Fixed point results for contraction mapping in cubic terms of metric via simulation function

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Abstract.

In this manuscript, we introduce new notion of modified weak contraction in cubic terms of metric via simulation functions and establish fixed point results for this kind of operator in the framework of complete metric space. Our result is an extension of δ-weak contraction in complete metric space which generalize some existing results. Moreover, example is given to illustrate the attained sequel.

2020 Mathematics Subject Classification: 47H10, 54H25.

***Key words:** and phrases: simulation function, modified weak contraction, fixed point.*

1 Introduction

Fixed point is very powerful tool in several spheres of analysis. In 2013, Murthy and Prasad [1] generalize weak contraction by making combinations of σ(x, y). In 2015, Khojasteh et al. [2] estabilshed some fixed point results for Z-contraction via simulation function. Afterwards, Argoubi et al. [5] inroduced a contraction comprising simulation function for a metric enriched with partial order. In 2017, Jain et al. [6] proved some results for commuting mappings comprising cubic terms of metric. In 2020, Arora et al. conferred common fixed point results for modified β-admissible contraction and almost Zcontraction in the edge of metric space and G-metric space (see [8], [10]). Afterwards, Debnath et al. [7] discussed the existence and uniqueness of common fixed point theorems for Kannan, 1 Reich and Chatterjea type pair of self-maps in the context of complete metric space. In 2021, Arora [9] established some common fixed point results for four self maps in the context of Gs-metric space via CLR property. Now, let us recall some definitions which will be significant in the sequel.

Definition 1.1. [2] Let (X, σ) be a metric space, f is a self-mapping on X and ζ ∈ Z. We say that f is a Z-contraction with respect to ζ, if

$$\zeta(\sigma(fx, fy), \sigma(x, y)) \geq 0, \text{ for all } x, y \in X.$$

Hang and Dung [3] investigated the concept of generalized F-contraction and proved useful fixed point results for such kind of functions.

Definition 1.2. [3] Let (Z, σ) be a metric space and f : Z → Z be a self-mapping. f is called a generalized F-contraction on (X, σ) if there exist F ∈ θ and δ > 0 such that

$$\sigma(fx, fy) > 0 \Rightarrow \delta + F(\sigma(fx, fy)) \leq F(max\{\sigma(x, y), \sigma(x, fx), \sigma(y, fy), \frac{\sigma(x,fy)+\sigma(y,fx)}{2}, \frac{\sigma(f^2x,x)+\sigma(f^2x,fy)}{2}, \sigma(f^2x, fx), \sigma(f^2x, y), \sigma(f^2x, fy)\}).$$

Definition 1.3. [4] Let (Z, σ) be a metric space and T : X → X be a function. T is known as F-weak contraction on (Z, σ) if there exist F ∈ θ and γ > 0 such that for all x, y ∈ Z ,

$$\sigma(T x, T y) > 0 \Rightarrow \gamma + F(\sigma(T x, T y)) \leq F(\max\{\sigma(x, y), \sigma(T x, x), \sigma(y, T y), \frac{\sigma(x,Ty)+\sigma(y,Tx)}{2}\}).$$

Definition 1.4. [4] Let (Z, σ) be a metric space and T : Z → Z be a function. T is known as δ-weak contraction on (Z, σ) if there exist δ : [0, ∞) → [0, ∞) such that for all x, y ∈ Z ,

$$\sigma(T x, T y) \leq \sigma(x, y) - \delta(\sigma(x, y)).$$

Definition 1.5. [5] The function μ : [0, ∞) × [0, ∞) → R is said to be a simulation function, if the following properties hold:

(μ1) μ(0, 0) = 0; (μ2)

(μ2) μ(a, b) < a − b for all a, b > 0;

(μ3) If $\{a_n\}$ and $\{b_n\}$ are sequences in (0, ∞) such that

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = l$, then $\lim_{n\to\infty} \sup\ \mu(a_n, b_n) < 0$.

Let Π be the class of all simulation functions μ : [0, ∞) × [0, ∞) → R. In this paper , we established fixed point results by making several combinations of σ(x, y) in the context of complete metric space by virtue of simulation function. As application, various analogous results in fixed point theory are easily deduced by conferring different values to s.

