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ON GENERALIZED τ–DERIVATIONS OF PRIME RINGS

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ABSTRACT

Let R be a prime ring and σ, τ be endomorphisms of R. In the present paper we study the commutativity of prime ring R admitting a generalized (σ, τ) - derivation F satisfying one of the following properties: (i) F(xy) - d(x)d(y) = 0, (ii) F(xy) + d(x)d(y) = 0, (iii) F[x, y] - [x, y] ϵ Z(R), (iv) F[x, y] + [x, y] ϵ Z(R), (v) F(x o y) – x o y ϵ Z(R) and (vi) F(x o y) + x o y ϵ Z(R), for all x, y in some appropriate subset of R .

Keywords: Prime ring, Generalized (σ, τ) - derivation, (σ, τ) - derivation, Ideal, Right ideal.

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I. INTRODUCTION

Throughout the paper R will denote an associative ring with centre Z(R). A ring R is said to be prime (resp. semiprime) if aRb = (0) implies that either a = 0 or b = 0 (resp. aRa = (0) implies that a = 0). For any x, y ϵ R we shall write [x, y] = xy - yx and x o y = xy + yx. An additive mapping d : R → R is called a derivation if d(xy) = d(x)y + xd(y) for all x, y ϵ R. In 1991, Bresar [6] introduced the notion of generalized derivation. An additive mapping F : R → R is called a generalized derivation on R if there exists a derivation d : R → R such that F(xy) = F(x)y + xd(y) for all x, y ϵ R. Let (σ, τ) be endomorphisms of R. An additive mapping d : R → R is called a (σ, τ)-derivation if d(xy) = d(x) (y) + (x)d(y) for all x, y ϵ R. An additive mapping F : R → R is called a generalized (σ, τ) - derivation on R if there exists a (σ, τ) - derivation d : R → R such that F(xy) = F(x)σ(y) + τ(x)d(y) for all x, y ϵ R. We shall call a generalized (σ, I) - derivation as a generalized σ-derivation where I is the identity automorphism of R. Similarly a generalized (I, τ)-derivation as a generalized τ-derivation.

There has been considerable interest in commuting and centralizing maps in prime and semiprime rings (see for examples Bell and Martindale [5], Bell H. E. [4] and Brear [7] where further references can be found). Daif and Bell [8] proved that if a semiprime ring R admits a derivation d such that either d([x, y]) + [x, y] = 0 or d([x, y]) - [x, y] = 0, holds for all x, y in a nonzero ideal I of R, then R is necessarily commutative. Hongan [9] generalized the above result considering R satisfying the conditions d([x, y]) + [x, y] ϵ Z(R) and d([x, y]) - [x, y] ϵ Z(R), for all x, y ϵ I. Being inspired by the result Ashraf et. al. [1] have studied the situation with derivation replaced by generalized derivation. Later Ali et. al. [2] explored the commutativity of a prime ring admitting a generalized derivation. Motivated by the above observations, we explore the commutativity of a prime ring admitting a generalized τ-derivation F satisfying any one of the following conditions:

(i) F([x, y]) - [x, y] ϵ Z(R)

(ii) F([x, y]) + [x, y] ϵ Z(R)

(iii) F(x o y) - (x y) ϵ Z(R) and

(iv) F(x o y) + (x o y) ϵ Z(R), for all x, y in some appropriate subsets of R.

Throughout the paper, we make some extensive use of the basic commutator and anti-commutators idetities [x, yz] = y[x, z] + [x, y]z, [xy, z] = [x, z]y + x[y, z] ,(x o yz) = y(x o z) + [x, y]z = (x o y)z - y[x, z] and (xy o z) = (x o z)y + x[y, z] = x(y o z) - [x, z]y.

II. MAIN RESULTS

We begin with the following known result which will be used frequently to prove our theorems.

Lemma 2.1 ([10]). If a prime ring R contains a nonzero commutative right ideal I, then R is commutative.

Theorem 2.1 Let R is a prime ring and I a nonzero ideal of R. Suppose that τ is an automorphism of R. If R admits a generalized τ-derivation F associated with a nonzero τ-derivation d, such that F(xy) = d(x)d(y), for all x, y ϵ I, then R is commutative.

Proof: By assumption, we have

$$F(xy) = F(x)y + \tau(x)d(y) = d(x)d(y), \text{ for all } x, y \in I. \quad (2.1)$$

Replacing y by y^2 in (2.1), we obtain

$$F(x)y^2 + \tau(x)d(y)y + \tau(x)\tau(y)d(y) = d(x)d(y)y + d(x)\tau(y)d(y), \text{ for all } x, y \in I.$$

Using (2.1), the relation reduces to

$$\tau(x)\tau(y)d(y) = d(x)\tau(y)d(y), \text{ for all } x, y \in I. \quad (2.2)$$

