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Common Fixed-Point Theorems For Weakly Compatible Mapping

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Abstract:

This paper's goal is to gather some fresh data regarding common fixed points. In this study, we define a common fixed point for pairs of weakly compatible mappings that satisfy a generalized O-weak, after first proving a point of coincidence for a pair of mappings. The main result includes a condition and contraction. Next, we demonstrate that the fixed point in the main result is unique. Finally, an application supporting our findings is provided.

Keywords: *Common fixed point; point of coincidence; weakly compatible;*

Introduction and Preliminaries:

The Banach Contraction Principle, sometimes referred to as the Banach fixed point theorem asserts that each fixed point on a complete metric space has a distinct contraction map. This idea is widely applied as a fundamental tool to solve problems in both the pure and applied sciences.

In 1969, Boyd and Wong [3] replaced the constant k in Banach contractive condition by an upper semi-continuous function as follows:

Let (Z^*, Δ) be a complete metric space and O: $[0, \infty) \rightarrow [0, \infty)$ be upper semi continuous from the right such that $0 \leq O(t) < t$ for all $t > 0$. If T: $Z^* \rightarrow Z^*$ satisfies

$$\Delta(T(x), T(y)) \leq O(\Delta(x, y)) \quad \text{for all } x, y \in X,$$

then it has unique fixed-point $x \in Z^*$ and $\{T^n x\}$ converges to x for all $x \in Z^*$.

In essence, necessary requirements for the existence of fixed points are involved in fixed point theorems. Jungck [5] may have been the first to generalize Banach's contraction condition and use the idea of commutative pairs of mappings to achieve a unique fixed point. Jungck [5] first proposed the idea of compatible mappings in 1986. The concept of compatible mappings was expanded to include a broader class of mappings known as weakly compatible mappings by Jungck[5]in1996.

Alber and Guerre-Dela Briere [2] first proposed the idea of weak contraction in 1997. We now provide some fundamental definitions and findings that help support our main finding.

Definition 1.1: A metric space where every Cauchy sequence converges to a point in the space is called a complete metric space.

Definition 1.2: Two self-maps η and p of a metric space (Z, Δ) are called compatible if

$$\lim_{n\to\infty} \Delta(\eta\rho x_n, \rho\eta x_n) = 0,$$

Whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} \eta x_n = \lim_{n\to\infty} \rho x_n = t$ for some $t \in Z$

Definition 1.3: Two self-mappings η* and p*of a metric space (Z*, Δ) are said to be commuting if $\eta^* p^* x = p^* \eta^* x$ for all $x \in Z^*$.

Definition 1.4: Two self-mapping η* and p* of a metric space (Z*, Δ) are called weakly compatible if they commute at their coincidence point.

Definition 1.5: A mapping η*: $Z^* \rightarrow Z^*$ is said to be a weak contraction if for all $a, b \in Z^*$, there exists a function O: $[0, \infty) \rightarrow [0, \infty)$ with $O(t) > 0$ and $O(o) = 0$ such that

$$\Delta(\eta^* x, \eta^* y) \leq \Delta(x, y) - O(\Delta(x, y))$$

Here (Z*, Δ) be the metric space.

In this paper we proved common fixed for four self-mapping over a given metric space.

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Main Result

Let p* be a self – mapping on Z* and we assume that (Z*, Δ) be metric space .also let b:Z*×Z*→ $[0,\infty)$ $is\ a\ function\ and \psi\epsilon\Psi$ .Then we can say

(i) (Z*, Δ) is an b-complete metric space;

(ii) P*is an b-acceptable mapping;

(iii) P* is a alter b-ψ- rational contraction on Z*;

(iv) P*is an b-continuous mapping on Z*;

(v) There is z*∈Z* like so b (z*, p*z*) ≥1.

