^2\right)^{\frac{243}{89}}}\left[-205590\,Cr - 30030C^2 r^3\right] \tag{23}$$

Thus the extrema of ρ occur at the centre if

$$\rho' = 0 \quad \Rightarrow \quad r = 0 \tag{24}$$

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$$\frac{8\pi G}{c^2}(\rho'')_{r=0} = -\frac{9240C^2}{7921B^2} \tag{25}$$

The right hand side of this equation will be –ve for positive as well negative value of B and C. Thus , the expressions of right hand side of (22) and (24) are negative showing thereby that the density ρ is maximum at the centre and monotonically decreasing..

In view of (21) and (24), we observe that central pressure and central density are maximum at the centre and monotonically decreasing with the increase of radial coordinate r.

The square of adiabatic sound speed at the centre,

$\frac{1}{c^2}\left(\frac{dp}{d\rho}\right)_{r=0}$, is given by

$$\frac{1}{c^2}\left(\frac{dp}{d\rho}\right)_{r=0} = \frac{1}{(1+A)^2}\frac{(5984A^2+6816A+1890)}{2310} \tag{26}$$

$$\leq 1 \text{ and } (+ve)$$

If A > -9/32 (27)

In view of (12) and (13) the ratio of pressure-density is given by

$$\frac{p}{\rho c^2} = \frac{(801C+441C^2r^2) + \dfrac{23A(1+Cr^2)^{\frac{23}{89}}\{65C^2r^2+89C\}}{\left\{1+A(1+Cr^2)^{\frac{23}{89}}\right\}}}{3204C + 924C^2r^2} \tag{28}$$

IV. BOUNDARY CONDITIONS

The solutions so obtained are to be matched over the boundary with Schwarzschild's exterior solution

$$ds^2 = \left(1-\frac{2GM}{c^2R}\right)c^2\,dt^2 - \left(1-\frac{2GM}{c^2R}\right)^{-1}dR^2 - R^2d\theta^2 - R^2\sin^2\theta\,d\phi^2 \tag{29}$$

Where M is the mass of the ball as determined by the external observer and R is the radial coordinate of the exterior region. The usual boundary conditions are that the first and second fundamental forms are continuous over the boundary $r = r_b$ or equivalently $R=R_b$.

Applying the boundary conditions we get the values of the arbitrary constants in terms of Schwarzschild's parameters $u = \frac{GM}{c^2R_b}$ and R_b

We obtain two values of constants A.

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$$A=\frac{(1-2u)^{\frac{1}{2}}\left(1+Cr_b^{\,2}\right)^{\frac{3}{89}}k^2-1}{\left[1+\frac{89\left\{(1-2u)^{\frac{1}{2}}-1\right\}}{\left\{65-89(1-2u)^{\frac{1}{2}}\right\}}\right]^{\frac{23}{89}}}=\frac{(1-2u)^{\frac{1}{2}}\left(1+Cr_b^{\,2}\right)^{\frac{3}{89}}k^2-1}{\left[1+Cr_b^{\,2}\right]^{\frac{23}{89}}}$$

(30)

$$A=\frac{89u(1+Cr_b^{\,2})-42Cr_b^{\,2}(1-2u)^{\frac{1}{2}}}{88Cr_b^{\,2}(1+Cr_b^{\,2})^{\frac{23}{89}}(1-2u)^{\frac{1}{2}}-89u(1+Cr_b^{\,2})^{\frac{112}{89}}} \quad (31)$$

From equation (29) and (30) we obtain the value of k as

$$k=\sqrt{\frac{\left\{\frac{89u(1+Cr_b^{\,2})-42Cr_b^{\,2}(1-2u)^{\frac{1}{2}}}{88Cr_b^{\,2}(1+Cr_b^{\,2})^{\frac{23}{89}}(1-2u)^{\frac{1}{2}}-89u(1+Cr_b^{\,2})^{\frac{112}{89}}}\right\}\left(1+Cr_b^{\,2}\right)^{\frac{23}{89}}+1}{\sqrt{1-2u}\left(1+Cr_b^{\,2}\right)^{\frac{3}{89}}}} \quad (32)$$

where $k=\frac{R_b}{r_b}$

$$B=k\left[1+\frac{89\left\{(1-2u)^{\frac{1}{2}}-1\right\}}{\left\{65-89(1-2u)^{\frac{1}{2}}\right\}}\right]^{\frac{12}{89}}$$

(33)

$$C=\frac{89\left\{(1-2u)^{\frac{1}{2}}-1\right\}}{\left\{65-89(1-2u)^{\frac{1}{2}}\right\}r_b^{\,2}}$$

(34)

Surface density is given by

$$\frac{8\pi G}{c^2}\rho_b\, r_b^{\,2}=\frac{1}{3R_b^{\,2}}\left\{77(1-2u)^{\frac{1}{2}}-77+95u\right\}>0 \quad (35)$$

Provided, $u < .30$

Central red shift is given by

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$$Z_0 = \left[\frac{B^2}{1+A} - 1\right] \tag{36}$$

The surface red shift is given by

$$Z_b = \left[(1-2u)^{-0.5} - 1\right] \tag{37}$$

Table 1: The central values of pressure, density, pressure density ratio, square of sound speed, red shift for different values of u.

