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STUDY OF LIMIT AND CONTINUITY OF FUNCTIONS OF SEVERAL VARIABLES

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I DEFINITIONS AND EXAMPLES OF A REAL-VALUED FUNCTION OF SEVERAL VARIABLES

We extend the definition of a function of one variable to functions of two or more variables. You will recall that a function was a rule which assigned a unique value to each input value. It is going to be similar for two or more variables. The only difference is that the input is not a value anymore, it is several values.

Definition: Suppose D is a set of n-tuples of real numbers $(x_1, x_2, x_3, \ldots, x_n)$. A real-valued function f on D is a rule that assigns a unique (single) real number $y = f(x_1, x_2, x_3, \ldots, x_n)$ to each element in D.

A real-valued function of **two** variables is a function whose domain is a subset of the plane $\mathbb{R}^2$ and whose range is a subset of $\mathbb{R}$, or the real numbers.

For example, consider the functions below

$$z = f(x, y) = 2x^2 + y^2$$

$$w = g(x, y, z) = 2xe^{yz}$$

$$h(x_1, x_2, x_3, x_4) = 2x_1 - x_2 + 4x_3 + x_4$$

The function $z = f(x, y)$ is a function of two variables. It has independent variables x and y, and the dependent variable z.

Likewise, the function $w = g(x, y, z)$ is a function of three variables. The variables x, y and z are **independent variables** and w is the **dependent variable**. The function h is similar except there are four independent variables.

The operations we performed with one-variable functions can also be performed with functions of several variables.

For example, for the two-variable functions f and g:

$$(f \pm g)(x, y) = f(x, y) \pm g(x, y)$$

$$(f.g)(x,y) = f(x,y).g(x,y)$$

$$\left(\frac{f}{g}\right)(x,y) = \frac{f(x,y)}{g(x,y)}, \textit{provided } g(x,y) \neq 0.$$

Remark: In general we will not consider the composition of two multi-variable functions.

Key words: Limit of function , Domain, Range of the function, Level of the curve ,

Domain and range of functions of several variables

The domain of a function can be either

- **Specified by the problem, i.e. specific restrictions are given**

For instance, $f(x;\ y) = x^2 + y^2$ such that $-1 < x < 1 ; -1 < y < 1$

- **Assumed to be all points for which the function is valid**

For instance $f(x;\ y) = \sqrt{4 - x^2 - y^2}$

Hence unless the domain is given, assume the **domain** is the set $\boldsymbol{D}$ of all points for which the equation is defined, and the **range** is set $\boldsymbol{R}$ of values that f takes on, that is $\boldsymbol{R} = \{f(x_1,x_2,x_3,\ldots,x_n)/(x_1,x_2,x_3,\ldots,x_n) \in \boldsymbol{D}\}$.

For example, consider the functions

$$f(x,y) = 3x^2 + y^2 \text{ and } g(x,y) = \frac{1}{\sqrt{xy}}$$

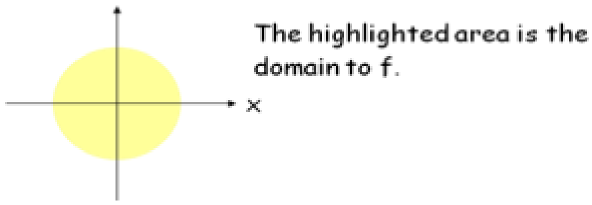
The domain of $f(x,y)$ is the entire $xy-$plane. Every ordered pair in the $xy-$ plane will produce a real value for f.

The domain of $g(x,y)$ is the set of all points (x,y) in the $xy-$plane such that the product xy is greater than 0. This would be all the points in the first quadrant and the third quadrant.

Example 1: Find the domain of the function $f(x,y) = \sqrt{25 - x^2 - y^2}$

Solution: The domain of $f(x,y)$ is the set of all points that satisfy the inequality: $25 - x^2 - y^2 \geq 0 \qquad or \qquad 25 \geq x^2 + y^2$

You may recognize that this is similar to the equation of a circle and the inequality implies that any ordered pair on the circle or inside the circle $x^2 + y^2 = 25$ is in the domain.



Example 2: Find the domain of the function $g(x, y, z) = \sqrt{x^2 + y^2 + z^2 - 16}$

Solution: Note that g is a function of three variables, so the domain is NOT an area in the xy −plane. The domain of g is a solid in the 3-dimensional coordinate system.