2. Main Results

Definition 2.1. Let (Z, σ) be a metric space, S : Z → Z be a mapping and μ ∈ Π. Then, S is known as modified weak contraction in cubic terms with respect to μ if the following condition satified.

$$\mu[(1 + s\sigma(x, y))\sigma^2 (Sx, Sy), sM(x, y) + L(x, y) - \delta(L(x, y)] \geq 0, \qquad (2.1)$$

Where

$$L(x, y) = \max\{\sigma^2 (x, y), \sigma(x, Sx)\sigma(y, Sy), \sigma(x, Sy)\sigma(y, Sx), \frac{\sigma(x,Sx)\sigma(y,Sy)+\sigma(y,Sx)\sigma(y,Sy)}{2}\} \qquad (2.2)$$

And

$$M(x, y) = \max\{\frac{\sigma(x,Sx)\sigma(y,Sy)+\sigma(y,Sx)\sigma(y,Sy)}{2}, \sigma(x, Sx)\sigma(x, Sy)\sigma(y, Sx), \sigma(x, Sy)\sigma(y, Sx)\sigma(y, Sy) \} \qquad (2.3)$$

where s ≥ 0 be a real number and δ : [0, ∞) → [0, ∞) is a continuous function with δ(t) = 0 iff t = 0 and δ(t) > 0 for every t > 0.

Before starting our Theorem, we will give some lemmas that have an important role for proving our main result.

Lemma 2.2. Let (Z , σ) be a metric space and S be modified weak contraction in cubic terms with respect to μ such that the sequence $\{x_n\}$ in Z is defined as x_{n+1} = Sx_n for all n ∈ N . Let $x_n \neq x_{n+1}$ for each n ∈ N . Then,

$$\lim_{n\to\infty} \sigma(x_n, x_{n+1}) = 0.$$

Proof. Firstly, we have to show that $\{\rho_n\}$ is non increasing sequence and convegent. Now, two cases arise.

Case 1. If n is even, substituting x = x_{2n}, y = x_{2n}+1 in (2.1) , we get

μ[(1+sσ(x_{2n}, x_{2n+1}))σ^2 (Sx_{2n}, Sx_{2n+1}), sM(x_{2n}, x_{2n+1})+L(x_{2n}, x_{2n+1})−δ(L(x_{2n}, x_{2n+1})] ≥ 0, (2.4)

where

L(x_{2n}, x_{2n+1}) = max{σ^2(x_{2n}, x_{2n+1}),σ(x_{2n}, Sx_{2n})σ(x_{2n+1}, Sx_{2n+1}),σ(x_{2n}, Sx_{2n+1})σ(x_{2n+1}, Sx_{2n}) ,

$\frac{\sigma(x_{2n}, Sx_{2n})\sigma(x_{2n}, Sx_{2n+1})+\sigma(x_{2n+1}, Sx_{2n})\sigma(x_{2n+1}, Sx_{2n+1})}{2}$} (2.5)

and

M(x_{2n}, x_{2n+1}) = max {$\frac{\sigma^2(x_{2n}, Sx_{2n})\sigma(x_{2n+1}, Sx_{2n+1})+\sigma(x_{2n}, Sx_{2n})\sigma^2(x_{2n+1}, Sx_{2n+1})}{2}$, σ(x_{2n}, Sx_{2n}) σ(x_{2n}, Sx_{2n+1}) σ(x_{2n+1}, Sx_{2n}) , σ(x_{2n}, Sx_{2n+1}) σ(x_{2n+1}, Sx_{2n}) σ(x_{2n+1}, Sx_{2n+1}) }. (2.6)

Using the given assumption of lemma, (2.4), (2.5) and (2.6) reduces to μ[(1+sσ(x_{2n}, x_{2n+1}))$\sigma^2(x_{2n+1}, x_{2n+2})$, sM($x_{2n}$, x_{2n+1}) + L(x_{2n}, x_{2n+1}) − δ(L(x_{2n}, x_{2n+1})] ≥ 0, (2.7) where

$L(x_{2n}, x_{2n+1}) = \max \{\sigma^2(x_{2n}, x_{2n+1}), \sigma(x_{2n}, x_{2n+1})\sigma(x_{2n+1}, x_{2n+2}),$
$\sigma(x_{2n}, x_{2n+2})\sigma(x_{2n+1}, x_{2n+1}), \frac{\sigma(x_{2n},x_{2n+1})\sigma(x_{2n},x_{2n+2})+\sigma(x_{2n+1},x_{2n+1})\sigma(x_{2n+1},x_{2n+2})}{2}$

