ASYMPTOTIC ESTIMATES OF LEVEL CROSSINGS OF A RANDOM ALGEBRAIC POLYNOMIAL

[1]Dipty Rani Dhal , [2]Dr.P.K.Mishra

[1,2] Department of Mathematics, ITER, SOA, BBSR, Odisha, (India)

ABSTRACT

This paper provides asymptotic estimates for the expected number of real zeros and k-level crossings of a random algebraic polynomial of the form

$a0(n\text{-}1\ c_{0})^{1/2} + a1(n\text{-}1\ c_{1})^{1/2} x + a2(n\text{-}1\ c_{2})^{1/2} x^2 + \ldots + a_{n-1}(n\text{-}1\ c_{n-1})^{1/2} x^{n-1}$,

where a_J *(J=0, 1, 2, ... , n-1) are independent standard normal random variables and k is constant independent of x . It is shown that these asymptotic estimates are much greater than those for algebraic polynomials of the form* $a0 + a1\,x + a2\,x^2 + \ldots + a_{n-1}x^{n-1}$.

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I INTRODUCTION

Let (Ω , A , Pr) be a fixed probability space and let $\{ a_J(\omega) \}_{j=0}^{n-1}$ be a sequence of independent random variables defined on Ω . The random algebraic polynomial was introduced in the pioneer work of Littlewood and Offord [5] and [6] as

$$Q(x) \equiv Q_n(x, \omega) = \sum_{j=0}^{n-1} aj(\omega)\ x^j$$

and since then has been greatly studied . Denote by $N_K(\alpha , \beta)$ the number of real roots of the equation p(x)= K in the interval (α , β) and by $EN_k(\alpha , \beta)$ its expected value. In particular it is shown (for example see kac [4] or wilkins [8]) that if the coefficients are assumed to have a standard normal distribution and n is sufficiently large , $EN_0(-\infty,\infty) \sim (2/\pi) \log n$. Recently (see farahmand [3]) , it was shown that this asymptotic value remains valid for $EN_k(-\infty,\infty)$ as long as k is bounded. For k large such that $k^2/n \to 0$ as $n \to \infty$, $EN_k(-\infty,\infty)$ asymptotically reduced to $(1/\pi) \log(n/k^2)$ in (-1, 1) while it remains the same as for k=0 in $(-\infty, -1) \cup (1, \infty)$. In contrast, a random trigonometric polynomial

$$T(x) \equiv T_n(x, \omega) = \sum_{j=0}^{n-1} a_j(\omega) \cos j\theta \text{ has more roots in } (0, 2\pi).$$

In fact $EN_k(0, 2\pi) \sim 2n/\sqrt{3}$ for $k = o(\sqrt{n})$. Motivated by the interesting results obtained in Littlewood and offord [6] we considered the case when the coefficients $a_j(\omega)$ have variance $1/j!$. It is presumably the case, possibly under some mild conditions for K and for n sufficiently large, that $EN_k(-\infty,\infty)$ Is $o(\sqrt{n})$. This author ,however ,was unable to make any substantial progress towards this conjecture. Instead in this paper we study the polynomials

$$P(x) \equiv P_n(x, \omega) = \sum_{j=0}^{n-1} a_j(\omega) \binom{n-1}{j}^{1/2} x^j$$

This is indeed, the same as saying that the j th coefficient of Q(x) has variance $\binom{n-1}{j}$. Besides the mathematical interest, as reported in Edelman and Kostlan [2 ,page 11] these polynomials have some relationship with physics [1]. We prove the following theorems. Theorem 1 was known to Edelman and Kostlan, however to be complete we give a proof here. Theorem3, and its comparison with theorem1, shows that for p(x) there are as many extrema as the number of zero crossings. Therefore unlike Q(x), all the oscillations of p(x) are between two zero crossings, asymptotically in expectation.