The expression under the radical must be nonnegative, resulting in the inequality:

$x^2 + y^2 + z^2 - 16 \geq 0$ or $x^2 + y^2 + z^2 \geq 16$

This implies that any ordered triple outside of the sphere centered at the origin with radius 4 is in the domain.

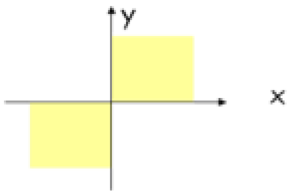
Example 3: Find the domain of the function $h(x, y) = \ln(xy)$

Solution:

We know the argument of the natural log must be greater than zero.

So, $x \cdot y > 0$

This occurs in $quadrant\ I$ and $quadrant\ III$. The domain is highlighted below. Note the $x - axis$ and the $y - axis$ are **NOT** in the domain.



Example 4:

For each of the following functions, evaluate $f(3,\ 2)$ and find the domain.

(a) $f(x,y) = \frac{\sqrt{x+y+1}}{x-1}$ (b) $f(x,y) = x\ell n\,(y^2 - x)$

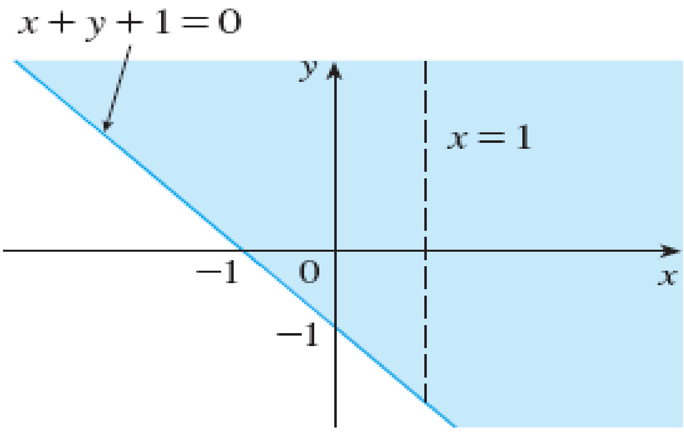
Solution:

(a) $f(3,2) = \frac{\sqrt{3+2+1}}{3-2} = \frac{\sqrt{6}}{2}$

The expression for f makes sense if the denominator is not 0 and the quantity under square root sign is nonnegative. Hence the domain of f is

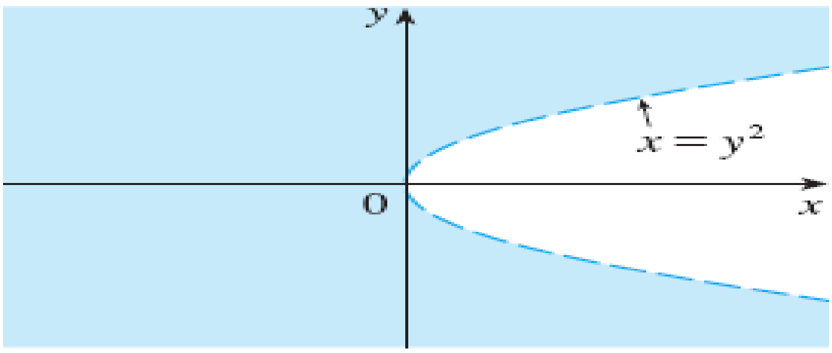
$D = \{(x,y)|x + y + 1 \geq 0,\ x \neq 1\}.$

The inequality $x + y + 1 \geq 0,\ \ or\ y \geq -x - 1,$ describes the points that lie on or above the line $y = -x - 1,\ while\ x \neq 1$ means the points on the line $x = 1$ must be excluded from the figure.



(b) $(3,\ 2) = 3\ell n\ (2^2 - 3) = 3\ell n 1 = 0.$

Since $\ell n(y^2 - x)$ is defined only when $y^2 - x > 0,$ that is, $x < y^2,$ the domain of f $is\ D = \{(x,y): x < y^2\}.$ This is the set of points to the left of the parabola $x = y^2.$



(c) **Example-5:** Find the domain and range of $g(x,y) = \sqrt{9 - x^2 - y^2}.$

Solution: The domain of g is

$$D = \{(x,y)|9 - x^2 - y^2 \geq 0\} = \{(x,y)|x^2 + y^2 \leq 9\}$$

which is a disk with center $(0,0)$ and radius 3.

