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REVIEW OF BARRIER FUNCTION METHODS WITH KKT CONDITIONS

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ABSTRACT

Optimization is the process by which we get the maximum and minimum value of the function. In this paper we are dealing with Barrier Function methods with KKT conditions. Barrier function method converts constrained optimization problem into unconstrained optimization problem and we get the feasible solution of the original problem. The paper aims to introduce the concept of Barrier Function Method. Then we will apply KKT multipliers in Barrier method. We will also discuss Sequential Unconstrained Minimization Techniques. In this Barrier method starts with the low value of barrier parameter and by the decreasing value of this parameter we get the sequence of unconstrained problems and solving this we will find the solution of original problem.

Keywords: Barrier Function Method, KKT Conditions, Unconstrained Optimization.

I. INTRODUCTION

In Barrier method a penalty like term is added to the given objective function for the motive that the current solution remains interior to the feasible domain and a “Barrier” is created on the boundary of the feasible region. This method starts at feasible but sub-optimal points and creates optimality as penalty parameter approaches to zero. By this method we get the sequence of feasible points whose limit is an optimal solution to the original problem. The main motive of the Barrier Method is to remain the point x of any approximation to the feasible region border. However, barrier term cannot be defined for equality constraints. The most popular Barrier Functions are:

- Inverse Barrier Function $b(x) = -\sum_{i=1}^{s} \frac{1}{g_i(x)}$
- Logarithmic Barrier Function $b(x) = -\sum_{i=1}^{s} \log(-g_i(x))$

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II. CONCEPT OF BARRIER FUNCTION

Barrier function that defined in the interior of the feasible region is a continuous function in constrained optimization. If a point approaches to the boundary of the feasible region of an optimization problem then the value of the barrier function on that point increase to infinity.

As we take constrained optimization problem

P: Minimize $f(x)$

Subject to $g_i(x) \leq 0, i = 1,2,\ldots,s$

$$\phi = \{R^n | g_i(x) \leq 0, i = 1,2,\ldots,s \}$$

To solve this problem by Barrier Method the barrier function is

$b(x) = \sum_{i=1}^{s} \xi\{g(x)\}$, ξ is a continuous function that satisfies

* $\xi(y) \geq 0$ if $y < 0$

* $\xi(y) = \infty$ if $\lim_{x} \max\{y(x)\} \to 0$

So the unconstrained optimization problem is

Minimize $f(x) + \mu b(x)$

Subject to $g(x) \leq 0 \quad x \in R^n$.

Now we take an example and solve by Barrier Method

Example: Minimize $z = x$

Subject to $-x + 2 \leq 0$

Sol. We have $f(x) = x$

And $g(x) = -x + 2$

Inverse Barrier function for this problem is

$$b(x) = \frac{-1}{(-x+2)}$$

The unconstrained problem for this problem is

$$B(x) = f(x) + \mu b(x)$$

$$= x - \frac{\mu}{(-x+2)}$$

$$\frac{\partial B}{\partial x} = 1 - \frac{\mu}{(-x+2)^2}$$

Now $\frac{\partial B}{\partial x} = 0$

$1-\frac{\mu}{(-x+2)^2}=0$, $1=\frac{\mu}{(-x+2)^2}$, $(-x+2)^2=\mu$

$(-x+2)=\pm\sqrt{\mu}$, $-x=\pm\sqrt{\mu}-2$, $x=2\pm\sqrt{\mu}$

Now take

$\underset{\mu\to 0}{lt}\, x=\underset{\mu\to 0}{lt}\, 2\mp\sqrt{\mu}$, $x^*=2$

So the feasible solution of the given problem is 2 .

$f(x)=2$ is the optimal solution of the problem.

III. BARRIER CONVERGENCE THEOREM

Consider the constrained problem:

Minimize $f(x)$

Subject to $g_i(x)\le 0$, $i=1,2,...,s$ $\quad x\in X$.

Here $f:R^n\to R$ and $g:R^n\to R^s$ are continuous functions and X be non empty closed set in R^n. By barrier method the unconstrained problem is

$$B(\mu)=f(x)+\mu\sum_{i=1}^{s}\xi\{g_i(x)\}$$

let $\{\mu_k\}$ be the sequence of the parameters that satisfies $0<\mu_{k+1}<\mu_k$ so $\mu_k\to 0$ as $k\to\infty$. Let $\{x^k\}$ be the sequence of the solution of $B(\mu_k)$ for $k=1,2,...,\infty$. Suppose for the unconstrained problem there exist a solution x^* for which $N_\in(x)\cap\{x\,|g(x)<0\}\neq 0$ for $\in>0$. Then any limit point $\bar{x}$ of $\{x^k\}$ is solution of the original problem.