THEOREM 1. *If the coefficients a_j of p(x) are independent standard normal random variables, then*

$$EN_0(-\infty,\infty) = \sqrt{n-1}$$

THEOREM 2. *Denote $\gamma(n) = (2n-1)!!/(2n)!!$ and assume $K^2\gamma(n) \to 0$. With the same assumptions as in theorem 1 for the coefficients of p(x), we have $EN_K(-\infty,\infty) \sim \sqrt{n-1}$*

II A FORMULA FOR THE EXPECTED NUMBER OF REAL ROOTS

Let (2.1) $A^2 = \text{var}\{P(X) - K\}$,

(2.2) $B^2 = \text{var}\{P'(x)\}$,

(2.3) $C = \text{cov}[\{P(x) - K\}, P'(x)]$

And $\eta = -CK / A\sqrt{A^2B^2 - C^2}$

Then by using the expected number of level crossings given by Cramer and Leadbetter [1 , page 285] for our equation P(x) – k = 0 , we can obtain

$$EN_K(\alpha , \beta) = \int_{\alpha}^{\beta} \frac{B\sqrt{1 - C^2 / A^2B^2}}{A} \phi\left(- \frac{K}{A}\right)[2\phi(\eta) + \eta\{2\phi(\eta) - 1\}]\, dx$$

Where as usual

$$\phi(t) = (2\Pi)^{-1/2} \int_{-\infty}^{t} \exp\left(- y^2 / 2\right) dy$$

and $\phi(t) = (2\Pi)^{-1/2} \exp(-t^2 / 2)\, dy$

Let $\Delta^2 = A^2B^2 - C^2$ and $\operatorname{erf}(x) = \int_0^x \exp(-t^2)\, dt$; then we can write the extension of a formula obtained by Rice [7] for the case of k=0 as

(2.4) $EN_K(\alpha , \beta) = I_1(\alpha , \beta) + I_2(\alpha , \beta)$

(2.5) $$I_1(\alpha , \beta) = \int_{\beta}^{\alpha} \frac{\Delta}{\Pi A^2} \exp\left(- \frac{B^2K^2}{2\Delta^2}\right) dx$$

(2.6) $$I_2(\alpha , \beta) = \int_{\beta}^{\alpha} \frac{\sqrt{2}\, KC}{\Pi A^3} \exp\left(- \frac{K^2}{2A^2}\right) erf\left(\frac{KC}{\sqrt{2}\, A\Delta}\right) dx$$

PROOF OF THEOREM 1

We need the following , where (3.2) and (3.3) are obtained by differentiation of (3.1) and

(3.1) $$A^2 = \sum_{J=0}^{n-1} \binom{n-1}{j} x^{2J} = \left(x^2+1\right)^{n-1}$$

(3.2) $$B^2 = \sum_{j=0}^{n-1} \binom{n-1}{j} x^{2j-2} = (n-1)(x^2+1)^{n-3}\left(nx^2 - x^2 + 1\right)$$

(3.3) $$C = \sum_{J=0}^{n-1} j \binom{n-1}{j} x^{2j-1} = (n-1)x\left(x^2+1\right)^{n-2}$$

Therefore , since from (3.1) – (3.3)

(3.4) $\Delta^2 = A^2B^2 - C^2 = (n-1)(x^2+1)^{2n-4}$ from (2.4) - (2.6) we obtain

$$EN_0(0, \infty) = \Pi^{-1} \int_0^\infty \frac{\Delta}{A^2} dx$$

(3.5) $$= \frac{\sqrt{n-1}}{\Pi} \int_0^\infty \frac{dx}{x^2+1} = \frac{\sqrt{n-1}}{2}$$

This gives the proof of theorem 1 . It is , indeed , interesting to note that from(3.5)

$$\frac{\sqrt{n-1}}{\Pi\left(x^2+1\right)}$$

is the destiny function of the number of real zeros of P(X) ; see also [2 , page 12] . Obtaining a closed form of the above density function is uncommon . An asymptotic result , for most cases , is the best that can be achieved .

PROOF OF THEOREM 2

Because | erf u | < $\sqrt{\Pi}/2$, it follows that from (2.6), (3.1),and (3.3) that

$$0 \leq I_2(-\infty,\infty) \leq \frac{|K|}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} \frac{C}{A^3} \exp\left(-\frac{K^2}{2A^2}\right) dx$$

$$= \frac{2}{\sqrt{\Pi}} \int_0^{|K|/\sqrt{2}} e^{-v^2} dv$$

if v = $|K|/(A\sqrt{2})$. Therefore, because erf u $\leq$ u when u $\geq$ 0

(4.1) $$0 \leq I_2(-\infty,\infty) \leq \frac{2}{\sqrt{\Pi}} erf\left(\frac{K}{\sqrt{2}}\right) \leq \min\left\{1, \frac{\sqrt{2}|K|}{\sqrt{\Pi}}\right\}$$