Replacing x by zx in (2.2) and using (2.2), we have

$$d(z)\tau(x)\tau(y)d(y) = 0, \text{ for all } x, y, z \in I. \quad (2.3)$$

Replace x by xr in (2.3) to get $d(z)\tau(x)\tau(r)\tau(y)d(y) = 0$, for all x, y, z ϵ I and r ϵ R i.e. $d(z)\tau(x)R\tau(y)d(y) = (0)$, for all x, y, z ϵ I. The primeness of R yields that $d(z)\tau(x) = 0$ or $\tau(y)d(y) = 0$, for all x, y, z ϵ I. If $d(z)\tau(x) = 0$, for all x, z ϵ I, then $d(z)R\tau(x) = (0)$, for all z ϵ I. Since I is a nonzero ideal of R and primeness of R yields that d(z) = 0, for all z ϵ I. This implies that $d(zr) = \tau(z)d(r) = 0$, for all z ϵ I and r ϵ R i.e. $\tau(z)Rd(r) = (0)$. Again I is a nonzero ideal of R and primeness of R yields that d(r) = 0, for all r ϵ R, which is a contradiction. On the other hand if $\tau(y)d(y) = 0$, for all y ϵ I, then linearization gives

$$\tau(x)d(y) + \tau(y)d(x) = 0, \text{ for all } x, y \in I \quad (2.4)$$

Replace y by zy to get

$$\tau(x)d(z)y + \tau(x)\tau(z)d(y) + \tau(z)\tau(y)d(x) = 0, \text{ for all } x, y \in I. \quad (2.5)$$

Comparing (2.4) and (2.5), we get

$$\tau(x)d(z)y + \tau(x)\tau(z)d(y) - \tau(z)\tau(x)d(y) = 0, \text{ for all } x, y, z \in I. \quad (2.6)$$

Replace y by yr, we obtain

$$\tau(x)d(z)yr + [\tau(x), \tau(z)]d(y)r + [\tau(x), \tau(z)]\tau(y)d(r) = 0, \text{ for all } x, y, z \in I, r \in R. \quad (2.7)$$

Application of (2.6) in (2.7) yields that $[\tau(x), \tau(z)]\tau(y)d(r) = 0$, for all x, y, z ϵ I and r ϵ R. Now replace y by ys to get $[\tau(x), \tau(z)]\tau(y)\tau(s)d(r) = 0$, for all x, y, z ϵ I and r, s ϵ R i.e. $[\tau(x), \tau(z)]\tau(y)Rd(r) = (0)$, for all x, y, z ϵ I and r ϵ R. Thus primeness of R implies that either $[\tau(x), \tau(z)]\tau(y) = 0$ or d(r) = 0, for all x, y, z ϵ I and r ϵ R. Hence, [x, z]y = 0 implies [x, z] = 0, for all x, z ϵ I. Since I is a nonzero ideal of a prime ring R, then R is commutative by Lemma 2.1.

Theorem 2.2 Let R be a prime ring and I be a nonzero ideal of R. Suppose that τ is an automorphism of R. If R admits a generalized τ-derivation F with associated nonzero τ-derivation d such that $F(xy) + d(x)d(y) = 0$, for all x, y ϵ I, then R is commutative.

Proof: If R satisfies the assumption $F(xy) + d(x)d(y) = 0$, for all x, y ϵ I, then generalized derivation (-F) also satisfies $(-F)(xy) - d(x)d(y) = 0$, for all x, y ϵ I and hence proof follows from Theorem 2.1.

Theorem 2.3 Let R be a prime ring and I be a nonzero right ideal of R. Suppose that τ is an automorphism of R and R admits a generalized τ-derivation F with associated nonzero τ-derivation d such that $d(Z(R)) \neq (0)$. If $F([x, y]) - [x, y] \in Z(R)$ for all x, y ϵ I, then R is commutative.

Proof: Since $d(Z(R)) \neq (0)$, there exists c ϵ Z(R) such that $d(c) \neq 0$. Thus $d(c) \in Z(R)$. By assumption, we have

$$F([x, y]) - [x, y] \in Z(R), \text{ for all } x, y \in I. \tag{2.8}$$

Replacing y by yc in (2.8), we have

$$(F([x, y]) - [x, y])c + [\tau(x), \tau(y)]d(c) \in Z(R), \text{ for all } x, y \in I. \tag{2.9}$$

This implies that $[[\tau(x), \tau(y)]d(c), r] = 0$, for all x, y ϵ I and r ϵ R. That is, $[[\tau (x), \tau(y)], r]d(c) = 0$, for all x, y ϵ I and r ϵ R. Since R is prime and $d(c) \neq 0$, we find that $[[\tau(x), \tau(y)], r] = 0$, for all x, y ϵ I and r ϵ R. Replacing y by yx, we have

$$[\tau(x), \tau(y)][\tau(x), r] + [[\tau(x), \tau(y)], r]\tau(x) = 0, \text{ for all } x, y \in I, r \in R \tag{2.10}$$

In view of the fact that $[[\tau(x), \tau(y)], r] = 0$, relation (2.10) yields that $[\tau(x), \tau(y)][\tau(x), r] = 0$, for all x, y ϵ I and r ϵ R. Replace r by ry to obtain $[\tau(x), \tau(y)]r[\tau(x), \tau(y)] = 0$, for all x, y ϵ I and r ϵ R, that is, $[\tau(x), \tau(y)]R[\tau(x), \tau(y)] = 0$, for all x, y ϵ I. The primeness of R yields that $[\tau(x), \tau(y)] = 0$, for all x, y ϵ I. Which implies that $[x, y] = 0$, for all x, y ϵ I. So I is a commutative right ideal. Hence application of Lemma 2.1 completes the proof of the theorem.