Sl.No.	u	$(\frac{8\pi G}{c^4} p r_b^{\,2})_{r=0}$	$(\frac{8\pi G}{c^4} \rho r_b^{\,2})_{r=0}$	$(\frac{p}{\rho c^2})_{r=0}$	$(\frac{1}{c^2}(\frac{dp}{d\rho}))_{r=0}$	$(Z)_0$
1	0.01	0.0004	0.0853	0.00504	0.3816	0.0153
2	0.02	0.0017	0.1747	0.0101	0.3893	0.0316
3	0.04	0.0076	0.3676	0.0208	0.4056	0.0669
4	0.06	0.0186	0.5844	0.0319	0.4228	0.1066
5	0.08	0.0362	0.8332	0.0455	0.4413	0.1520
6	0.10	0.0629	1.1266	0.0558	0.4610	0.2047
7	0.12	0.1024	1.4856	0.0689	0.4825	0.2670
8	0.16	0.2550	2.587	0.0985	0.5323	0.4390

Table 2 : By assuming the surface density $\rho_b = 2\times10^{14}\ g/cm^3$,the variation of maximum Neutron star mass , Radius R_b, central red shift Z_0 and surface red shift $Z_b = \left[(1-2u)^{-0.5} - 1\right]$ with u .

Sl. No.	u	$\frac{8\pi G}{c^2}\rho_b r_b^{\,2}$	$\frac{M}{M_\Theta}$	R_b in km	Z_0	Z_b

1	.01	0.0426	0.019	2.884	0.0153	0.010
2	.02	0.0842	0.055	4.077	0.0316	0.020
3	.04	0.1642	0.155	5.756	0.0669	0.042
4	.06	0.2390	0.284	7.026	0.1066	0.066
5	.08	0.3077	0.435	8.070	0.1520	0.0911
6	.1	0.3692	0.604	8.953	0.2047	0.1180
7	.12	0.4222	0.785	9.700	0.2670	0.1470
8	.16	0.4953	1.165	10.796	0.4390	0.2126

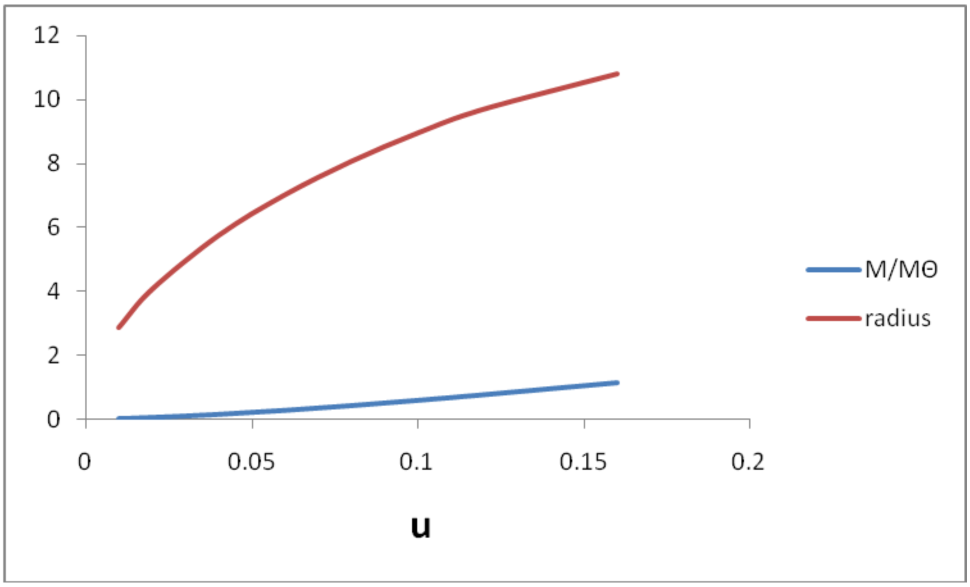

Fig 1: The variation of mass and radius of star with u.

