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STRONG DOMINATING χ- COLOR NUMBER ON SUM AND CARTESIAN PRODUCT OF GRAPHS

T.Ramachandran[1], D.Udayakumar[2], A.Naseer Ahmed[3]

[1]*Department of Mathematics, M.V.M Government Arts College for Women, Dindigul, (India)*

[2]*Department of Mathematics, Government Arts College, Karur, (India)*

[3] *Department of Mathematics, K.S.R College of Arts and Science, Thiruchengode, (India)*

ABSTRACT

Let G = (V, E) be a graph. We define strong dominating χ- color number of a graph G as the maximum number of color classes which are strong dominating sets of G, and is denoted by $sd_\chi(G)$, where the maximum is taken over all χ -coloring of G. This paper determines the exact values of strong dominating χ- color number of sum and Cartesian product of graphs.

Keywords: Dominating χ-color number, Fan graph, Grid graph, Helm graph, Ladder graph, Prism graph.Stacked Book graph, Strong dominating χ -color number,

1. INTRODUCTION

In this paper, we consider finite, connected, undirected and simple graph $G = (V(G), E(G))$ with vertex set $V = V(G)$ and edge set $E = E(G)$. The number of vertices $|V(G)|$ of a graph G is called the order of G and the number of edges $|E(G)|$ of a graph G is called the size of G. The order and size is denoted by n and m respectively. In graph theory, coloring and dominating are two important areas which have been extensively studied. The fundamental parameter in the theory of graph coloring is the chromatic number $\chi(G)$ of a graph G which is defined to be the minimum number of colors required to color the vertices of G in such a way that no two adjacent vertices receive the same color. If $\chi(G) = k$, we say that G is $k -$ chromatic [1]. For any vertex $v \in V(G)$, the open neighborhood of v is the set $N(v) = \{u | uv \in E(G)\}$ and the closed neighborhood is the set $N[v] = N(v) \cup \{v\}$. Similarly, for any set $S \subseteq V(G)$, $N(S) = \cup_{v \in S} N(v) - S$ and $N[S] = N(S) \cup S$. A set S is a dominating set if $N[S] = V(G)$.The minimum cardinality of a dominating set of G is denoted by $\gamma(G)$[4]. A set $D \subseteq V$ is a dominating set of G, if for every vertex $x \in V - D$ there is a vertex $y \in D$ with $xy \in E$ and D is said to be strong dominating set of G, if it satisfies the additional condition $deg(x) \leq deg(y)$ [2]. The strong domination number $\gamma_{st}(G)$ is defined as the minimum cardinality of a strong dominating set. A set $S \subseteq V$ is called weak dominating set of G if for every vertex $u \in V - S$, there exists vertex $v \in S$ such that $uv \in E$ and $deg(u) \geq deg(v)$. The weak domination number $\gamma_w(G)$ is defined as the minimum cardinality of a weak dominating set and was introduced by Sampathkumar and Pushpa Latha [3]. The number of maximum degree vertices of G is denoted by $n_\Delta(G)$ and the number of minimum degree vertices of G is denoted by $n_\delta(G)$. All the basic notations of graph theory are referred in [4].

II. TERMINOLOGIES

We start with more formal definition of dominating-χ-color number of G [5]. Let G be a graph with $\chi(G) = k$. Let $C = V_1, V_2, ..., V_k$ be a k-coloring of G. Let d_C denotes the number of color classes in C which are dominating sets of G. Then $d_\chi(G) = \max_C d_C$ where the maximum is taken over all the k-colorings of G, is called the dominating-χ-color number of G. Instead of dominating set in the definition of dominating-χ-color number, if we consider strong dominating set, then it is called strong dominating χ-color number, if we consider weak dominating set, then it is called weak dominating-χ-color number[6]. A grid graph $G_{m,n}$ [7], Helm graph H_n [8], Fan graph $F_{m,n}$ [9], Ladder graph L_n [9], Prism graph Y_n [9] and stacked book graph $B_{m,n}$ [10], are some of the sum and Cartesian product of graphs.

III. STRONG DOMINATING -χ- COLOR NUMBER ON SUM OF GRAPHS

The new parameter, strong dominating χ- color number and weak dominating -χ- color number defined by us in [11]. The following are some of our observed propositions,

Proposition:

1. For any graph G, $1 \le sd_\chi(G) \le d_\chi(G)$
2. For any graph G, $1 \le wd_\chi(G) \le d_\chi(G)$
3. For any star graph $K_{1,n}$, $sd_\chi(K_{1,n}) = 1$
4. For any complete graph K_n; $sd_\chi(K_n) = n$
5. For any cycle C_n; $sd_\chi(C_n) = \begin{cases} 3, & \text{if } n \equiv 3 (\text{mod} 6) \\ 2, & \text{otherwise} \end{cases}$
6. If W_n, is any wheel with n > 4, then $sd_\chi(W_n) = 1$.
7. For any path P_n, $sd_\chi(P_n) = \begin{cases} 1 & \text{if } n = 3 \\ 2 & \text{if } n \neq 3 \end{cases}$

Theorem 3.1:

For any graph G, $sd_\chi(G) \le n_\Delta(G)$ and $wd_\chi(G) \le n_\delta(G)$.

