Differentiation Formulae For $\overline{H}$ -Function

Neelam Pandey and Jai Prakash Patel

Department of Mathematics
Govt. Model Science College, Rewa, (M.P.), India, 486001.
Email :pandeypadra@gmail.com, jpnt.84@gmail.com

Abstract

In the present paper we derive a differentiation formulae for $\overline{H}$-function which was introduced and studied in a paper by Devra and Raithie $[5, 107-113]$. Special cases inclue known and new formulae for special function such as H-function, generalized Wright hypergeometric function.

Mathematics Subject Classification: $33A20, 26A33$
Keywords: $\overline{H}$-function, H-function, Wright hypergeometric function.

1 Introduction

Special Function have contributed a lot to the Science and Engineering. Inayat-Hussain $(1987a)$ introduced generalization form of Fox's H-function which is popularly known as $\overline{H}$-function.
Now $\overline{H}$-function stands on fairly firm footing through the research contribution of various authors like Inayat-Hussain $(1987b)$, Rathie (1993), Gupta & Soni (2005), Chaurasia & singh (2010), Agrawal & Mehar (2012), Marko, Pandey & Sukla (2013).
The $\overline{H}$-Function is defined and represented in the following manner $[2, 10]$.

$$\overline{H}^{m,n}_{p,q}[z] = \overline{H}^{m,n}_{p,q}\left[z\Big|^{(a_j,A_j;\alpha_j)_{1,n}\ ,\ (a_j,A_j)_{n+1,p}}_{(b_j,B_j)_{1,m}\ ,\ (b_j,B_j;\beta_j)_{m+1,q}}\right] = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\theta(s)\ z^s\, ds. \tag{1}$$

where $\theta(s)$ is given by

$$\theta(s) = \frac{\Pi_{j=1}^{m}\Gamma(b_j - B_j s)\ \Pi_{j=1}^{n}\Gamma^{\alpha_j}(1 - a_j + A_j s)}{\Pi_{j=m+1}^{q}\Gamma^{\beta_j}(1 - b_j + B_j s)\ \Pi_{j=n+1}^{p}\Gamma(a_j - A_j s)}. \tag{2}$$

Also
(i) $z \neq 0$.
(ii) $i = \sqrt{(-1)}$.
(iii) m,n,p,q are integers satisfying $0 \leq m \leq q$, $0 \leq n \leq p$.
(iv) L is a suitable contour in the complex plane.
(v) An empty product is interpreted as unity.
(vi) $A_j, j = 1, ..., p; B_j, j = 1, ..., q; \alpha_j, j = 1, ..., n; \beta_j, j = 1, ..., q$ are real positive numbers.
(vii) $a_j, j = 1, ...p$ and $b_j, j = 1, ..., q$ are all complex numbers.

The nature of contour L, sufficient conditions of convergence of defining integral (1) and other details about the $\overline{H}$-Function can be seen in the paper [8].

The behaviour of the $\overline{H}$-function for small values of $|z|$ follows easily from a result given by [12]:

$$\overline{H}^{m,n}_{p,q}[z] = O(|z|^{\alpha});$$

where

$$\alpha = \min_{1\leq j\leq m} Re\left(\frac{b_j}{\alpha_j}\right), |z| \to 0.$$

$$\Omega = \sum_{j=1}^{m} |\beta_j| + \sum_{j=1}^{n} |\alpha_j A_j| - \sum_{j=m+1}^{q} |\beta_j B_j| - \sum_{j=n+1}^{p} A_j > 0 \textit{ and } 0 < |z| < \infty.$$

If $f(x)$ is a linear polynomial in x and D represents $\frac{d}{dx}$ then

$$f(D\,x)\; x^{\alpha} = f(\alpha+1)\; x^{\alpha} \tag{3}$$

The ${}_p\overline{\psi}_q(z)$ function will be defined and represented [8] as follows

$${}_p\overline{\psi}_q\left[{}^{(a_j,A_j;\alpha_j)_{1,p}}_{(b_j,B_j;\beta_j)_{1,q}}; z\right] = \overline{H}^{1,p}_{p,q+1}\left[-z/{}^{(1-a_j,A_j;\alpha_j)_{1,p}}_{(0,1),(1-b_j,B_j;\beta_j)_{1,q}}\cdot\right] \tag{4}$$

The function ${}_p\overline{\psi}_q(z)$ is termed as generalized Wright hypergeometric function because it gives ${}_p\psi_q$ for $A_j = 1, j = 1, ..., p; B_j = 1, j = 1, ..., q$ in it.

