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Fixed Point Theorems and its Applications in Fuzzy Metric Spaces

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Abstract:
Main aim of this paper is to prove some fixed point theorems in Fuzzy Metric spaces through rational inequality. Our results extends and generalizes the results of many other authors existing in the literature. Some applications are also given in support of our results.

Keywords: *Fuzzy Metric Space, Rational Expression, Integral Type, Control Function.*

1. Introduction

The foundation of fuzzy mathematics is laid by Lofti A Zadeh [3] with the introduction of fuzzy sets in 1965. This foundation represents a ambiguous in everyday life. Subsequently several authors have applied various from general topology of fuzzy sets and developed the concept of fuzzy space. In 1975, Kramosil and Michalek [5] introduced concept of fuzzy metric spaces. In 1988, Mariusz Grabiec [4] extended fixed point theorem of Banach and eldestien to fuzzy metric spaces in the sense of Kramosil and Michalek. In 1994, George et al. [1] modified the notion of fuzzy metric spaces with the help of continuous t-norms. A number of fixed point theorem have been obtained by various authors in fuzzy metric space by using the concept of compatible map, implicit relation, weakly compatible map, R weakly compatible map. (See section: [7-14]). Also R.K. Saini and Vishal Gupta [9, 10] proved some fixed points theorems.

On expansion type maps and common coincidence points of R-Weakly commuting fuzzy maps, in Fuzzy Metric Space. The present paper extends and generalizes the results of Mariusz Grabeic [4] and also many other authors existing in the literature.

2. Preliminaries

In this section, we define some important definition and results which are used in sequel.

Definition 1 ([3]): Let X' be any set. A fuzzy set A in X' is a function with domain X' and values in [0, 1].

Definition 2 ([2]): A binary operation $*: [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norms if ([0, 1]), $*$) is an abelian topological monoid with the unit 1 such that $a * b \le c * d$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0, 1]$.

Definition 3 ([5]): A triplet $(X', M', *)$ is a fuzzy metric space if X' is a an arbitrary set, $*$ is continuous t-norm and M' is a fuzzy set on $X'^2 \times (0, \infty)$ satisfying the following conditions, for all $x, y, z \in X'$, such that $t, s \in (0, \infty)$.

1. $M'(x, y, t) > 0$
2. $M'(x, y, t) = 1$ iff $x = y$
3. $M'(x, y, t) = M'(y, x, t)$
4. $M'(x, y, t) * M'(y, z, s) \le M'(x, z, t + s)$.
5. $M'(x, y, .) : [0, \infty) \to [0, 1]$ is continuous.

Then M' is called a fuzzy metric on X' and $M'(x, y, t)$ denotes the degree of nearness between x and y with respect to t.

Definition 4 ([4]): Let $(X', M', *)$ is a fuzzy metric space then a sequence $[x_n] \in X'$ is said to be convergent to a point $x \in X'$ if $\lim_{n\to\infty} M'(x_n, x, t) = 1$ for all $t > 0$.

Definition 5 ([4]): Let $(X', M', *)$ is a fuzzy metric space then a sequence $[x_n] \in X'$ is called Cauchy sequence if $\lim_{n\to\infty} M'(x_{n+p}, x_n, t) = 1$ for all $t > 0$ and $p > 0$.

Definition 6 ([4]): Let $(X', M', *)$ is a fuzzy metric space then an Fuzzy Metric space in which every Cauchy sequence is convergent is called complete. It is called compact, if every sequence contains a convergent subsequence.

Lemma 1 ([4]): For all, x, y $\in$ X′, M′(x, y, ·) is non-decreasing.

Lemma 2 ([11]): If there exist k $\in$ (0, 1) such that M′(x, y, kt) $\geq$ M(x, y, t) for all x, y $\in$ X′ and t $\in$ (0, ∞), then x = y.

Now we prove our main result.

3. Main Results

Theorem 1: Let (X′, M′, $*$) be a complete fuzzy metric space and f: X′ $\to$ X′ be a mapping satisfying

$$M'(x, y, t) = 1 \tag{1.1}$$

and

$$M'(fx, fy, kt) \geq \lambda(x, y, t) \tag{1.2}$$

where

$$\lambda(x,y,t) = \min\left\{\frac{M'(y, fy,t)[1+M'(x, fx,t)]}{[1+M(x,y,t)]}, M(x,y,t)\right\} \tag{1.3}$$

for all x, y $\in$ X′ and k $\in$ (0, 1). Then f has a unique fixed point.

Proof: Let us consider x $\in$ X′ be any arbitrary point in X′. Now construct a sequence $[x_n] \in$ X′ such that $fx_n = x_{n+1}$ for all n $\in$ N.

Claim: $\{x_n\}$ is a Cauchy sequence.