So p* has a fixed point

Theorem 1: Let η*, p*, ψ and λ be four self-mappings on a complete metric space (Z*, Δ) satisfying the following conditions:

(A1) $\eta^*(Z^*) \subset \lambda(Z^*)$, $p^*(Z^*) \subset \psi(Z^*)$;

(A2) $(1 + r\,\Delta(\psi x, \lambda x))\,\Delta(\eta^* x, p^* y)^2$

$$\leq r.max\{\tfrac{1}{2}\left(\Delta(\Psi x,\eta x)^2\Delta(\lambda y,p*y)+\Delta(\Psi x,\eta x)\Delta(\lambda y,p*y)\right),\Delta(\psi\,x,\eta*x)\Delta(\psi x,p*y)\Delta$$
$$(\lambda y,\eta*x),\Delta(\Psi x,p*y)\Delta(\lambda y,\eta*x)\Delta(\lambda y,p^*y)\}$$
$$+m(\psi x,\lambda y)-O(m(\psi x,\lambda y))$$

for all η*, y ∈ Z*, were

$$m(\psi x, \lambda y) = \max\{\Delta(\psi x, \lambda y)^2, \Delta(\psi x, \eta^* x)\,\Delta(\lambda y, p^* y), \Delta(\psi x, p^* y)\,\Delta(\lambda y, \eta^* x),$$
$$\frac{1}{2}[\Delta(\psi x, \eta^* x)\,\Delta(\psi x, p^* y) + \Delta(\lambda y, \eta^* x)\,\Delta(\lambda y, p^* y)]\},$$

p* ≥ 0 is a real number and O : [0, ∞) → [0, ∞) is a continuous function such that O(t) = 0 iff t = 0 and O(t) > 0 for all t > 0 and with the conditions (A1) and (A2) these four self-mapping of complete metric space (Z*, Δ) also satisfy that one of the subspaces ψZ*, λZ*, η*Z* and p*Z* be closed. Then

(i) ψ and η* have a point at coincidence;

(ii) λ and p* have a point of coincidence.

and also, if (ψ, η*) and (λ, p*) are weakly compatible, then η*, p*, ψ and λ have a unique common fixed point.

Proof: Let $x_0 \in Z^*$ be an arbitrary point from (A1), we can find an x_1 such that

$\eta^*(x_0) = \lambda(x_1) = y_0$ and for this x_1 one can find an $x_2 \in Z^*$ such that

$$P^*(x_1) = (x_2) = y_1$$

Continuing in this way, one can construct a sequence such that

$y_{2n} = \eta^*(x_{2n}) = \lambda(x_{2n+1})$, $y_{2n+1} = p^*(x_{2n+1}) = \psi(x_{2n+1})$ for all $n \geq 0$ and $\{y_n\}$ is a Cauchy Sequence in Z^*.

Now let us assume that ψZ^* is closed subspace of (Z^*, Δ). As we know that subspace of a complete metric space is complete if and only if it is closed. Hence subspace ψZ^* is complete subspace of complete metric space (Z^*, Δ).

As $\psi Z^*$is a complete subspace of Z^*. Then there exists $z^* \in Z^*$ such that

$$y_{2n+1} = p^*(x_{2n+1}) = \psi(x_{2n+1}) \to z^*$$

$n \to \infty$ consequently, we can find $w \in Z^*$ such that $\psi w = z^*$. Further, a Cauchy sequence $\{y_n\}$ has a convergent subsequence $\{y_{2n+1}\}$ and so the sequence $\{y_n\}$ converges and hence a subsequence $\{y_{2n}\}$ also converges. Thus, we have

$$y_{2n+1} = \eta^*(x_{2n}) = \lambda(x_{2n+1}) \to z^* \text{ as } n \to \infty.$$

Letting $x = w$ and $y = z^*$ in (A2), we get

$[1 + r\Delta(\psi w, \lambda z^*)]\ \Delta(\eta^* w, p^* z^*)^2 \leq r \max \{\frac{1}{2}\ [\Delta(\psi w, \eta^* w)^2\ \Delta(\lambda z^*, p^* z^*) + \Delta(\psi(w, \eta^* w)$