Proof. Let $\bar{x}$ be the limit point of the sequence $\{x^k\}$.As $f(x)$ and $g(x)$ are continuous functions so

$\underset{k\to\infty}{lt}\, f(x^k)=f(\bar{x})$

And $\underset{k\to\infty}{lt}\, g(x^k)=g(\bar{x})\le 0$ (a)

So $\bar{x}$ is the feasible solution of the given problem.

If $g(\bar{x})<0$ then by the barrier function is defined as:

$b(x^k)\ge 0$

So $\underset{k\to\infty}{lt}\, \mu_k b(x^k)=0$ ($\because \mu_k\to 0$ for $k\to\infty$) (b)

If the feasible solution $\bar{x}$ lies on the boundary on the region then $\underset{k\to\infty}{lt}\, b(x^k)=+\infty$.

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Now $\underset{k\to\infty}{lt}\inf\left\{f\left(x^k\right)+\mu_k b\left(x^k\right)\right\}=\underset{k\to\infty}{lt}f\left(x^k\right)+\underset{k\to\infty}{lt}\inf\mu_k b\left(x^k\right)$

$\geq f\left(\bar{x}\right)$ (by (a) and (b))

As we want to prove that $\bar{x}$ is the solution of the original problem, for this we have to prove that $\bar{x}$ is the global minimizer.

Suppose $\bar{x}$ is not global minimizer, there would exist a feasible vector x^* such that

$f\left(x^*\right)< f\left(\bar{x}\right)$

That can be approached arbitrary closely through the interior of the feasible set. So there exist a point $\tilde{x}$ in the domain of the barrier function such that

$$f\left(\tilde{x}\right)< f\left(\bar{x}\right) \tag{c}$$

By the def. of x^k

$f\left(x^k\right)+\mu_k b\left(x^k\right)\leq f\left(\tilde{x}\right)+\mu_k b\left(\tilde{x}\right)\quad\forall k$

Taking limit as $k\to\infty$ then

$\underset{k\to\infty}{lt}\left(f\left(x^k\right)+\mu_k b\left(x^k\right)\right)\leq\underset{k\to\infty}{lt}\left(f\left(\tilde{x}\right)+\mu_k b\left(\tilde{x}\right)\right)$

$f\left(\bar{x}\right)< f\left(\tilde{x}\right)$($\because \mu_k\to 0$ for $k\to\infty$)

It is the contradiction of (c).

It shows that $\bar{x}$ is the global minimizer of the given problem. So $\bar{x}$ is the solution of the original problem..

IV. KARUSH-KUHN-TUCKER MULTIPLIERS IN BARRIER METHOD

Consider the problem Minimize $f\left(x\right)$

Subject to $g_i\left(x\right)\leq 0,\, i=1,2,\ldots,s \quad x\in X=R^n$.

The barrier function is defined as

$$b\left(x\right)=\sum_{i=1}^{s}\xi\left(g_i\left(x\right)\right)$$

Then the unconstrained problem is

$f\left(x\right)+\mu b\left(x\right)$

Or $f\left(x\right)+\mu\sum_{i=1}^{s}\xi\left(g_i\left(x\right)\right)$ (1)

Here f, g and ξ are continuously differentiable with $\xi\left(y\right)\geq 0$ If $y<0$ and $\underset{y\to 0^-}{lt}\xi\left(y\right)=\infty$ So we have

$$\nabla b\left(x\right)=\sum_{i=1}^{s}\xi'\left(g_i\left(x\right)\right)\nabla g_i\left(x\right)$$

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Let $\bar{x}$ be the limit point of the sequence $\{x^k\}$ of the solutions of $b(x)$ for $k = 1,2,\ldots,\infty$. Then by convergence theorem $\bar{x}$ is the optimal solution of the original problem.