(2.8)

and

$M(x_{2n}, x_{2n+1}) = \max\{\frac{\sigma(x_{2n},x_{2n+1})\sigma(x_{2n+1},x_{2n+2})+\sigma(x_{2n},x_{2n+1})\sigma^2(x_{2n+1},x_{2n+2})}{2}, \sigma(x_{2n},$
$x_{2n+1})\, \sigma(x_{2n}, x_{2n+2}) + \sigma(x_{2n+1}, x_{2n+1})\, \sigma(x_{2n}, x_{2n+2})\, \sigma(x_{2n+1}, x_{2n+1}), \sigma(x_{2n+1}, x_{2n+2})\}.$
(2.9)

Let $p_{2n} = \sigma(x_{2n}, x_{2n+1})$. Now, (2.7), (2.8) and (2.9) reduces to

$\mu[(1 + sp_{2n})\, p_{2n+1}, sM(x_{2n}, x_{2n+1}) + L(x_{2n}, x_{2n+1}) - \delta(L(x_{2n}, x_{2n+1})] \geq 0,$

(2.10)

Where

$L(x_{2n}, x_{2n+1}) = \max\{p_{2n}^2 , p_{2n+1}p_{2n}, 0, \frac{p_{2n}(\sigma(x_{2n},x_{2n+2}))}{2}, 0 \}$ (2.11)

and

$M(x_{2n}, x_{2n+1}) = \max\{\frac{p_{2n}^2 p_{2+1} + p_{2n}p_{2n}^2+1}{2}, 0,0\}.$ (2.12)

On account of triangle inequality, we infer that

$\sigma(x_{2n}, x_{2n+1}) + \sigma(x_{2n+1}, x_{2n+2}) \geq \sigma(x_{2n}, x_{2n+1}).$

Therefore,
$\sigma(x_{2n}, x_{2n+1}) \leq p_{2n+1} + p_{2n}.$

Taking (2.11) into account, we obtain

$L(x_{2n}, x_{2n+1}) \leq \{\max p_{2n}^2 , p_{2n+1}p_{2n}, 0, \frac{p_{2n}(p_{2n},p_{2n+1})}{2}, 0 .$ (2.13)

If $p_{2n+1} \geq p_{2n}$ then (2.10) yields that

$$0 \leq \mu[sp_{2n+1}^2 , sp_{2n+1}^2 - \delta(p_{2n+1}^2)]$$
$$< sp_{2n+1}^2 - \delta(p_{2n+1}^2) - sp_{2n+1}^2 .$$

Therefore,

$$sp_{2n+1}^2 + \delta(sp_{2n+1}^2) < sp_{2n+1}^2 ,$$

a contradiction.

Hence , $p_{2n+1} \leq p_{2n}$. Similarly, we can prove that $p_{2n+1} \leq p_{2n+1}$ when n is odd natural number. Consequently, p_n is decreasing sequence.

Therefore, there exist q ≥ 0 such that

$$\lim_{n\to\infty} p_n = q .$$

Assume that q > 0, then substituting x = x_n and y = x_{n+1} in the inequality (2.1), (2.2) and (2.3), we get

$$\mu[(1 + s\sigma (x_n, x_{n+1})) \sigma^2 (Sx_n, Sx_{n+1}), sM(x_n, x_{n+1})+L(x_n, x_{n+1}) - \delta(L(x_n, x_{n+1})] \geq 0, \quad (2.14)$$

where

$$L(x_n, x_{n+1}) = \max \{\sigma^2 (x_n, x_{n+1}), \sigma(x_n, Sx_n)\sigma(x_{n+1}, Sx_{n+1}), \sigma(x_n, Sx_{n+1})\sigma(x_{n+1}, Sx_n),$$
σ(xn, Sxn)σ(xn, Sxn+1) + σ(xn+1, Sxn)σ(xn+1, Sxn+1)