Moreover , it follows from (2.5), (3.1), (3.2) and (3.4) that

$$I_1(-\infty,\infty) = \frac{\sqrt{n-1}}{\Pi} \int_{-\infty}^{\infty} \frac{e^{-s}}{x^2+1} dx .$$

In which s= $K^2(nx^2-x^2+1) / \{2(x^2+1)^{n-1}\}$. If x = tanθ , we find that

$$I_1(-\infty,\infty) = \frac{2\sqrt{n-1}}{\Pi} \int_0^{\Pi/2} e^{-s} d\theta$$

in which s = $(K^2/2)\{(n-1)\sin^2\theta + \cos^2\theta\}\cos^{2n-4}\theta$. Therefore ,

(4.2) $$I_1(-\infty,\infty) = \frac{2\sqrt{n-1}}{\Pi} \int_0^{\Pi/2} \{1-(1-e^{-s})\} d\theta = \sqrt{n-1} - R ,$$

In which from [4 , page 369] ,

$$0 \le R = \frac{2\sqrt{n-1}}{\Pi} \int_0^{\Pi/2} \left(1 - e^{-s}\right) d\theta$$

$$\le \frac{2\sqrt{n-1}}{\Pi} \int_0^{\Pi/2} s d\theta = K^2 \beta_n$$

$$\beta_n = \frac{\sqrt{n-1}(3n-4)(2n-5)!!}{(2n-2)(2n-4)!!}$$

$$= \sqrt{n-1}\gamma\left(n-2\right)\frac{3n-4}{2n-2}$$

A straight forward algebraic calculation shows that $\beta_{n+1} < \beta_n$ when $n \ge 2$. We conclude that that $0 \le R \le K^2 \beta_2 = K^2/2$ and then that $R = o(\sqrt{n})$ because $K^2 = o\{\gamma^{-1}(n)\}$ and $\gamma(n) \sim \sqrt{n\Pi}$. When this last result is combined with (4.1) , (4.2) and (2.4) , it is clear that theorem 2 is true . In fact , we have actually proved the better result that

$$-\frac{K^2}{2} \le EN_K(-\infty, \infty) - \sqrt{n-1} \le \min\left\{1, \frac{\sqrt{2}|K|}{\sqrt{\Pi}}\right\},$$

from which we can also infer not only Theorem 2 but also that

$$EN_K(-\infty, \infty) = \sqrt{n-1} + O(1)$$

When K is bounded .

EXTREMA

The expected number of extrema of P(x) , denoted by EM(-∞,∞) ,is simply the expected number of real zeros of $P'(x) = \sum_{j=1}^{n-1} a_j j \binom{n-1}{j}^{1/2} x^{j-1}$

Therefore we apply EN_0 (-∞,∞) for P'(x) . To this end , by successive differentiation of (2.2)

We obtain

$$A^2 = \sum_{j=0}^{n-1} j^2 \binom{n-1}{j} x^{2j-2}$$

(5.1) $$= (n-1)(x^2+1)^{n-3}(nx^2 - x^2 + 1), ,$$

$$C = \sum_{j=1}^{n-1} j^2 (j-1) \binom{n-1}{j} x^{2j-3}$$

(5.2) $$= (n-1)(n-2)x(x^2+1)^{n-4}\{(n-1)x^2 + 2\}$$

And $$B^2 = \sum_{j=1}^{n-1} j^2 (j-1)^2 \binom{n-1}{j} x^{2j-4}$$

(5.3) $$= (n-1)(n-2)(x^2+1)^{n-5}\{(n-1)(n-2)x^4 + 4(n-2)x^2 + 2\}$$

therefore , from (5.1) – (5.3) we obtain

(5.4) $$\frac{\Delta}{A^2} = \frac{\sqrt{(n-2)\{(n-1)(n-2)x^4 + 2(n-1)x^2 + 2\}}}{(x^2+1)\{(n-1)x^2+1\}}$$

From (5.4) it then follows that , for all non-zero x

$$\lim_{n \to \infty} \frac{\Delta}{A^2\sqrt{n-2}} = \frac{1}{x^2+1}$$

Then since $\Delta / A^2\sqrt{n-2} \le \sqrt{2} / \left(\sqrt{x^2+1}\right)$ for all real x , the dominated convergent theorem for Lebesgue integrals shows that

$$\int_{-\infty}^{\infty} \frac{\Delta}{A^2\sqrt{n-2}} dx = \int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \Pi$$

theorem 3 is then an intermediate consequence of this result .

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