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A STUDY OF ELEMENTARY FUNCTIONS AND THEIR APPLICATIONS IN FUNCTIONS OF COMPLEX VARIABLES

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I INTRODUCTION

Complex analysis, traditionally known as the theory of functions of a complex variable, is the branch of mathematical analysis that investigates functions of complex numbers. It is useful in many branches of mathematics, including algebraic geometry, number theory, analytic combinatorics, applied mathematics; as well as in physics, including hydrodynamics and thermodynamics and also in engineering fields such as nuclear, aerospace, mechanical and complex analysis as "one of the most beautiful as well as useful branches of Mathematics ".Complex analysis is particularly concerned with analytic functions of complex variable. Because the separate real and imaginary parts of any analytic function must satisfy Laplace's equation, complex analysis is widely applicable to two-dimensional problems in physics.

In mathematics, an elementary function is a function of one variable which is the composition of a finite number of arithmetic operations (+ – × ÷), exponentials, logarithms, constants, and solutions of algebraic equations (a generalization of *n*th roots).

The elementary functions include the trigonometric and hyperbolic functions and their inverses, as they are expressible with complex exponentials and logarithms.

Some elementary functions, such as roots, logarithms, or inverse trigonometric functions, are not entire functions and may be multi valued. Complex functions are, of course ,quite easy to come by—they are simply ordered pairs of real valued functions of two variables. That it is those complex functions that are differentiable .

Properties of elementary complex functions like exponential function ,trigonometric and these functions will become real functions when z=x a real number .These functions play a very important role in the applications.

Definitions

Complex exponential function e^z

Let $z = x + iy.$

Complex exponential function is written as e^z or exp.(z) it is defined as follows

$$e^z = e^x.e^{iy} = e^x(\cos y + i\sin y)$$

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Definition

Let a function f(z) be drivable at every point z in an $\in$ neighborhood of z_o .

i.e., $f^1(z)$ exists for all z such that $|z-z_0|<\in$ where $\in> 0.$

then f(z) is said to be analytic at z_o.

Definition : Cauchy –Rieman Equations:

An analytical function is derivable

$f(z)=u(x,y)+iv(x,y)$ to exist for all values of z in domain R are

(i) $\frac{\partial u}{\partial x},\frac{\partial u,}{\partial y},\frac{\partial v}{\partial x},\frac{\partial v}{\partial y}$ are continuous functions x and y in R .

(ii) $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$

Definition: let D be a domain of complex numbers .

If f(z) is analytic at every z $\in$ D, f(z) is said to be analytic in the domain D .

If f(z) is analytic at every point z on the complex plane ,f(z) is said to be an entire function or integral function.

Properties:

1) When y=0 we have $z=x \text{ and } \cos y=1, \sin y=0$

If $x=0, e^{iy}=e^0(\cos y+i\sin y)=(\cos y+i\sin y) \quad where\ y\ is\ real.$

2) $e^z=e^x.e^{iy}=e^x(\cos y+i\sin y)=u+iv\ (say)$

Then $u=e^x\cos y \quad and \quad v=e^x\sin y$

$$\therefore \frac{\partial u}{\partial x}=e^x\cos y, \frac{\partial u}{\partial y}=-e^x\sin y$$

$$\frac{\partial v}{\partial x}=e^x\sin y, \frac{\partial v}{\partial y}=e^x\cos y$$

$$\therefore \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\ and\ \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

$\therefore$ Cauchy-riemann euations are satisfied.

Function is analytic for all z i.e., the function is “entire” .

Also $u_x, u_{y,} v_x, v_y$ are continuous for all x

$\therefore$ exponential function is analytic for all z.

Hence e^z is an entire function.

3) $\frac{d}{dz}(e^z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x \cos y + ie^x \sin y = e^x(\cos y + i\sin y) = e^z$

4) $e^{z_1}.e^{z_2} = e^{z_1+z_2}$ where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

proof:
$$e^{z_1}.e^{z_2} = e^{x_1}(\cos y_1 + i\sin y_1).e^{x_2}(\cos y_2 + i\sin y_2)$$
$$e^{x_1+x_2}(\cos(y_1 + y_2) + i\sin(y_1 + y_2)) = e^{z_1+z_2}$$

5) $\left|e^{iy}\right| = \left|\cos y + i\sin y\right| = \sqrt{\cos^2 y + \sin^2 y} = 1$

Trigonometric functions:

1) We have $e^{iy} = \cos y + i\sin y$ and $e^{-iy} = \cos y - i\sin y$

By adding and subtracting ,we get
$2\cos y = e^{iy} + e^{-iy}$ and
$2i\sin y = e^{iy} - e^{-iy}$

2) $\sin^2 z + \cos^2 z = 1$
3) $1 + \tan^2 z = \sec^2 z$
4) $1 + \cot^2 z = \cos ec^2 z$
5) $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$
6) $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$
7) $\cos^2 z - \sin^2 z = \cos 2z$
8) $2\sin z \cos z = \sin 2z$
9) $\sin(-z) = -\sin z$
10) $\cos(-z) = \cos z$

Derivates :

(i) $\frac{d}{dz}(\sin z) = \cos z$

(ii) $\frac{d}{dz}(\cos z) = -\sin z$

(iii) $\frac{d}{dz}(\tan z) = \sec^2 z$

(iv) $\frac{d}{dz}(\cot z) = \cos ec^2 z$

(v) $\frac{d}{dz}(\sec z) = \sec z \tan z$

(vi) $\frac{d}{dz}(\cos ecz) = -\cos ecz \cot z$

Hyperbolic functions :

For real x , we define $\sinh x = \frac{e^x - e^{-x}}{2}, \cosh x = \frac{e^x + e^{-x}}{2}$

$$\sinh z = \frac{e^z - e^{-z}}{2} . \cosh z = \frac{e^z + e^{-z}}{2}$$

Relation ship between complex, trigonometric and hyperbolic function.