The range of g is $\{z/z = \sqrt{9 - x^2 - y^2}, (x,y) \in D\}$.

Since z is a positive square root, ≥ 0 . Also

$$9 - x^2 - y^2 \leq 9 \Rightarrow \sqrt{9 - x^2 - y^2} \leq 3$$

So the range is $\{z/0 \leq z \leq 3\} = [0,3]$.

II GRAPHS AND LEVEL CURVES OF FUNCTIONS OF SEVERAL VARIABLES

2.1 Graphs of functions of several variables

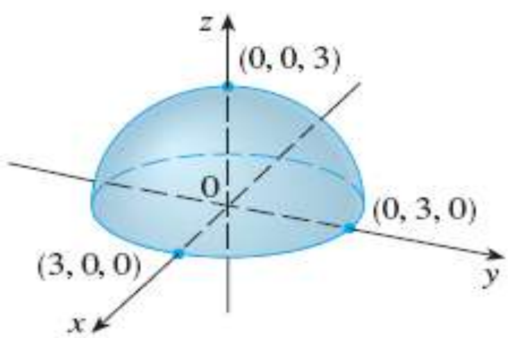
Definition: If f is a function of two variables with domain D, then the graph of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ *and* (x, y) is in D.

Example 1: Sketch the graph of the function $f(x, y) = 6 - 3x - 2y$.

Solution: The graph of f has the equation $z = 6 - 3x - 2y$, *or* $3x + 2y + z = 6$, which represents a plane. To graph the plane we first find the intercepts. Putting $y = z = 0$ in the equation, we get $x = 2$ as the x −intercept is 3 and the z −intercept is 6. This helps us to sketch the portion of the graph that lies in the first octant.

Example 2: Sketch the graph of $g(x, y) = \sqrt{9 - x^2 - y^2}$.

Solution: The graph has equation $z = \sqrt{9 - x^2 - y^2}$. We square both sides of this equation to obtain $z^2 = 9 - x^2 - y^2$, *or* $x^2 + y^2 + z^2 = 9$, which we recognize as an equation of the sphere with centre at origin and radius 3. But since $z \geq 0$, the graph of g is just the top half of this sphere.

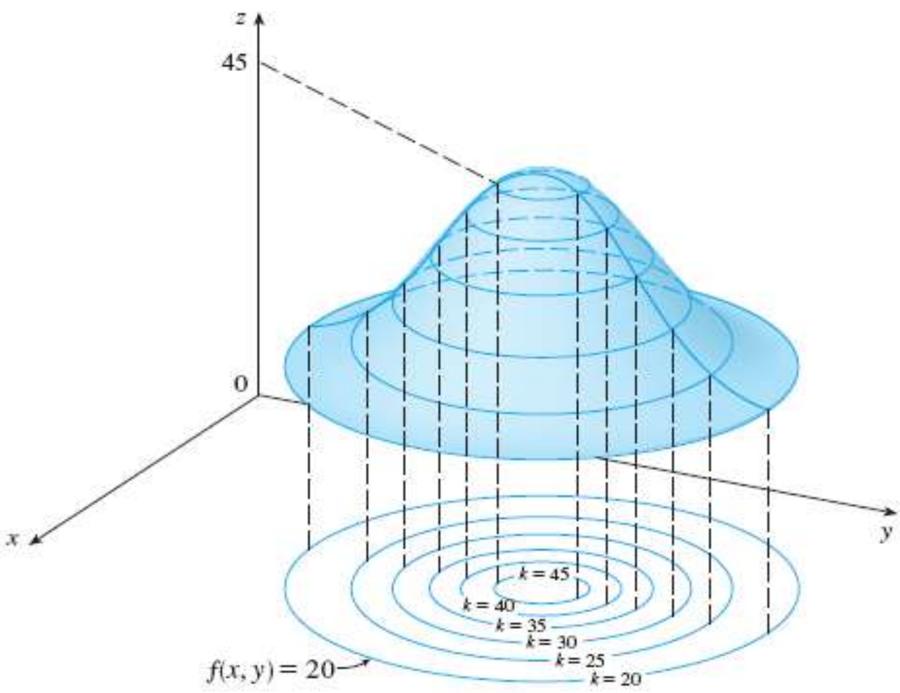


Graph of $g(x, y) = \sqrt{9 - x^2 - y^2}$

2.2 Level Curves of functions of several variables

Definition: The level curves of a function f of two variables are the curves with equations $f(x, y) = k,$ where k is a constant (in the range of f).