An important special case of ${}_p\overline{\psi}_q(z)$ that generalizes several special functions of practical importance is given as

$${}_p\overline{F}_q\left[{}^{(a_j,1;\alpha_j)_{1,p}}_{(b_j,1;\beta_j)_{1,q}}; z\right] = \frac{\prod_{j=1}^{q}\{\Gamma(b_j)\}^{\beta_j}}{\prod_{j=1}^{p}\{\Gamma(a_j)\}^{\alpha_j}} \overline{H}^{1,p}_{p,q+1}\left[-z/{}^{(1-a_j,1;\alpha_j)_{1,p}}_{(0,1),(1-b_j,1;\beta_j)_{1,q}}\cdot\right] \tag{5}$$

The function ${}_p\overline{\psi}_q(z)$ reduces to the well known ${}_p\psi_q(z)$ for $\alpha_j = 1, ..., p; \beta_j = 1, j = 1, ..., q$ in it.

In this paper for the sake of brevity we shall use the following contracted notation for $\overline{H}$-function in (1).

where

A stand for $(a_j, A_j; \alpha_j)_{1,n}$, **B** stand for $(b_j, B_j)_{1,m}$

C stand for $(a_j, A_j)_{n+1,p}$, **D** stand for $(b_j, B_j; \beta_j)_{m+1,q}$.

2 Main Results

$$[(ax^v+b)D-\lambda_1]...[(ax^v+b)D-\lambda_r]\{(ax^v+b)^{\alpha}\overline{H}^{m,n}_{p,q}[z(ax^v+b)^h/{}^{\mathbf{A},\mathbf{C}}_{\mathbf{B},\mathbf{D}}]\}$$

$$= a^r(ax^v+b)^{\alpha}\overline{H}^{m,n+r}_{p+r,q+r}[z(ax^v+b)^h/{}^{(c_j-\alpha,h;1)_{1,r},\mathbf{A},\mathbf{C}}_{\mathbf{B},\mathbf{D},(c_j-\alpha+1,h;1)_{1,r}}], \tag{6}$$

provided $h > 0$ and a, b are complex numbers, r is a positive integers $.\lambda = ac_i$ where a and c_i are not simultaneously zero, and $i = 1, .., r$.

Proof: Taking L.H.S. of (6) and using definition of $\overline{H}$-function (1) and (2), operating under the integral sign using (3),the expression becomes

$$a^r(ax^v+b)^\alpha \frac{1}{2\pi i}\int_L \theta(s)[\alpha+hs+1-c_1]...[\alpha+hs+1-c_r]z^s(ax^v+b)^{hs}ds.$$

Expressing $(\alpha+hs+1-c_j)$as

$$(\alpha+hs+1-c_j) = \frac{\Gamma(\alpha+hs+2-c_j)}{(\alpha+hs+1-c_j)}, for j = 1, ..., r$$

and interpreting with the help of (1), the result follows.
Differentiation under the integral sign used in the proof is valid,provided
(i) $\theta(s)z^s(ax^v+b)^{\alpha+hs}$ is continuous function of x and s.
(ii) $\theta(s)(\alpha+hs+1-c_1)...(\alpha+hs+1-c_r)z^s(ax^v+b)\alpha+hs$ is continuous function of x and s.
(iii) $\int_L \theta(s)z^s(ax^v+b)^{hs}ds$ converges.
(iv) $\int_L \theta(s)(\alpha+hs+1-c_1)...(\alpha+hs+1-c_r)z^s(ax^v+b)^{\alpha+hs}\alpha^{hs}$ is converges uniformly with respect to x.
The conditions on continuity are clearly satisfied and conditions on convergence are satisfied when the L.H.S. of (6) exists.