Proof

If the graph G has the number of $n_\Delta(G)$ vertices which has maximum degree, then by the definition of strong dominating $\chi -$ color number $sd_\chi(G)$, it is clear that strong dominating set should contain atleast one vertex that has maximum degree Δ. Since $n_\Delta(G)$ is the number of vertices which has maximum degree of graph. It is possible to have atmost $n_\Delta(G)$ strong dominating set. Therefore $sd_\chi(G) \le n_\Delta(G)$. Similarly, we can easily prove $wd_\chi(G) \le n_\delta(G)$.

There is an open problem mentioned in [11], that characterize the class of graphs with $sd_\chi(G) = 1$ and $wd_\chi(G) = 1$. The following theorem gives the necessary condition for that.

Theorem 3.2:

a. If G has an isolated vertex, then $sd_\chi(G) = 1$

b. If G has an isolated vertex, then $wd_\chi(G) = 1$

c. If G has $n_\Delta(G) = 1$, then $sd_\chi(G) = 1$

d. If G has $n_\delta(G) = 1$, then $wd_\chi(G) = 1$

Proof

If the graph G has an isolated vertex, v, then the vertex v is dominated by only one vertex, say u. So it is clear that the color class which has a vertex u, is the dominating set. There fore $d_\chi(G) = 1$.

a) By proposition 1, $sd_\chi(G) \leq d_\chi(G)$, then $sd_\chi(G) = 1$.

b) By proposition 2, $wd_\chi(G) \leq d_\chi(G)$, then $wd_\chi(G) = 1$.

c) If the graph G has the number vertices which has maximum degree, $n_\Delta(G) = 1$. By the definition of strong dominating $\chi -$ color number $sd_\chi(G)$, it is very clear that strong dominating set should contain atleast one vertex that has maximum degree Δ. Since $n_\Delta(G) = 1$, It is possible to have only one strong dominating set. Therefore $sd_\chi(G) = 1$.

d) If the graph G has the number vertices which has minimum degree, $n_\delta(G) = 1$. By the definition of weak dominating $\chi -$color number $wd_\chi(G)$, it is very clear that weak dominating set should contain atleast one vertex that has minimum degree δ. Since $n_\delta(G) = 1$, it is possible to have only one weak dominating set. Therefore $wd_\chi(G) = 1$.

We proved the following Lemma 3.2 in [11].

Lemma 3.2

Let G_1 and G_2 be two graphs. The sum G_1+G_2 is the graph having vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$ together with all the edges joining the points of $V(G_1)$ to the points of $V(G_2)$. Then

1. $sd_\chi(G_1+G_2) = sd_\chi(G_1) + sd_\chi(G_2)$ if $|V(G_1)| - \Delta(G_1) = |V(G_2)| - \Delta(G_2)$.

2. $sd_\chi(G_1 + G_2) = sd_\chi(G_1)$ if $|V(G_1)| - \Delta(G_1) < |V(G_2)| - \Delta(G_2)$.

Instead of discussing on both strong and weak, we are going to prove only strong dominating χ- color number in the following theorems to avoid the repeating arguments.

Theorem 3.3: Let G be a Fan graph $F_{m,n} (n \neq 3)$. Then $sd_\chi(G) = \begin{cases} 1 & \text{if } m < n-2 \\ 2 & \text{if } n-2 < m \\ 3 & \text{if } m = n-2 \end{cases}$

Proof

By the definition of Fan graph, $F_{m,n} = \overline{K_m} + P_n$

Now by Lemma – 3.2, it is clear that,

If $m = n - 2$, $for\ n \neq 3$ then $sd_\chi(\overline{K_m} + P_n) = sd_\chi(\overline{K_m}) + sd_\chi(P_n) = 1 + 2 = 3$

If $m < n - 2$, $for\ n \neq 3$ then $sd_\chi(\overline{K_m} + P_n) = sd_\chi(\overline{K_m}) = 1$

If $m > n - 2$, $for\ n \neq 3$ then $sd_\chi(\overline{K_m} + P_n) = sd_\chi(P_n) = 2$

Theorem 3.4: Let G be a Fan graph $F_{m,3}$. Then $sd_\chi(G) = \begin{cases} 1 & \text{if } m \neq 1 \\ 2 & \text{if } m = 1 \end{cases}$

Proof

By the definition of Fan graph, $F_{m,3} = \overline{K_m} + P_3$

Now by Lemma – 3.2,

When $m = 1$, that is, $for\ n = 3, m = n - 2 = 1$ then,

$sd_\chi(\overline{K_m} + P_3) = sd_\chi(\overline{K_m}) + sd_\chi(P_3) = 1 + 1 = 2$

When $m \neq 1$, that is, $for\ n = 3\ , m \neq n - 2 = 1$ then,

$sd_\chi(\overline{K_m} + P_3) = sd_\chi(\overline{K_m}) = 1$ or $sd_\chi(\overline{K_m} + P_3) = sd_\chi(P_3) = 1$

IV. STRONG DOMINATING -χ- COLOR NUMBER ON CARTESIAN PRODUCT OF GRAPHS

We proved the following Lemma 4.1 in [11].