$\Delta(\lambda z^*, p^* z^*)^2]$,

$\Delta(\psi w, \eta^* w)\ \Delta(\psi w, p^* z^*)\ \Delta(\lambda z^*, \eta^* w), \Delta(\psi w, p^* z^*)\ \Delta(\lambda z^*, \eta^* w)\ \Delta(\lambda z^*, p^* z)\}$

$+ m(\psi w, \lambda z^*) - O(m(\psi w, \lambda z^*))$,

where

$m(\psi w, \lambda z^*) = \max \{\Delta(\psi w, \lambda z^*)^2, \Delta(\psi w, \eta^* w)\ \Delta(\lambda z^*, p^* z^*), \Delta(\psi w, p^* z^*)\ \Delta(\lambda z^*, \eta^* w),$

$\frac{1}{2}[\Delta(\psi w, \eta^* w)\Delta(\psi w, p^* z^*) + \Delta(\lambda z^*, \eta^* w)\ \Delta(\lambda z^*, p^* z^*)]\}$.

Since

$M(\psi w, \lambda z^*) = \max \{\Delta(z^*, z^*)^2, \Delta(z^*, \eta^* w)\ \Delta(p^* z^*, p^* z^*), \Delta(z^*, z^*)\ \Delta(z^*, \eta^* w),$

$$\frac{1}{2}[\Delta(z^*, \eta^* w)\ \Delta(z^*, z^*) + \Delta(z^*, \eta^* w)\ \Delta(p^* z^*, p^* z^*)]\} = 0,$$

[As $\Delta(x, x) = 0$ according to the definition of metric space]

$[1 + r\Delta(z^*, z^*)]\ \Delta(\eta^*w, z^*)^2 \le r \max \{\frac{1}{2}[\Delta(z^*, \eta^*w)^2\ \Delta(z^*, z^*) + \Delta(z^*, \eta^*w)\ \Delta(z^*, z^*)^2],$

$\Delta(z^*, \eta^*w)\ \Delta(z^*, z^*)\ \Delta(z^*, \eta^*w), \Delta(^*z, z^*)\ \Delta(z^*, \eta^*w)\ \Delta(z^*, z^*)\} + \phi(0).$

This implies that $\eta^*w = z^*$ and so $\eta^*w = \psi w = z^*$. Therefore, w is a coincidence point of ψ and η^*.

Since $z^* = \eta^*w \in \eta^*Z^* \subset \lambda Z^*$, there exists $v \in Z^*$ such that $z^* = \lambda v$.

Now we claim that $p^*v = z^*$. Letting as $x = x_{2n}$ and $y = v$ in (A2) we get

$[1 + r\Delta(\psi x_{2n}, \eta^* \lambda v)]\Delta(\eta^*x_{2n}, p^*v)^2$

$\le r \max \{\frac{1}{2}[\Delta(\psi x_{2n}, \eta^*x_{2n})^2\ \Delta(\lambda v, p^*v) + \Delta(\psi x_{2n}, \eta^*x_{2n})\ \Delta(\lambda v, p^*v)^2],$

$\Delta(\psi\ x_{2n}, \eta^*\ x_{2n})\ \Delta(\psi\ x_{2n}, p^*v)\Delta(\lambda z^*, \eta^*x_{2n}), \Delta(\psi x_{2n}, p^*v)\ \Delta(\lambda v, \eta^*x_{2n})\ \Delta(\lambda v, p^*v)\}$

$+m\ (\psi x_{2n}, \lambda v) - O\ (m\ (\psi x_{2n}, \lambda v)),$

where

$m\ (\psi x_{2n}, \lambda v) = \max\ \{\Delta(\psi x_{2n}, \lambda v)^2, \Delta(\psi x_{2n}, \eta^*x_{2n})\ \Delta(\lambda v, p^*v), \Delta(\psi x_{2n}, p^*v)\ \Delta(\lambda v, \eta^*x_{2n},),$

$\frac{1}{2}[\Delta(\psi x_{2n}, \eta\ ^*x_{2n})\ \Delta(\psi\ x_{2n}, p^*v) + (\lambda v, \eta^*\ x_{2n})\ \Delta(\lambda v, p^*v)]\} = 0.$

Therefore,

$[1 + r\Delta(z^*, z^*)]\ \Delta(z^*, p^*v)^2 \le r \max\ \{1/2[0 + 0], 0, 0\} + 0 - O\ (0).$

This gives $z^* = p^*v$ and hence $z^* = p^*v = \lambda v$.