(2.15)

and

$$M(x_n, x_{n+1}) = \max \{\frac{\sigma^2(x_n,Sx_n)\sigma(x_{n+1},Sx_{n+1})+\sigma(x_n,Sx_n)\sigma^2(x_{2+1},Sx_{n+1}}{2}, \sigma(x_n, Sx_n)\, \sigma(x_n, Sx_{n+1})$$
$$\sigma(x_{n+1}, Sx_n) , \sigma(x_n, Sx_{n+1})\, \sigma(x_{n+1}, Sx_n)\, \sigma(x_{n+1}, Sx_{n+1})\} \quad (2.16)$$

By using the given assumption of lemma, (2.14), (2.15) and (2.16) yields that

$$\mu[(1 + s\sigma(x_n, Sx_{n+1}))\sigma^2(x_{n+1}, x_{n+2}) , sM(x_n, x_{n+1}) + L(x_n, x_{n+1}) - \delta(L (x_n, x_{n+1})] \geq 0,$$
(2.17)

where

$L(x_n, x_{n+1}) = \max\{\sigma^2(x_n, x_{n+1}), \sigma(x_n, x_{n+1})\sigma(x_{n+1}, x_{n+2}), \sigma(x_n, x_{n+2})\sigma(x_{n+1}, x_{n+1}), \frac{\sigma(x_n, x_{n+1})\sigma(x_n, x_{n+2})+\sigma(x_{n+1}, x_{n+1})\sigma(x_{n+1}, x_{n+2})}{2}\}$

and

$M(x_n, x_{n+1}) = \max\{\frac{\sigma(x_n, x_{n+1})\sigma(x_{n+1}, x_{n+2})+\sigma(x_n, x_{n+1})\sigma^2(x_{n+1}, x_{n+2})}{2}, \sigma(x_n, x_{n+1})\sigma(x_n, x_{n+2})\sigma(x_{n+1}, x_{n+1}), \sigma(x_n, x_{n+2})\sigma(x_{n+1}, x_{n+1})\sigma(x_{n+1}, x_{n+2})\}$.

Making $n \to \infty$ in (2.17), we obtain

$$0 \leq \mu[(1 + sq)q^2), sq^3 + q^2 - \delta(q^2)]$$

$$< sq^3 + q^2 - \delta(q^2) - (1 + sq)q^2).$$

Therefore,

$$(1 + sq)q^2 < (1 + sq)q^2 - \delta(q^2).$$

Due to positivity of q , $\delta(q^2) \leq 0$.

Further, using the property of δ, we get q = 0. Consequently,

$$\lim_{n\to\infty} \sigma(x_n, x_{n+1}) = 0.$$

Lemma 2.3. Let (Z, σ) be a metric space and S be modified weak contraction in cubic terms with respect to μ. Let $\{x_n\}$ be the sequence in Z such that $x_{n+1} = Sx_n$ with $x_n \neq x_{n+1}$ for all n ∈ N . Then the picard sequence $\{x_n\}$ is bounded.

Proof. Let $x_0 \in X$ be any point. Let us suppose that $\{x_n\}$ is not a bounded sequence. Therefore, there exists a subsequence $\{x_{m_l}\}$ such that $m_1 = 1$ and m_{l+1} is smallest integer for every positive integer l such that

$$\sigma(x_{m_{l+1}}, x_{m_l}) > 1$$

and

$$\sigma(x_n, x_{m_l}) \leq 1$$

for $m_l \leq n \leq m_{l+1} - 1$.

On account of triangle inequality, we get

$$1 \leq \sigma(x_{m_{l+1}}, x_{m_l})$$
$$\leq \sigma(x_{m_{l+1}}, x_{m_{l+1}} - 1) + \sigma(x_{m_{l+1}} - 1, x_{m_l})$$

$$\leq \sigma(x_{m_{l+1}}, x_{m_{l+1}} - 1) + 1.$$

Making $l \to \infty$ and using lemma 2.2, we obtain

$$\lim_{l\to\infty} \sigma(x_{m_{l+1}}, x_{m_l}) = 1.$$

Since, S is modified weak contraction in cubic terms with respect to μ ∈ Π, therefore by using μ_3, we have