Lemma 4.1

Let $G_1 \times G_2$ be the Cartesian product of two graphs G_1 and G_2. If $\chi(G_1) > \chi(G_2)$, then $sd_\chi(G_1 \times G_2) = \chi(G_1 \times G_2)$ if and only if $sd_\chi(G_1) = \chi(G_1)$.

As an application of the Lemma 4.1, we are going to prove the following theorems.

Theorem 4.1: Let $B_{m,n}$ be the stacked book graph, then $sd_\chi(B_{m,n}) = \begin{cases} 1 & \text{if } n = 3 \\ 2 & \text{if } n \neq 3 \end{cases}$

Proof

For all m, $\chi(S_{m+1}) = 1$ and for all n, $\chi(P_n) = 2$.

And for all m, $sd_\chi(S_{m+1}) = 1$ and for $n = 3$, $sd_\chi(P_n) = 1$.

By the definition of stacked book graph, $B_{m,n} = S_{m+1} \times P_n$ where S_m is a star graph and P_n is the Path graph on n nodes, it is clear that if $n = 3$, then $n_\Delta(B_{m,n}) = 1$ and if $n \neq 3$, then $n_\Delta(B_{m,n}) > 1$.

If $n = 3$, then $n_\Delta(B_{m,n}) = 1$, that implies by the theorem 3.2(c), $sd_\chi(B_{m,n}) = 1$.

If $n \neq 3$, then $n_\Delta(B_{m,n}) > 1$. But the chromatic number is 2 and each color class have atleast one vertex that has maximum degree Δ. So, both the color classes are strong dominating sets of $B_{m,n}$. There fore $sd_\chi(B_{m,n}) = 2$

Theorem 4.2: Let $G_{m,n}$ be a Grid graph.

Then $sd_\chi(G_{m,n}) = \begin{cases} 1 & \text{if } (n = 3 \text{ or } 1) \text{ and } (m = 1 \text{ or } 3) \\ 2 & \text{otherwise} \end{cases}$

Proof

Let G be a Grid graph $G_{m,n}$. By definition of Grid graph, $G_{m,n}$ is Cartesian product $P_m \times P_n$ of path graphs on m and n vertices.

If either $n = 3$ or 1 and $m = 1$ or 3, then $n_\Delta(G_{m,n}) = 1$, that implies by the theorem 3.2(c), $sd_\chi(B_{m,n}) = 1$. Otherwise, $n_\Delta(B_{m,n}) > 1$. But the chromatic number is 2 and each color class have atleast one vertex that has maximum degree Δ. So, both the color classes are strong dominating sets of $B_{m,n}$. There fore $sd_\chi(B_{m,n}) = 2$

Theorem 4.3: Let G be a n −ladder graph. Then $sd_\chi(G) = 2$

Proof

Since n −ladder graph is $2 \times n$ grid graph $G_{2,n}$. By Theorem 4.1, $sd_\chi(G) = 2$

Theorem 4.4: Let G be a Helm graph H_n. Then $sd_\chi(G) = \begin{cases} 2 & \text{if } n = 4 \\ 1 & \text{otherwise} \end{cases}$

Proof

For n = 4, H_n is the graph obtained from a 4 – Wheel graph by adjoining a pendant edge at each node of the cycle. And it is clear that, each vertex in the wheel has degree 4. Hence
$sd_\chi(G) = 2$.

For n ≠ 4, H_n is the graph obtained from a n – Wheel graph by adjoining a pendant edge at each node of the cycle. And it is clear that, centre vertex in the wheel has degree n and the outer vertices of the cycle of the wheel have degree 4. Hence $sd_\chi(G) = 1$

Theorem 4.5: Let Y_n be the Prism graph, then$sd_\chi(Y_n)$= $\begin{cases} 3, & \text{if } n \equiv 3(\text{mod}6) \\ 2, & \text{otherwise} \end{cases}$

Proof

For $n \geq 3$, prism Y_n is the Cartesian product $C_n \times k_2$ where C_n is a cycle on n-vertices and K_2 is the complete graph on 2-vertices.

For all n, $\chi(Y_n)$ =$\chi(C_n)$ and for all n, $sd_\chi(Y_n) = sd_\chi(C_n)$.

Thus $sd_\chi(Y_n)$= $\begin{cases} 3, & \text{if } n \equiv 3(\text{mod}6) \\ 2, & \text{otherwise} \end{cases}$

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