Using the definition of hyperbolic functions we get

(a) $\cosh(iz) = \cos z$
(b) $\sinh(iz) = i \sin z$
(c) $\cos(iz) = \cosh z$
(d) $\sin(iz) = i \sinh z$
(e) $\tan(iz) = i \tanh z$

Derivates of hyperbolic functions :

From the definitions of hyperbolic functions ,the derivates

(i) $\frac{d}{dz}(\sinh z) = \cosh z$
(ii) $\frac{d}{dz}(\cosh z) = \sinh z$
(iii) $\frac{d}{dz}(\tanh hz) = \sec h^2 z$
(iv) $\frac{d}{dz}(\coth hz) = -\cos ech^2 z$
(v) $\frac{d}{dz}(\coth z) = -\sec hz \tanh z$
(vi) $\frac{d}{dz}(\cos echz) = -\cos echz \coth z$

Applications of elementary functions.

We consider applications of first order differential equations. Its begin with a discussion of exponential growth and decays.

Application 1.

Growth and decay

Since the application in this section deal with function of time, we will denote the independent variable by t. f Q is a function of t, and then Q' will denote the derivative of Q with respect to Exponential Growth and Decay.

One of the most common mathematical models for a physical process is the exponential model, in which it is assumed that the rate of change of a quantity Q at a time t is proportional to its value at that time; thus

$Q' = \alpha Q$ .... (1)

Where α is constant proportionality. The General solution of (1) is

$Q = ce^{\alpha t}$

Then (1) satisfies:

1) Growth exponentially if $\alpha > 0$

2) Decays if $\alpha < 0$

o t, thus $Q' = \frac{dQ}{dt}$

Example:-

1) The number N of bacteria in a culture grew at a rate proportional to N, The value of N was initially 100 and increased to332 in one hour .what was the value of N after $1\frac{1}{2}$ hours.

Solution:- The differential equation to be solved is $\frac{dN}{dt} = k\ N$ ----(1)

Separating the variables, $\frac{dN}{dt} = k\ dt$

Integrating , $\log N = kt + \log c \Rightarrow \frac{N}{c} = e^{kt} \Rightarrow ce^{kt}$ --------(2)

When t=0,we have N=100 so that =c i.e, $N = ce^{kt}$ ------(3)

When t=3600 sec ,N=332 i.e, $332 = 100e^{3600k}$ ------(4)

When $t = \frac{3}{2}$ hours =5400 sec,we have $N = 100e^{5400k}$ ----(5)

From (4) $e^{5400k} = (\frac{332}{100})^{3/3}$, substituting in (5) ,we get

$N = 100(\frac{332}{100})^{3/2} = 605.$

Application 2:

Radio Active Materials or Rate of Decay :

Experimental evidences show that radioactive material decays at a rate, proportional to mass of the material present. According to this model the mass Q(t) of the radioactive material present at a time t satisfies(1), where α is a negative constant whose value for any given material must be determined by experimental observation. It is customary to replace the negative (-ve) constant α by-k, where k is appositive number that we will call the Decay constant of the material.

Thus (1) becomes: $Q' = -KQ$ -----(1)

If the mass of the material present at $t = t_{0\ is\ Q_0}$.Then the mass present at time t is the solution of $Q' = -KQ, Q(t_0) = Q_0$

Then with α=-k, the solution of this initial value problem is

$Q' = Q_0 e^{-k(t-t_0)}$ ……… (2)

Example :-

Uranium disintegrates at a rate proportional to the amount present at any instant .If M_1 *and* M_2 are grams of uranium that are at times T_1 *and* T_2 respectively, find the half-life of uranium.

Solution:- Let the mass of uranium at any rate t be m grams .then the equation of disintegration of uranium is $\frac{dm}{dt} = -um,$ where u is a constant .

Integrating $\log(m) = c - (ut)$------(1)

Initially when t=0,m=M(say) so that $c = \log M$

$\therefore$ (1) becomes $ut = \log M - \log m$----(2)

Also When $t = T_1$, $m = M_1 and\ t = T_2$, $m = M_2$

$\therefore$ From (2) ,we get $uT_1 = \log M - \log M_1$ ------(3)

and $uT_2 = \log M - \log M_2$------(4)

Substracting (3) from (4) ,we get $u(T_2 - T_1) = \log M_1 - \log M_2$

i.e , $u = \dfrac{\log(\frac{M_2}{M_1})}{(T_2 - T_1)}$ -------(5)

let the mass reduce to half its initial value in time T.

i.e, when $t = T, m = \frac{1}{2}M$

$\therefore$ From(2), we get $uT = \log M - \log(\frac{M}{2}) = \log 2$

Thus , $T = \frac{1}{u}\log 2 = \frac{(T_2 - T_1)}{\log(M_1/M_2)}(by(5))$.

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