$0 \leq \lim_{l \to \infty} sup\mu[(1+s\sigma(x_{m_{l+1}} - 1, x_{m_l} - 1))\sigma^2(Sx_{m_{l+1}} - 1, Sx_{m_l} - 1)$, $sM(x_{m_{l+1}} - 1, x_{m_l} - 1) + L(x_{m_{l+1}} - 1, x_{m_l} - 1) - \delta(L(x_{m_{l+1}} - 1, x_{m_l} - 1))] < 0$,(2.18) where

$L(x_{m_{l+1}} - 1, x_{m_l} - 1) = \max \sigma^2(x_{m_{l+1}} - 1, x_{m_l} - 1), \sigma(x_{m_{l+1}} - 1, Sx_{m_l} - 1)\sigma(x_{m_l} - 1, Sx_{m_l}), \sigma(x_{m_{l+1}} - 1, Sx_{m_l} - 1)\sigma(x_{m_l} - 1, Sx_{m_l+1} - 1),$
$\frac{\sigma(x_{m_{l+1}}-1, Sx_{m_{l+1}}-1)\sigma(x_{m_{l+1}}-1, Sx_{m_l}-1)+\sigma(x_{m_l}-1, Sx_{m_{l+1}}-1)\sigma(x_{m_l}-1, Sx_{m_l}-1)}{2}\}$

And

$M(x_{m_{l+1}} - 1, x_{m_l} - 1) = \max\{ \sigma(x_{m_{l+1}} - 1, Sx_{m_l+1} - 1)\,\sigma(x_{m_{l+1}} - 1, Sx_{m_l} - 1), \sigma(x_{m_l} - 1, Sx_{m_l+1} - 1), \sigma(x_{m_{l+1}} - 1, Sx_{m_l} - 1)\,\sigma(x_{m_l} - 1, Sx_{m_l+1} - 1)\,\sigma(x_{m_l} - 1, Sx_{m_l} - 1),$
$\frac{\sigma^2(x_{m_{l+1}}-1, Sx_{m_l+1}-1)\sigma(x_{m_l}-1, Sx_{m_l}-1)+\sigma(x_{m_{l+1}}-1, Sx_{m_l+1}-1)\sigma^2(x_{m_l}-1, Sx_{m_l}-1)}{2}\}$. (2.18)

Contradiction proves our result.

Lemma 2.4. Let (Z, σ) be a metric space and S be modified weak contraction in cubic terms with respect to μ. Let $\{x_n\}$ be the sequence in Z such that $x_{n+1} = Sx_n$ with $x_n \neq x_{n+1}$ for all n ∈ N . Then the picard sequence $\{x_n\}$ is a Cauchy sequence.

Proof. Let us suppose that $\{x_n\}$ is not a Cauchy sequence. Let $\delta_1 > 0$ be however small for which we have sequences $\{n_l\}$ and $\{m_l\}$ such that

$$\sigma(x_{n_l}, x_{m_l} - 1) < \delta_1$$

and

$$\sigma(x_{n_l}, x_{m_l}) \geq \delta_1,$$

for each $m_\ell > n_\ell > \ell$.

On account of triangle inequality, we get

$$\delta_1 \leq \sigma(x_{n_\ell}, x_{m_\ell})$$

$\leq \sigma(x_{n_\ell}, x_{m_\ell - 1}) + \sigma(x_{m_\ell - 1}, x_{m_\ell})$. (2.19)

Making $\ell \rightarrow \infty$ in (2.19),we get

$$\lim_{\ell\to\infty} \sigma(x_{n_\ell}, x_{m_\ell}) = \delta_1 \quad (2.20)$$

Again using triangle inequality, we obtain

$$\sigma(x_{n_\ell}, x_{n_\ell+1}) \geq | \sigma(x_{m_\ell}, x_{n_\ell+1}) - \sigma(x_{n_\ell}, x_{m_\ell}) | .$$

Letting $\ell \rightarrow \infty$ in the above inequality and taking lemma 2.2, (2.20) into account, we get

$$\lim_{\ell\to\infty} (x_{n_\ell}, x_{n_\ell+1}) = \delta_1. \quad (2.21)$$

With the help of triangle inequality, we obtain

$$|\sigma(x_{m_\ell}, x_{m_\ell+1}) \geq | \sigma(x_{n_\ell}, x_{m_\ell+1}) - \sigma(x_{n_\ell}, x_{m_\ell}) | .$$