- ❖ A level curve $f(x, y) = k,$ is the set of all points in the domain of f at which f takes on the given value. In other words, it shows where the graph of f has height k.
- ❖ The level curves $f(x, y) = k$ are just the traces of the graph of f in the horizontal plane $z = k$ projected down to the $xy-$ plane.



Example 1: Sketch the level curves of the function $f\ (x,y) = 6 - 3x - 2y$ for the value $k = -6,\ 0,\ 6,\ 12.$

Solution:

The level curves are $6 - 3x - 2y = k$ $\quad or \quad$ $3x + 2y + (k - 6) = 0$

This is a family of lines with slope $\frac{-3}{2}$. The four particular level curves with $k = -6,\ 0,\ 6,$ and 12 are $3x + 2y - 12 = 0,$ $3x + 2y - 6 = 0,$ $3x + 2y = 0$, and $3x + 2y + 6 = 0.$

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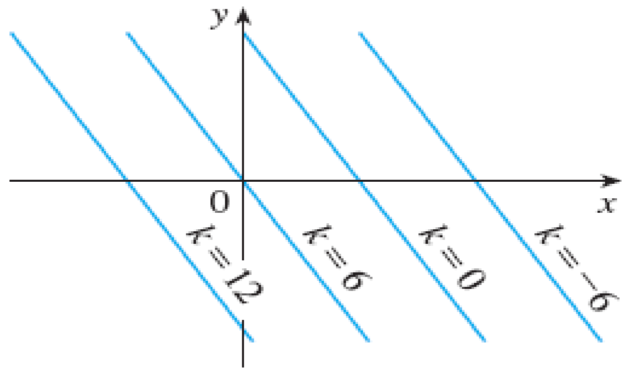


Counter map of $f\ (x,y) = 6 - 3x - 2y$.

Example 2:

Sketch the level curves of the function

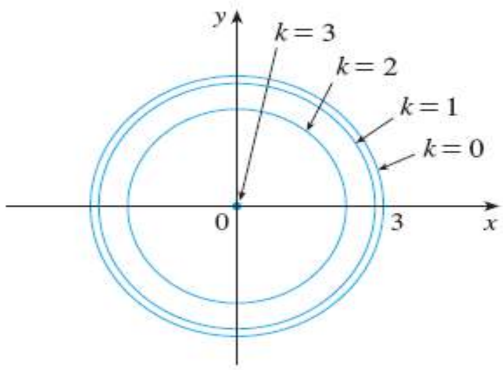
$$g\ (x,y) = \sqrt{9 - x^2 - y^2}\ \ for\ \ k = 0, 1, 2, 3$$

Solution:

The level curves are

$$\sqrt{9 - x^2 - y^2} = k\ \ or\ \ x^2 + y^2 = 9 - k^2$$

This is a family of concentric circles with center $(0,0)$ and radius $\sqrt{9 - k^2}$.



III LIMITS AND CONTINUITY OF FUNCTIONS OF SEVERAL VARIABLES

Definition: Let f be a function of two variables whose domain D includes points arbitrarily close to(a, b). Then we say that the limit of $f\ (x,y)$ as (x, y) approaches (a, b) is L and we write

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if for every number $\epsilon > 0$, there is a corresponding number $\delta > 0$ such that

if $(x, y) \in D$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$, then $|f(x,y) - L| < \epsilon$.

Remark:

If $f(x,y) \to L_1$ *as* $(x,y) \to (a,b)$ a long a path C_1 and $f(x,y) \to L_2$ *as* $(x,y) \to (a,b)$ a long a path C_2, where $L_1 \neq L_2$, then $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

Example-1: Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x^2 + y^2}$ does not exist.

Solution: Let $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$. First let's approach $(0,0)$ a long the x – axis. Then $y = 0$ gives $f(x,0) = \frac{x^2}{x^2} = 1$ for all $x \neq 0$, *so* $f(x,y) \to 1$ *as* $(x,y) \to (0,0)$ a long the x – axis.

We now approach along the y – axis by putting $x = 0$. The

$f(0,y) = \frac{y^2}{y^2} = -1$ *for all* $y \neq 0$, so $f(x,y) \to -1$ *as* $(x,y) \to (0,0)$ a long the y – axis.

Since f has two different limits a long two different lines, the given limit does not exist.