Theorem 2.4 Let R be a prime ring and I be a nonzero right ideal of R. Suppose that τ is an automorphism of R and R admits a generalized τ-derivation F with associated nonzero τ-derivation d such that $d(Z(R)) \neq (0)$. If $F([x, y]) + [x, y] \in Z(R)$, for all x, y ϵ I, then R is commutative.

Proof: If R satisfies the assumption $F([x, y]) + [x, y] \in Z(R)$, for all x, y ϵ I, then generalized τ-derivation (-F) also satisfies $(-F)([x, y]) - [x, y] \in Z(R)$, for all x, y ϵ I and hence proof follows from Theorem 2.3.

Theorem 2.5 Let R is a prime ring and I a nonzero right ideal of R. Suppose that τ is an automorphism of R and R admits a generalized τ-derivation F with associated nonzero τ-derivation d such that $d(Z(R)) \neq (0)$. If $F(x \circ y) - (x \circ y) \in Z(R)$, for all x, y ϵ I, then R is commutative.

Proof: By assumption, we have

$$F(x \circ y) - (x \circ y) \in Z(R), \text{ for all } x, y \in I \tag{2.11}$$

Since $d(Z(R)) \neq (0)$, there exists $c \in Z(R)$ such that $d(c) \neq 0$ and $d(c) \in Z(R)$. Replacing y by yc in (2.11), we have

$$(F(x \text{ o } y) - x \text{ o } y)c + (\tau(x) \text{ o } \tau(y))d(c) \in Z(R), \text{ for all } x, y \in I. \tag{2.12}$$

That is, $(\tau(x) \text{ o } \tau(y))d(c) \in Z(R)$, for all $x, y \in I$. Since $d(c) \neq 0$ and R is prime, it follows that $(\tau(x) \text{ o } \tau(y)) \in Z(R)$, for all $x, y \in I$. Thus $[(\tau(x) \text{ o } \tau(y)), r] = 0$, for all $x, y \in I$. Substitute yx for y, we obtain $(\tau(x) \text{ o } \tau(y))[\tau(x), r] = 0$, for all $x, y \in I$ and $r \in R$. Replacing r by sr, we find that $(\tau(x) \text{ o } \tau(y))R[\tau(x), r] = (0)$, for all $x, y \in I$ and $r \in R$. Now primeness of R, for each $x \in I$ gives either $(\tau(x) \text{ o } \tau(y)) = 0$ or $[r, \tau(x)] = 0$, for all $y \in I$ and $r \in R$. Let $I_1 = \{x \in I : \tau(x) \text{ o } \tau(y) = 0, \text{ for all } y \in I\}$ and $I_2 = \{ x \in I : [r, \tau(x)] = 0, \text{ for all } r \in R\}$. Then I_1 and I_2 are both additive subgroups of I whose union is I. Hence either $I_1 = I$ or $I_2 = I$. If $I_1 = I$, then $(x \text{ o } y) = 0$ for all $x, y \in I$. Now replace y by yz, to get $(x \text{ o } yz) = (x \text{ o } y)z - y[x, z] = 0$, which gives $y[x, z] = 0$, for all $x, y, z \in I$. Thus $yR[x, z] = 0$, for all $x, y, z \in I$. Since I is a non zero right ideal of R, primeness of R yields that $[x, z] = 0$, for all $x, z \in I$. Thus I is commutative and the application of Lemma 2.1 gives that R is commutative. On the other hand if $I_2 = I$, then $[r, \tau(x)] = 0$, for all $r \in R$ and $x \in I$. Substitute xs for x, we get $\tau(x)[r, \tau(s)] = 0$, for all $x \in I$ and $r, s \in R$. Since I is a non zero right ideal of R, $[r, \tau(s)] = 0$, for all $r, s \in R$. Hence in both the case R is commutative.

Using the same techniques with necessary variations, we get the following:

Theorem 2.6 Let R be a prime ring and I be a nonzero right ideal of R. Suppose that R admits a generalized τ-derivation F associated with nonzero τ-derivation d such that $d(Z(R)) \neq (0)$. If $F(x \text{ o } y) + (x \text{ o } y) \in Z(R)$, for all $x, y \in I$, then R is commutative.

The following example demonstrates that the above results do not hold for arbitrary rings.

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