Letting $\ell \rightarrow \infty$ in the above inequality and using lemma 2.2, (2.20), we get

$$\lim_{\ell\to\infty} \sigma (x_{n_\ell}, x_{m_\ell+1}) = \delta_1. \quad (2.22)$$

Now, using triangle inequality, we get

$$| \sigma(x_{n_\ell}, x_{n_\ell+1}) + \sigma(x_{m_\ell}, x_{m_\ell+1}) | \geq | \sigma(x_{n_\ell+1}, x_{m_\ell+1}) - \sigma(x_{n_\ell}, x_{m_\ell}) | .$$

Letting $\ell \rightarrow \infty$ in the above inequality and using lemma 2.2, (2.20), we get

$$\lim_{\ell\to\infty} \sigma (x_{m_\ell+1}, x_{n_\ell+1}) = \delta_1.$$

Substituting x = x_{n_ℓ} and y = x_{m_ℓ}, in (2.1), we get

$$\mu[(1 + s\sigma(x_{n_\ell}, x_{m_\ell}))\sigma^2(Sx_{n_\ell}, Sx_{m_\ell}), sM(x_{n_\ell}, x_{m_\ell}) + L(x_{n_\ell}, x_{m_\ell}) - \delta(L(x_{n_\ell}, x_{m_\ell})] \geq 0, \quad (2.24)$$

$$L(x_{n_\ell}, x_{m_\ell}) = \max\{\sigma^2(x_{n_\ell}, x_{m_\ell}), \sigma(x_{n_\ell}, Sx_{n_\ell})\sigma(x_{m_\ell}, Sx_{m_\ell}), \sigma(x_{n_\ell}, Sx_{m_\ell})\sigma(x_{m_\ell}, Sx_{n_\ell}), \frac{\sigma(x_{n_\ell}, Sx_{n_\ell})\sigma(x_{n_\ell}, Sx_{m_\ell}) + \sigma(x_{m_\ell}, Sx_{n_\ell})\sigma(x_{m_\ell}, Sx_{m_\ell})}{2}\} \quad (2.25)$$

and

$$M(x_{n_\ell}, x_{m_\ell}) = \max\{\frac{\sigma^2(x_{n_\ell}, Sx_{n_\ell})\sigma(x_{m_\ell}, Sx_{m_\ell}) + \sigma(x_{n_\ell}, Sx_{n_\ell})\sigma(x_{m_\ell}, Sx_{m_\ell})}{2},$$
$$\sigma(x_{n_\ell}, Sx_{n_\ell})\sigma(x_{n_\ell}, Sx_{m_\ell})\sigma(x_{m_\ell}, Sx_{n_\ell}), \sigma(x_{n_\ell}, Sx_{m_\ell})\sigma(x_{m_\ell}, Sx_{n_\ell})\sigma(x_{m_\ell}, Sx_{m_\ell})\}. \quad (2.26)$$

Using the given assumption of lemma, (2.24), (2.25) and (2.26) reduces to

$\mu[(1 + s\sigma(x_{n_\ell}, x_{m_\ell})\sigma^2(x_{n_\ell+1}, x_{m_\ell+1}), sM(x_{n_\ell}, x_{m_\ell}) + L(x_{n_\ell}, x_{m_\ell}) - \delta(L(x_{n_\ell}, x_{m_\ell})] \geq 0,$

(2.27)

where

$L(x_{n_\ell}, x_{m_\ell}) = \max\{\frac{\sigma(x_{n_\ell},x_{n_\ell+1})\sigma(x_{m_\ell},x_{m_\ell+1})+\sigma(x_{m_\ell},x_{n_\ell+1})\sigma(x_{m_\ell},x_{m_\ell+1})}{2}\}$

$(x_{n_\ell}, x_{m_\ell}), \sigma(x_{m_\ell}, x_{m_\ell+1})\sigma(x_{m_\ell}, x_{m_\ell+1}), \sigma(x_{n_\ell}, x_{m_\ell+1})\sigma(x_{m_\ell}, x_{n_\ell+1}),$

$$\frac{\sigma(x_{n_\ell}, x_{n_\ell+1})\sigma(x_{m_\ell}, x_{m_\ell+1}) + \sigma(x_{m_\ell}, x_{n_\ell+1})\sigma(x_{m_\ell}, x_{m_\ell+1})}{2}\}$$

and

$M(x_{n_\ell}, x_{m_\ell}) = \max\{\frac{\sigma^2(x_{n_\ell},x_{n_\ell+1})\sigma(x_{m_\ell},x_{m_\ell+1})+\sigma(x_{n_\ell},x_{n_\ell+1})\sigma^2(x_{m_\ell},x_{m_\ell+1})}{2},$