Therefore, v is a coincidence point of λ and p^*.

Since $\eta^*z^* = \eta^*(\psi w) = \psi(\eta^*w) = \psi z^*$,

$P^*z^* = p^*(\lambda v) = \lambda(p^*v) = \lambda z^*$.

Now, we show that $\eta^*z^* = z^*$, for this, letting $x = z^*$ and $y = x_{2n+1}$ in (A2), we get

$[1 + r\Delta(\psi z^*, \lambda\ x_{2n+1})]\Delta(\eta^*z^*, p^*x_{2n+1})^2$

$\le r \max\ \{\frac{1}{2}[\Delta(\psi z^*, \eta^*z^*)^2\ \Delta(z^*, z^*) + \Delta(\psi z^*, \eta^*z^*)\ \Delta(z^*, z^*)^2],$

$\Delta(\psi z^*, \eta^*z^*)\ \Delta(\psi z^*, z^*)\ \Delta(z^*, \eta^*z^*), \Delta(\psi z^*, z^*)\ \Delta(z^*, \eta^*z^*)\ \Delta(z^*, z^*)]\}$

$+m\ (\psi z^*, z^*) - O\ (m\ (\psi z^*, z^*)),$

where

$m\ (\psi z^*, z^*) = \max\ \{\psi z^*, z^*)^2, \Delta(\psi z^*, \eta^*z^*)\ \Delta(z^*, z^*), \Delta(\psi z^*, z^*)\ \Delta(z^*, \eta^*z^*),$

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$\frac{1}{2}$ [Δ(ψz*, η*z*) Δ(ψz*, z*) + Δ(z*, η*z*) Δ(z*, z*)]}

= Δ(η*z*, z*)^2

Therefore, we have

[1 + rΔ(η*z*, z*)] Δ(η*z*, z*)^2 ≤ r max {$\frac{1}{2}$ [0 + 0], 0, 0} + Δ(η*z*, z*)^2 – O (Δ(η*z*, z*)^2).

Thus, we get Δ(η*z*, z*)^2 = 0. This implies that η*z* = z*. Hence η*z* = ψz* = z*.

Next, we claim that p*z *= z*. Now letting x = x_{2n} and y = z* in (A2) we get

[1 + rΔ(ψ x_{2n}, λz*)] Δ(η *x_{2n}, p*z*)^2

≤ r max {$\frac{1}{2}$ [Δ(ψ x_{2n}, n x_{2n})^2 Δ(λz*, p*z*) + Δ(ψ x_{2n}, η* x_{2n}) Δ(λz*, p*z*)^2],

Δ(ψ x_{2n}, η* x_{2n}) Δ(ψ x_{2n}, p*z*) Δ(λz*, η* x_{2n}), Δ(ψ x_{2n}, p*z*) Δ(λz*, η *x_{2n}), Δ(λz*, p*z*)} + m (ψx_{2n}, λz*) – O (m (ψx_{2n}, λz*),

where

m(ψx_{2n}, λz*) = max{Δ(ψ x_{2n}, λz*)^2, Δ(ψ x_{2n}, η* x_{2n}) Δ(λz*, p*z*), Δ(ψ x_{2n}, p*z*) Δ(λz*, η* x_{2n}), $\frac{1}{2}$ [Δ(ψ x_{2n}, η* x_{2n}) Δ(ψ x_{2n}, p*z*) + Δ(λz*, η* x_{2n}) Δ(λz*, p*z*)]}

= Δ(z*, p*z*)^2.