Special case:

When c_j's are zeroes,(6) reduces to the following result

$$[(ax^v+b)D-\lambda_1]...[(ax^v+b)D-\lambda_r]\{(ax^v+b)^\alpha \overline{H}^{m,n}_{p,q}[z(ax^v+b)^h/^{\mathbf{A},\mathbf{C}}_{\mathbf{B},\mathbf{D}}]\}$$

$$= a^r(ax^v+b)^\alpha \overline{H}^{m,n+r}_{p+r,q+r}[z(ax^v+b)^h/^{(-\alpha,h;1)_{1,r},\mathbf{A},\mathbf{C}}_{\mathbf{B},\mathbf{D},(1-\alpha,h;1)_{1,r}}], \qquad (7)$$

provided $h > 0$ and a, b are complex numbers, r is a positive integers $.\lambda = ac_i$ where a and c_i are not simultaneously zero, and $i = 1, .., r$.
When the λ's are in the arithmetic progression (6) takes the following form-

$$[(ax^v+b)D-\lambda][(ax^v+b)D-\lambda+k]...[(ax^v+b)D-\lambda+(r-1)k]$$

$$.\{(ax^v+b)^{\alpha d+c}\overline{H}^{m,n}_{p,q}[z(ax^v+b)^{hd}/^{\mathbf{A},\mathbf{C}}_{\mathbf{B},\mathbf{D}}]\}$$

$$= a^r d^r(ax^v+b)^{\alpha d+c}\overline{H}^{m,n+1}_{p+1,q+1}[z(ax^v+b)^{hd}/^{(1-r-\alpha,h;1),\mathbf{A},\mathbf{C}}_{\mathbf{B},\mathbf{D},(1-\alpha,h;1)_{(1,r)}}], \qquad (8)$$

provided $d \neq 0, h > 0$ a and b are complex numbers, r is a positive integer, $\lambda = ac, k = ad$, a and c are not simultaneously zero.

Proof: Taking L.H.S. of (8) and using (1), where $\theta(s)$ is given by (2), we have operating under the integral sign using (3), the expression becomes

Expressing

$$(\alpha+hs)(\alpha+hs+1)...(\alpha+hs+r-1)=\frac{\Gamma(\alpha+hs+r)}{\Gamma(\alpha+hs)}$$

and interpreting with the help of (1), the result follows.
when $\lambda_1=\lambda_2=\lambda_3=....=\lambda_r=\lambda$ (6) becomes

$$[(ax^v+b)D]^r\{(ax^v+b)^{\alpha+c}.\overline{H}^{m,n}_{p,q}[z(ax^v+b)^h/^{\mathbf{A},\mathbf{C}}_{\mathbf{B},\mathbf{D}}]\}$$

$$=a^r(ax^v+b)^{\alpha+c}\overline{H}^{m,n+r}_{p+r,q+r}[z(ax^v+b)^h/^{(-\alpha,h;1)_{1,r},\mathbf{A},\mathbf{C}}_{\mathbf{B},\mathbf{D},(1-\alpha,h;1)_{1,r}}], \tag{9}$$

provided $h>0$, a and b are complex numbers, r is a positive integer and λ=ac , a and c are not simultaneously zero .

Specializing the parameters in (6) and using (4), (6) reduces to

$$[(ax^v+b)D-\lambda_1]...[(ax^v+b)D-\lambda_r]\{(ax^v+b)^\alpha.{}_p\overline{\psi}_q[^{(a_j,A_j;\alpha_j)_{1,p}}_{(b_j,B_j;\beta_j)_{1,q}};z(ax^v+b)^h]\}$$

$$=a^r(ax^v+b)_{p+r}\overline{\psi}_{q+r}[/^{(1-\alpha-c_j,h;1)_{1,r},(a_j,A_j;\alpha_j)_{1,p}}_{(\alpha-c_j,h;1)_{1,r},(b_j,B_j;\beta_j)_{1,q}};z(ax^v+b)^h], \tag{10}$$

provided $h>0, a$ and b are complex numbers, r is a positive integer. $\lambda=ac_i$ where a and c_i are not simultaneously zero, and $i=1,..,r$.
Further specializing the parameters and using (5),(6) reduces to

$$[(ax^v+b)D-\lambda_1]...[(ax^v+b)D-\lambda_r]\{(ax^v+b)\,{}_p\overline{F}_q[^{(a_j,1;\alpha_j)_{1,p}}_{(b_j,1;\beta_j)_{1,q}};z(ax^v+b)]\}$$