Table 3: The march of pressure, density, pressure-density ratio, square of adiabatic sound speed , adiabatic index , red shift within the ball corresponding to u=.16

r/r_b	$\frac{8\pi G}{c^4} p\, r_b^{\,2}$	$\frac{8\pi G}{c^4} \rho\, r_b^{\,2}$	$\frac{p}{\rho c^2}$	$\frac{1}{c^2}(\frac{dp}{d\rho})$	$\gamma = \frac{dp}{d\rho} / \frac{p}{\rho}$	Z

0.0	0.255	2.587	0.0985	0.5323	5.40	0.4390
0.1	0.245	2.519	0.0976	0.5322	5.45	0.4343
0.2	0.221	2.334	0.0948	0.5317	5.60	0.4211
0.3	0.186	2.074	0.0900	0.5309	5.89	0.4009
0.4	0.149	1.789	0.0833	0.5295	6.35	0.3759
0.5	0.112	1.515	0.0744	0.5274	7.08	0.3481
0.6	0.080	1.271	0.0635	0.5245	8.25	0.3193
0.7	0.053	1.064	0.0505	0.5205	10.30	0.2907
0.8	0.031	0.894	0.0354	0.5151	14.51	0.2631
0.9	0.014	0.756	0.0186	0.5083	27.32	0.2370
1.0	0.000	0.645	0.0000	0.4999	∞	0.2126

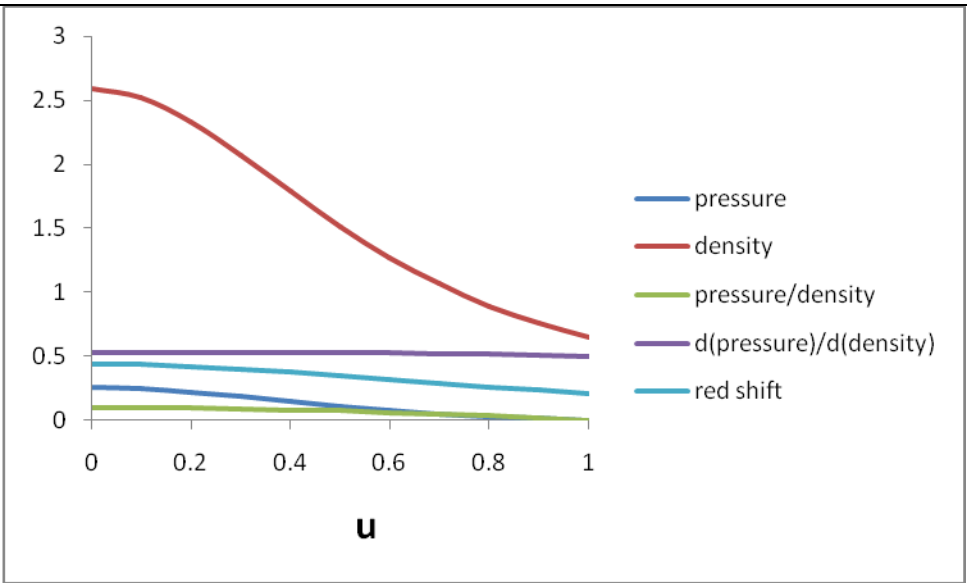

Fig 2: The variation of p , ρ , $\frac{p}{\rho c^2}$, $\frac{1}{c^2}\left(\frac{dp}{d\rho}\right)$ and Z from centre to surface for u = 0.16 .

VI. Application: Slowly Rotating Structures and their Application to the pulsars.

For slowly rotating structures like the Vela pulsars (rotation velocity about 70 *rad/sec)*, one can calculate the moment of inertia in the first-order approximation which appears in the Lense-Thrirring frame dragging effect. However, for the present case, it is very useful to apply an approximate, but very precise, empirical formula which is based on the numerical results obtained for a large number of theoretical equations of state (EOS) of dense nuclear matter. For the type of solution considered in the present study, the formula yields in the following form[15], [18] .

$$I = (2/5)(1 + y)MR^2; \quad (37)$$

where y is the dimensionless compactness parameter measured in units of M_Θ (in km)/km, i.e

$$y = (M/R) / M_\Theta \text{(in km) km}^{-1} \quad (38)$$

we can calculate the moment of inertia, for various super dense objects with the help of Equation (37),

VII. Discussions and Conclusions :

It has been observed that the physical parameters (p, ρ, $\frac{p}{\rho c^2}$, z) are positive at the centre and within the limit of realistic state of equation and monotonically decreasing for $0 < u \leq 0.16$. The causality condition is obeyed throughout within the ball. Thus, the solution is well behaved for all the values of u satisfying the inequality $0 < u \leq 0.16$. We now here present a model of super dense star based on the particular solution discussed above by assuming surface density; $\rho_b = 2 \times 10^{14}\ g/cm^3$. Corresponding to u = 0.16, the resulting well behaved model has maximum mass M=1.165M_Θ with radius R_b ≈ 10.796 km and Moment of inertia $1.59 \times 10^{45} gm\,cm^2$.

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