Hence, we get

[1 + r Δ(z*, p*z*)] Δ(z*, p*z*)^2 ≤ r. max {$\frac{1}{2}$ [0 + 0], 0, 0} + Δ(z*, p*z*)^2 – O (Δ(z*, p*z*)^{*2}).

gives z* = p*z* and so z* = p*z* = λz*. Therefore, z* is a common fixed point of η*, p*, ψ and λ.

Similarly, we can complete the proofs for the cases that subspaces λZ* or η*Z* or p*Z* is closed.

Now, we have to prove the uniqueness. Suppose z* and w are two common fixed points of η*, p*, ψ and λ with z* and w are distinct. Letting x = z* and y = w. We get

[1 + r Δ(ψz*, η*w)] Δ(η*z, p*w)^2

≤ r. max {$\frac{1}{2}$ (Δ(ψz*, η*z*)^2 Δ(λw, p*w) + Δ(ψz*, η*z) Δ(λw, p*w)^2,

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$\Delta(\psi z^*, \eta^* z^*)\ \Delta(\psi z^*, p^* z^*)\ \Delta(\lambda w, \eta^* z^*),$

$\Delta(\psi z^*, p^* w)\ \Delta(\lambda w, \eta^* z^*)\ \Delta(\lambda w, p^* w)\}$

$+ m(\psi z^*, \lambda w) - O(m(\psi z^*, \lambda w))$

$$m(\psi z^*, \lambda w) = \max\{\Delta(\psi z^*, \lambda w)^2, \Delta(\psi z^*, \eta^* z^*)\ \Delta(\lambda w, p^* w), \Delta(\psi z^*, p^* w)\ \Delta(\lambda w, \eta^* z^*),$$

$$\frac{1}{2}[\Delta(\psi z, \eta^* z^*)\ \Delta(\psi z^*, p^* w) + \Delta(\lambda w, \eta^* z^*)\ \Delta(\lambda w, p^* w)]\}$$

$$= 0$$

So

$[1 + r\,\Delta(\psi z^*, \eta^* w)]\ \Delta(\eta^* z^*, p^* w)^2$

$\leq r.\max\{0, 0, 0\} + m(\psi z^*, \lambda w) - O(m(\psi z^*, \lambda w)$

$= r.\max\{0, 0, 0\} + 0 - O(0)$

$= r \times 0 + 0 - 0$

$= 0$

$[1 + r\,\Delta(\psi z^*, \eta^* w)]\ \Delta(\eta^* z, p^* w)^2 \leq 0$

which implies that

$\Delta(\eta^* z^*, p^* w)^2 = 0.$

as z* and w are fixed point so

$\eta^* z^* = z^*$ $\qquad p^* w = w$

$\Delta(z^*, w)^2 = 0$

Hence z* = w

This proves the uniqueness of the fixed point.

Hence proved.

Example: Let Z = [2, 18] and Δ be a usual metric. Define self-mapping η, ρ,ψ and λon Z by

$$\eta x = \begin{cases} 2 & \text{if} \quad x = 2 \\ 12 & \text{if} \quad 2 < x \leq 5 \\ x - 3 & \text{if} \quad x > 5 \end{cases} \qquad \rho x = \begin{cases} 2 & \text{if} \quad x = 2 \\ 6 & \text{if} \quad x > 2 \end{cases}$$

$$\psi x = \begin{cases} 2 & \text{if} \quad x = 2 \\ 6 & \text{if} \quad 2 < x \leq 5 \\ 2 & \text{if} \quad x > 5 \end{cases} \qquad \lambda x = \begin{cases} x & \text{if} \quad x = 2 \\ 3 & \text{if} \quad x > 2 \end{cases}$$

Let us consider a sequence $\{y_n\}$ with $y_n = 2$. As we find that all the conditions of the main result are satisfied. So, this theorem is applicable in this example and hence 2 is the unique common fixed point of η, p, ψ and λ.

Conclusion:

We proved a common fixed-point theorem for pairs of weakly compatible mappings satisfying a generalized O-weak contraction condition and also a different distance condition of metric function. And also, we have given an example in support of our result.

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