$$=a^r\prod_{j=1}^{r}(\alpha-c_j)(ax^v+b)^r\,{}_{p+r}\overline{F}_{q+r}[^{(\alpha-c_j+1,1;1)_{1,r},(a_j,A_j;\alpha_j)_{1,p}}_{(\alpha-c_j,1;1)_{1,r},(b_j,B_j;\beta_j)_{1,q}};z(ax^v+b)^h], \tag{11}$$

provided $h>0, a$ and b are complex numbers, r is a positive integer.

When $a=1, b=0$,(6) (7) and (8) reduces to the results obtained By Devra and Rathie [2, pp107-113].
Now, if we put $\alpha_j=1$, for $j=1,...,n$ and $\beta_j=1$ for $j=m+1,...,q$ in equatios (6), (7), (8) and (9) to obtain the corresponding formulas for H-function of Fox.

$$[(ax^v+b)D-\lambda_1]...[(ax^v+b)D-\lambda_r]\{(ax^v+b)^\alpha H^{m,n}_{p,q}[z(ax^v+b)^h/^{(a_j,A_j)_{1,p}}_{(b_j,B_j)_{1,q}}]\}$$

$$=a^r(ax^v+b)^\alpha H^{m,n+r}_{p+r,q+r}[z(ax^v+b)^h/^{(c_j-\alpha,h)_{1,r},(a_j,A_j)_{1,p}}_{(b_j,B_j)_{1,q},(c_j-\alpha+1,h)_{1,r}}], \tag{12}$$

provided $h > 0$, a and b are complex numbers, r is a positive integers. $\lambda = ac_i$ where a and c_i are not simultaneously zero, and $i = 1, .., r$.

$$[(ax^v + b)D]^r\{(ax^v + b)^\alpha\, H^{m,n}_{p,q}[z(ax^v + b)^h/^{(a_j,A_j)_{1,p}}_{(b_j,B_j)_{1,q}}]\}$$

$$= a^r(ax^v + b)^\alpha\, H^{m,n+r}_{p+r,q+r}[z(ax^v + b)^h/^{(-\alpha,h)_{1,r},(a_j,A_j)_{1,p}}_{(b_j,B_j)_{1,q},(1-\alpha,h)_{1,r}}], \tag{13}$$

provided $h > 0$, a and b are complex numbers, r is a positive integer.

$$[(ax^v + b)D - \lambda][(ax^v + b)D - \lambda + k]...[(ax^v + b)D - \lambda + (r-1)k]\{(ax^v + b)^{\alpha d+c}$$

$$.H^{m,n}_{p,q}[z(ax^v + b)^h d/^{(a_j,A_j)_{1,p}}_{(b_j,B_j)_{1,q}}]\}$$

$$= a^r d^r(ax^v + b)^{\alpha d+c}\, H^{m,n+1}_{p+1,q+1}[z(ax^v + b)^{hd}/^{(1-r-\alpha,h),\,(a_j,A_j)_{1,p}}_{(b_j,B_j)_{1,q},\,(1-\alpha,h)}], \tag{14}$$

provided $d \neq 0, k \neq 0, h > 0$, a and b are complex number, r is a positive integer, $\lambda = ac, k = ad$, a and c are not simultaneously zero.

$$[(ax^v + b)D]^r\{(ax^v + b)^{\alpha+c}\, H^{m,n}_{p,q}[z(ax^v + b)^h/^{(a_j,A_j)_{1,p}}_{(b_j,B_j)_{1,q}}]\}$$

$$= a^r(ax^v + b)^{\alpha+c}\, H^{m,n+r}_{p+r,q+r}[z(ax^v + b)^h/^{(-\alpha,h)_{1,r},(a_j,A_j)_{1,p}}_{(b_j,B_j)_{1,q},(1-\alpha,h)_{1,r}}], \tag{15}$$

provided $h > 0$, a and b are complex numbers, r is a positive integer. **When**$a = 1, b = 0$ (12) ,(13)and(14) are the results given by Nair[7, $pp74 - 78$].
when $r = 1$, equation (11) reduces to