If $I = \{i : g_i(\bar{x}) = 0\}$ be the index set of active constraints then there exist a unique sequence of Lagrange's multipliers $\bar{u}_i, i = 1,2,\ldots,s$ such that

$$\nabla f(\bar{x}) + \sum_{i=1}^{s} \bar{u}_i \nabla g_i(\bar{x}) = 0$$

$$\bar{u}_i \geq 0 \text{ for } i = 1,2,\ldots,s$$

$$\text{And} \quad \bar{u}_i = 0 \text{ for } i \notin I \tag{2}$$

As x^k is the solution of the unconstrained problem

$$\nabla f(x^k) + \mu_k \sum_{i=1}^{s} \xi'[g_i(x^k)] \nabla g_i(x^k) = 0$$

$$\nabla f(x^k) + [u_k]_i \sum_{i=1}^{s} \nabla g_i(x^k) = 0 \tag{3}$$

Where $[u_k]_i = \mu_k \xi'[g_i(x^k)] \quad , i = 1,2,\ldots,s$

And $\mu_k \to 0$ and $\{x^k\} \to \bar{x}$ as $k \to \infty$

So $[u_k]_i = \mu_k \xi'\{g_i(x^k)\} \to 0$ for $i \in I$,

From (2) and (3) we have $[u_k]_i \to \bar{u}_i \quad \forall i \in I$.

So we can estimate the set of Lagrange multipliers by u_k when $\mu \to 0^+$, then these multipliers approaches to the optimal set of Lagrange multipliers $\bar{u}$.

V. SEQUENTIAL UNCONSTRAINED MINIMIZATION TECHNIQUE

We prefer to solve the sequence of barrier problems instead of solving just one, with low value of parameter μ , the reason is that the Hessian $\nabla^2 b(x,\mu)$ of the barrier function become ill-condition with the small value of μ which create difficulty for the solution of the problem. Barrier method starts with the low value of barrier parameter and by the decreasing value of this parameter we get the sequence of unconstrained problems. We solve the barrier problem by the following procedure:

Step: 1 Starting with x^k , solve the problem:

$$\text{Minimize} \quad f(x) + \mu b(x)$$

$$\text{Subject to} \quad g(x) < 0, \quad x \in X .$$

Let x^{k+1} be an optimal solution then

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Step: 2 If $\mu_k b\left(x^{k+1}\right) < \in$ stop; otherwise, take $\mu_{k+1} = \beta\mu_k$, take $k == k+1$ and start the procedure again.

Example: Minimize $x_1^2 + 2x_2^2$

Subject to $1 - x_1 - x_2 \le 0$, $(x_1, x_2) \in R^2$.

Sol. We have $f(x) = x_1^2 + 2x_2^2$

Subject to $g(x) = 1 - x_1 - x_2 \le 0$

Barrier function for this problem is defined as

$$b(x) = -\log(-g(x))$$

$$= -\log(x_1 + x_2 - 1)$$

The unconstrained problem is

Minimize $f(x) + \mu b(x)$

Subject to $g(x) \le 0$

We can write it as

$$B(x,\mu) = f(x) + \mu(-\log(x_1 + x_2 - 1))$$

$$= x_1^2 + 2x_2^2 - \mu(\log(x_1 + x_2 - 1))$$

Now $$\frac{\partial B}{\partial x_1} = 2x_1 - \frac{\mu}{x_1 + x_2 - 1}$$

And $$\frac{\partial B}{\partial x_2} = 4x_2 - \frac{\mu}{x_1 + x_2 - 1}$$

For $\frac{\partial B}{\partial x_1} = 0$ and $\frac{\partial B}{\partial x_2} = 0$

We have $$2x_1 - \frac{\mu}{x_1 + x_2 - 1} = 0 \qquad (4)$$

$$4x_2 - \frac{\mu}{x_1 + x_2 - 1} = 0 \qquad (5)$$

Subtract (4) from (5), we get

$$4x_2 - \frac{\mu}{x_1 + x_2 - 1} - 2x_1 + \frac{\mu}{x_1 + x_2 - 1} = 0$$

$$4x_2 - 2x_1 = 0, \quad 4x_2 = 2x_1, \quad 2x_2 = x_1$$

Put this value in (4), we get

$$2(2x_2) - \frac{\mu}{2x_2 + x_2 - 1} = 0, \quad 4x_2 - \frac{\mu}{3x_2 - 1} = 0, \quad 12x_2^2 - 4x_2 - \mu = 0$$

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$$x_2 = \frac{4 \pm \sqrt{16+48\mu}}{24}, \quad = \frac{1 \pm \sqrt{1+3\mu}}{6}, \quad = \frac{1+\sqrt{1+3\mu}}{6}$$

As $2x_2 = x_1$

So $$x_1 = 2\left(\frac{1+\sqrt{1+3\mu}}{6}\right) \quad = \frac{1+\sqrt{1+3\mu}}{3}$$

Now start the iterations with $\mu_1 = 1$ and the scalar $\beta = 0.1$ and $x^1 = (0,0)$