$\sigma(x_{n_\ell}, x_{n_\ell+1})\sigma(x_{n_\ell}, x_{m_\ell+1})\sigma(x_{m_\ell}, x_{n_\ell+1}), \sigma(x_{n_\ell}, x_{m_\ell+1})\sigma(x_{m_\ell}, x_{n_\ell+1}), \sigma(x_{m_\ell}, x_{m_\ell+1})\}.$

Letting $\ell \to \infty$ in (2.27) and using (2.20), (2.21), (2.22), (2.23), we get

$$0 \leq \mu[(1 + s\delta_1)\delta_1^2, s\max\{\tfrac{0+0}{2}, 0, 0\} + \delta_1^2 - \delta(\delta_1^2)]$$

$$= \mu[(1 + s\delta_1)\delta_1^2, \delta_1^2 - \delta(\delta_1^2)]$$

$$< \delta_1^2 - \delta(\delta_1^2) - (1 + s\delta_1)\delta_1^2,$$

which implies that.

$$(1 + s\delta_1)\delta_1^2 < \delta_1^2 - \delta(\delta_1^2).$$

which is a contradiction.

Consequently, $\{x_n\}$ is a cauchy sequence in X. □

Theorem 2.5. Let (Z, σ) be a complete metric space and S be modified weak contraction in cubic terms with respect to μ such that the sequence $\{x_n\}$ in X is defined as $x_{n+1} = Sx_n$ for all n∈N. Let $x_n \neq x_{n+1}$ for each n ∈ N. Then, S has a unique fixed point in X.

Proof. Since, (Z, σ) be a complete metric space and in lemma (2.4), we proved that $\{x_n\}$ is a cauchy sequence in X. Therefore, it yields that

$$\lim_{m\to\infty} x_m = \upsilon, \qquad (2.28)$$

for some $\upsilon \in Z$.

Now, we shall prove that υ is fixed point of S. Let us assume contrary that $\upsilon \neq S\upsilon$. So,

$\sigma(S\upsilon, \upsilon) > 0$.

Now, substituting x=x_m and y = υ in (2.1), we have

$$0 \leq \lim_{m\to\infty} \sup \mu[(1+s\sigma(x_m, \upsilon))\sigma^2(Sx_m, S\upsilon), sM(x_m, \upsilon)+L(x_m, \upsilon)-\delta(L(x_m, \upsilon)], \quad (2.29)$$

where

$$L(x_m, \upsilon) = \max\{\sigma^2(x_m, \upsilon), \sigma(x_m, Sx_m)\sigma(v, S\upsilon), \sigma(x_m, S\upsilon)\sigma(\upsilon, Sx_m), \frac{\sigma(x_m, Sx_m)\sigma(x_m, S\upsilon)+\sigma(v, Sx_m)\sigma(v, S\upsilon)}{2} \quad (2.30)$$

and

$$M(x_m, \upsilon) = max \frac{\sigma^2(x_m, Sx_m)\sigma(,S\upsilon) + \sigma(x_m, Sx_m)\sigma^2(\upsilon, S\upsilon)}{2},$$
$$\sigma(x_m, Sx_m)\sigma(x_m, S\upsilon)\sigma(\upsilon, Sx_m), \sigma(x_m, S\upsilon)\sigma(\upsilon, Sx_m)\sigma(\upsilon, S\upsilon). \quad (2.31)$$