$$[(ax^v + b)D - \lambda]\{(ax^v + b)^\alpha\, {}_p\overline{F}_q[^{(a_j,1;\alpha_j)_{1,p}}_{(b_j,1;\beta_j)_{1,q}}; z(ax^v + b)]\}$$

$$= a\,(\alpha - c)(ax^v + b)\, {}_{p+1}\overline{F}_{q+1}[^{(\alpha-c+1,1;1),\,(a_j,1;\alpha_j)_{1,p}}_{(\alpha-c_j,1;1),\,(b_j,1;\beta_j)_{1,q}}; z(ax^v + b)], \tag{16}$$

provided $h > 0$, a and b are complex numbers, $\lambda = ac$,a and c are not smultaneously zero.

Now, in (16) put $\alpha_j = \beta_j = 1$ to get the following result:

$$[(ax^v + b)D - \lambda]\{(ax^v + b)^\alpha\, {}_pF_q[^{(a_j)_{1,p}}_{(b_j)_{1,q}}; z(ax^v + b)]\}$$

$$= a(\alpha - c)(ax^v + b)^\alpha_{p+1}F^{q+1}[/^{(\alpha-c+1),\,(a_j)_{1,p}}_{(\alpha-c),\,(b_j)_{1,q}}; z(ax^v + b)], \tag{17}$$

provided a and b are complex numbers, $\lambda = ac$, a and c are not simultaneously zero.

Again by specializing the parameters in (17) a number of other results involving various other special functions can be obtain.

Acknowledgements. Our sincere thanks are due to Professor C.K.Sharma (A.P.S.University Rewa (M.P.)) whose numerous suggestions were responsible for bringing out the paper in its present form.

References

[1] Praveen Agrarwal and Chand Mehar : New Finite Integrals involving product of $\overline{H}$-function and Srivastava polynomial, *Asian Journal of Mathematics and statistics*, **5(4)** (2012), 142 - 149.

[2] R.G. Buschman and H.M. Srivastava : The $\overline{H}$-function associated with a certain class of Feyman integrals, *J.Phy.A.Math.Gen.* (1990), 4707 - 4710.

[3] V.B.L. Chaurasia and Jagdav : Fractional-Calculus results pertaining to special functions,*Int.J.Contemp.Math.Sci.*, **5(48)** (2010), 2381-2389.

[4] R.C. Singh Chandel and Vishwakarma : Fractional derivatives of certain generalized hypergeometric functions of several variables, *J.math.Ana.Appl.*, **184** (1994), 560 - 572.

[5] H.M. Devra and A.K. Rathie : *Vijnana Parishad Anusandhann Patrika*, **36** (1993), 107 - 113.

[6] Charles Fox: The G and H-functions as symmetrical Fourier kernels, *Trans.Amer.Math.Soc.*, **98** (1961), 395 - 429.

[7] K.C. Gupta and V.C. Jain : On the derivative of the H-function, *Proc.Nat.Acad.Sci.India*, **38** (1968), 162.

[8] K.C. Gupta and R.C. Soni : On a basic formula involving the product of $\overline{H}$-function, *J.Rajasthan Acad.Phy.Sci.*, **4(3)** (2006), 157-164.

[9] A.A. Inayat-Hussain : New properties of hypergeometric series derivable form Feynman integrals:I.Transformation and reduction formulae, *J.Phys.A.Math.Gen.*, **20** (1987a), 4109 - 4117.

[10] A.A. Inayat-Hussain : New properties of hypergeometric series derivable form Feynman integrals II. A generalization of the H-function, *J.Phys.A.Math.Gen.*, **20** (1987b), 4119 - 4128.

[11] D.S. Marko, S. Pandey and M. k.Sukla : On some new integrals relation of $\overline{H}$-function, *Res.J.*
Mathematical and statistical Sciences., **1(3)** (2013), 21 - 22.

[12] A.K. Rathie : A new generalization of generalized hypergeometric functions, , *Le Matematiche Fasc.II*, **52** (1997), 297 - 310.

[13] H.M. Srivastava and S.P. Goyal : fractional derivatives of H-function of several variables., *J.Math.Ana.App.*, **112** (1985), 641 - 651.