Example-2: If $f(x,y) = \frac{x.y}{x^2+y^2}$, does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist?

Solution: If $y = 0$, *then* $f(x,0) = \frac{0}{x^2} = 0$.Therefore $f(x,y) \to 0$ *as* $(x,y) \to (0,0)$ a long x– axis.

If $x = 0$, then $f(0,y) = \frac{0}{y^2} = 0$, so $f(x,y) \to 0$ *as* $(x,y) \to (0,0)$ a long the x – axis.

Although we have obtained identical limits along the axes that do not show that the given limit is 0. Let's now approach $(0,0)$ a long another line, say $y = x$.

For all $x \neq 0$, $f(x,x) = \frac{x^2}{x^2+x^2} = \frac{1}{2}$. Therefore $f(x,y) \to \frac{1}{2}$ *as* $(x,y) \to (0,0)$ along $y = x$.

Since we have obtained different limits along different paths, the given limit does not exist.

Example-3: Find $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$ if it exists.

Solution: Limit of the function a long line through the origin is 0. This doesn't prove that the given limit is 0, but the limits along the parabolas $y = x^2$ *and* $x = y^2$ also turn out to be 0, So we begin to suspect that the limit does exist and is equal to zero.

Let $\epsilon > 0$. We want to find $\delta > 0$ such that

if $0 < \sqrt{x^2+y^2} < \delta$ *then* $\left|\frac{3x^2y}{x^2+y^2} - 0\right| < \epsilon$, that is, if $0 < \sqrt{x^2+y^2} < \delta$, then $\frac{3x^2|y|}{x^2+y^2} < \epsilon$. But $x^2 \leq x^2 + y^2$ since $y^2 \geq 0$. So $\frac{x^2}{x^2+y^2} < 1$ and therefore

$$\frac{3x^2|y|}{x^2+y^2} \leq 3|y| = 3\sqrt{y^2} \leq 3\sqrt{x^2+y^2}$$

Thus if we choose $\delta = \frac{\epsilon}{3}$ *and let* $0 < \sqrt{x^2 + y^2} < \delta$, *then* $\left|\frac{3x^2y}{x^2+y^2} - 0\right| \leq 3\sqrt{x^2 + y^2} < 3\delta = 3\left(\frac{\epsilon}{3}\right) = \epsilon$

.

Hence, $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$.

Definition: A function f of two variables is called **continuous** at (a, b) if

$$\lim_{(x,y)\to(a,b)} = f(a, b).$$

We say f is continuous on D if f is continuous at every point (a, b) in D.

By properties of limits, the sums, differences, products, and quotients of continuous functions are continuous on their domains.

Example 4: Evaluate $\lim_{(x,y)\to(1,2)}(x^2y^3 - x^3y^2 + 3x + 2y)$.

Solution: Since $f(x, y) = x^2y^3 - x^3y^2 + 3x + 2y$ is a polynomial, it is continuous every where, so we can find the limit by direct substitution:

$$\lim_{(x,y)\to(1,2)}(x^2y^3 - x^3y^2 + 3x + 2y) = 1^2 \cdot 2^3 - 1^3 \cdot 2^2 + 3 \cdot 1 + 2 \cdot 2 = 11.$$

Example 5: Where is the function $f(x, y) = \frac{x^2-y^2}{x^2+y^2}$ continuous?

Solution: The function f is discontinuous at $(0,0)$ because it is not defined there. Since f is a rational function, it is continuous on its domain, which is the set $D = \{(x, y) | (x, y) \neq (0,0)\}$.

Example 6: Let $g(x, y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2} & if (x, y) \neq (0,0) \\ 0 & if\ (x, y) = (0,0) \end{cases}$

g is defined at $(0,0)$ but g is still discontinuous there because $\lim_{(x,y)\to(0,0)} g(x, y)$does exist.

Example 7: Let $f(x, y) = \begin{cases} \frac{3x^2y}{x^2+y^2} & if (x, y) \neq (0,0) \\ 0 & if\ (x, y) = (0,0) \end{cases}$

We know f is continuous for $(x, y) \neq (0,0)$ since it is equal to a rational function there.

From example 3, we have $\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0 = f(0,0)$.

Therefore f is continuous at$(0,0)$ and so it is continuous on $\Re^2$

IV CONCLUSION

By the help of limits function verifying of the function is continues and finding domain of the function and range of the function and finally finding the level and graph of the continues functions.

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