Due to given hypothesis of Theorem and using (2.28), we obtain

$$0 \leq \lim_{m\to\infty} \sup \mu[(1+s\sigma(\upsilon, \upsilon))\sigma^2(\upsilon, S\upsilon), sM(\upsilon, \upsilon)+L(\upsilon, \upsilon)-\delta(L(\upsilon, \upsilon)]<0, \quad (2.32)$$

where

$$L(\upsilon, \upsilon) = \max\{\sigma^2(\upsilon, \upsilon), \sigma(\upsilon, \upsilon)\sigma(\upsilon, S\upsilon), \sigma(\upsilon, S\upsilon)\sigma(\upsilon, \upsilon), \frac{\sigma(\upsilon,\upsilon)\sigma(\upsilon,S\upsilon)+\sigma(\upsilon,\upsilon)\sigma(\upsilon,S\upsilon)}{2}\}$$

and

$$M(\upsilon, \upsilon) = \max\{\frac{\sigma^2(\upsilon,\upsilon)\sigma(\upsilon,S\upsilon)+\sigma(\upsilon,\upsilon)\sigma^2(\upsilon,S\upsilon)}{2}, \sigma(\upsilon, \upsilon)\sigma(\upsilon,S\upsilon)\sigma(\upsilon,\upsilon), \sigma(\upsilon, S\upsilon), \sigma(\upsilon, \upsilon)\sigma(\upsilon, S\upsilon)\}$$

The inequality (2.32) is a contradiction.

Therefore, $\sigma(S\upsilon, \upsilon) = 0$, which implies that $S\upsilon = \upsilon$.

Hence, S has a fixed point υ in Z.

Now, we examine the uniqueness of attained fixed point of S.

Let us assume that $x = \upsilon$ and $y = \omega$ in(2.1),

we can derive that $\sigma(\upsilon, \omega) = 0$. Therefore, $\upsilon = \omega$.

Consequently, S has unique fixed point in Z.

Example 2.6. Consider Z = {0, 1, 2, 3} associated with the metric

$$d(x, y) = \begin{cases} 0 & if\ x=y \\ |x-y| & otherwise \end{cases}$$

for all x,y$\in Z$. Now, we define the mapping S : Z → Z by

$$Sx = \begin{cases} 1 & if\ x=3 \\ 0 & otherwise \end{cases}$$

Let δ : [0,+∞) → [0,+∞) be defined as $\delta(t) = \frac{t}{3}$. Indeed, for every value of $s > 0$ and $x, y \in Z$, it is simple to justify that the condition of inequality(2.1) fulfils. So, we guarantee the occurence of fixed point of S. Therefore, Theorem (2.5) holds.

Corollary 2.7. Let (Z, σ) be a complete metric space and S be self map which fulfils the condition

$$\mu[\sigma^2(Sx,Sy), L(x,y) - \delta(L(x,y)] \geq 0,$$

$$L(x,y) = \max\{ \sigma^2(x,y), \sigma(x,Sx)\sigma(y,Sy), \sigma(x,Sx)\sigma(y,Sx), \frac{\sigma(x,Sx)\sigma(x,Sy)+\sigma(y,Sx)\sigma(y,Sy)}{2}\}$$

where δ : [0, ∞) → [0, ∞) is a continuous function with $\delta(t)= 0$ iff t = 0 and δ(t) > 0 for every $(t)> 0$. Then, S has a unique fixed point in Z.

Proof By substituting s = 0 in Theorem 2.5, we will get the unique fixed point of S.

Corollary 2.8. Let (Z, σ) be a complete metric space and S be self map which fulfils the condition

$$\mu[(1 + 2\sigma(x,y))\sigma^2(Sx,Sy), 2M(x,y) + L(x,y) - \delta(L(x,y)] \geq 0,$$

where

$$L(x,y) = \max\{\sigma^2(x,y), \sigma(x,Sx)\sigma(y,Sy), \sigma(x,Sy)\sigma(y,Sx), \frac{\sigma(x,Sx)\sigma(x,Sy)+\sigma(y,Sx)\sigma(y,Sy)}{2}\}$$

and

$$M(x,y) = \max\{\frac{\sigma(x,Sx)\sigma(y,Sy)+\sigma(x,Sx)\sigma(y,Sy)}{2}, \sigma(x,Sx)\sigma(x,Sy)\sigma(y,Sx), \sigma(x,Sy)\sigma(y,Sx)\sigma(y,Sy)\}.$$

Where

δ : [0, ∞) → [0, ∞) is a continuous function with δ(t) = 0 iff t = 0 and δ(t) > 0 for every t > 0. Then, S has a unique fixed point in Z.

Proof. By substituting s = 2 in Theorem 2.5, we will get the unique fixed point of S.

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