Iteration: 1 $$x_1 = \frac{1+\sqrt{1+3(1)}}{3}, \quad = \frac{1+\sqrt{4}}{3}, \quad = \frac{1+2}{3}, \quad = \frac{3}{3}, \quad = 1$$

$$x_2 = \frac{1+\sqrt{1+3(1)}}{6}, \quad = \frac{1+\sqrt{4}}{6}, \quad = \frac{1+2}{6}, \quad = \frac{3}{6}, \quad = \frac{1}{2}$$

And $$g(x^2) = 1 - x_1 - x_2, \ = 1 - 1 - 0.5, \ = -0.5$$

$$\mu_1 b(x^2) = -\log(-g(x)), \ = -\log(-(-0.5)), \ = -\log(0.5)$$

$$= -(-0.301), \ = 0.301$$

Iteration: 2 $\mu_2 = \mu_1 \beta$, $= 1 \times 0.1$, $= 0.1$

$$x_1 = \frac{1+\sqrt{1+0.3}}{3}, \quad = \frac{1+\sqrt{1.3}}{3}, \quad = \frac{2.140175}{3}, \quad = 0.71339$$

$$x_2 = \frac{1+\sqrt{1.3}}{6}, \quad = \frac{2.140175}{6}, \quad = 0.35669$$

$$g(x^3) = 1 - 0.71339 - 0.35669, \ = -0.070$$

And $$\mu_2 b(x^3) = (-0.1)\log(-(-0.070)),$$

$= (-0.1)\log(0.070), = (-0.1)(-1.1549), = 0.1155$ **Iteration: 3** $\mu_3 = \mu_2 \beta$, $= 0.1 \times 0.1, = 0.01$

$$x_1 = \frac{1+\sqrt{1+0.03}}{3}, = \frac{1+\sqrt{1.03}}{3}, = \frac{2.014889}{3}, \ = 0.6716$$

$$x_2 = \frac{1+\sqrt{1.03}}{6} = \frac{2.014889}{6} = 0.3358$$

$$g(x^4) = 1 - 0.3358 - 0.6716$$
$$= -0.0074$$

$$\mu_3 b(x^4) = (-0.01)\log(-(-0.0074))$$
$$= (-0.01)\log(0.0074) = (-0.01)(-2.131) = 0.0213$$

Iteration: 4 $\mu_4 = \mu_3 \beta \quad = 0.01 \times 0.1 \quad = 0.001$

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$$x_1 = \frac{1+\sqrt{1.003}}{3} = \frac{2.001499}{3} = 0.6672$$

$$x_2 = \frac{2.001499}{6} = 0.3336$$

$$g(x^5) = 1 - 0.6672 - 0.3336 = -0.0008$$

$\mu_4 b(x^5) = (-0.001)\log(-(-0.0008)) = (-0.001)\log(0.0008) = (-0.001)(-3.097) = 0.0031$ **Iteration: 5** $\mu_5 = \mu_4 \beta = 0.001 \times 0.1 = 0.0001$

$$x_1 = \frac{1+\sqrt{1+0.0003}}{3} = \frac{1+\sqrt{1.0003}}{3} = \frac{2.00014}{3} = 0.6667$$

$$x_2 = \frac{2.00014}{6} = 0.3334$$

$$g(x^6) = 1 - 0.6667 - 0.3334 = -0.0001$$

$$\mu_5 b(x^6) = (-0.0001)\log(-(-0.0001)) = (-0.0001)\log(0.0001) = (-0.0001)(-4.) = 0.0004$$

So finally we get

Iter. k	μ_k	x^{k+1}	$g(x^{k+1})$	$\mu_k b(x^{k+1})$
1	1	$(1,0.5)$	-0.5	0.301
2	0.1	$(0.71339,0.35669)$	-0.07	0.1155
3	0.01	$(0.6716,0.3358)$	-0.0074	0.0213
4	0.001	$(0.6672,0.3336)$	-0.0008	0.0031
5	0.0001	$(0.6667,0.3334)$	-0.0001	0.0004

We get $\mu_k b(x^{k+1}) < \in$ at the $5th$ iteration, so we get the optimal solution is $x^* = (.6667,0.3334)$

Or $x^* = \left(\frac{2}{3},\frac{1}{3}\right)$

VI. CONCLUSION

Optimization is the technique by which we obtain the optimal solution of the given problem. By this paper we conclude that The Barrier Function Method is the powerful method that is used for those problems which have the robust feasible region. The solution of unconstrained problem approaches to the solution of the original problem from inside of the feasible region that is why this method is also called Interior Point Method. Barrier

Method also helps to estimate the Karush-Kuhn-Tucker Multipliers by which the optimal solution of the given problem can be obtained.

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