Let us take $x = x_{n-1}$ and $y = x_n$ in (1.2), we get

$$M'(x_n, x_{n+1}, kt) = M'(f x_{n-1}, f x_n, kt) \geq \lambda(x_{n-1}, x_n, t) \tag{1.4}$$

Now

$$\lambda(x_{n-1}, x_n, t) = \min\left\{\frac{M'(x_n, fx_n, t)[1+M'(x_{n-1}, fx_{n-1}, t)]}{[1+M(x_{n-1}, x_n, t)]}, M(x_{n-1}, x_n, t)\right\}$$

$$\lambda(x_{n-1}, x_n, t) = \min\left\{\frac{M'(x_n, x_{n+1}, t)[1+M'(x_{n-1}, x_n, t)]}{[1+M(x_{n-1}, x_n, t)]}, M(x_{n-1}, x_n, t)\right\}$$

$$\Rightarrow \lambda(x_{n-1}, x_n, t)$$

$$= \min\{M'(x_n, x_{n+1}, t), M'(x_{n-1}, x_n, t)\}$$

Now if $M'(x_n, x_{n+1}, t) \leq M'(x_{n-1}, x_n, t)$, then from equation (1.4)

$$M'(x_n, x_{n+1}, kt) \geq M'(x_n, x_{n+1}, t)$$

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Hence from lemma (2), our claim follows immediately. Now suppose $M'(x_n, x_{n+1}, t) \geq M'(x_{n-1}, x_n, t)$ then again from equation (1.4),

$$M'(x_n, x_{n+1}, kt) \geq M'(x_{n-1}, x_n, t)$$

Now by simple induction, for all n and t > o, we get

$$M'(x_n, x_{n+1}, kt) \geq M'\left(x, x_1, \frac{t}{k^{n-1}}\right) \quad (1.5)$$

Now for any positive integer 's', we have

$$M'(x_n, x_{n+s}, t) \geq M'\left(x_n, x_{n+1}, \frac{t}{s}\right) * \ldots(s)\ldots * M'\times\left(x_{n+p-1}, x_{n+p}, \frac{t}{s}\right)$$

By using equation (1.5), we get

$$M'(x_n, x_{n+s}, t) \geq M'\left(x, x_1, \frac{t}{sk^n}\right) * \ldots(s)\ldots * M'\times\left(x, x_1, \frac{t}{sk^n}\right)$$

Now taking $\lim_{n\to\infty}$ and using (1.1), we get

$$\lim_{n\to\infty} M'(x_n, x_{n+s}, t) = 1 \quad (1.6)$$

This implies, $\{x_n\}$ is a Cauchy sequence. Call the limit υ.

Claim: υ is a fixed point of f.

Consider

$$M'(\upsilon, f\upsilon, t) \geq M'(fx_n, f\upsilon, t) * M'(\upsilon, x_{n+1}, t)$$

$$\geq \lambda\left(x_n, \upsilon, \frac{t}{2k}\right) * M'(\upsilon, x_{n+1}, t) \quad (1.7)$$

Now

$$\lambda\left(x_n, \upsilon, \frac{t}{2k}\right) = \min\left\{\frac{M'\left(\upsilon, f\upsilon, \frac{t}{2k}\right)\left[1 + M'\left(x_n, fx_n, \frac{t}{2k}\right)\right]}{\left[1 + M'\left(\upsilon, x_n, \frac{t}{2k}\right)\right]}, M'\left(\upsilon, x_n, \frac{t}{2k}\right)\right\}$$

Taking $\lim_{n\to\infty}$ in above inequality and using (1.1), we get

$$\lambda\left(\upsilon, \upsilon, \frac{t}{2k}\right) = \min\left\{M'\left(\upsilon, f\upsilon, \frac{t}{2k}\right), 1\right\}$$

Now if $M'\left(\upsilon, f\upsilon, \frac{t}{2k}\right) \geq 1$ then $\lambda\left(\upsilon, \upsilon, \frac{t}{2k}\right) = 1$

Therefore from (1.7) and using definition 3, we get υ is a fixed point of f.

Now if $M'\left(\upsilon, f\upsilon, \frac{t}{2k}\right) \le 1$ then $\lambda\left(\upsilon, \upsilon, \frac{t}{2k}\right) = M'\left(\upsilon, f\upsilon, \frac{t}{2k}\right)$.

Hence from equation (1.7), we get

$$M'(\upsilon, f\upsilon, t) \ge M'\left(\upsilon, f\upsilon, \frac{t}{2k}\right) * M'(x_{n+1}, \upsilon, t) \tag{1.8}$$

Now taking $\lim_{n\to\infty}$ in (1.8) and using equation (1.1) and lemma (2), we get $f\upsilon = \upsilon$.

Uniqueness: Now we show that υ is a unique fixed point of f. Suppose not, then there exist a point $\omega \in X'$ such that $f\omega = \omega$. Consider

$$1 \ge M'(\omega, \upsilon, t) = M'(f\omega, \upsilon, t) \ge \lambda\left(\omega, \upsilon, \frac{t}{k}\right) \tag{1.9}$$

where

$$\lambda\left(\omega, \upsilon, \frac{t}{k}\right) = \min\left\{\frac{M'\left(\upsilon, f\upsilon, \frac{t}{k}\right)\left[1 + M'\left(\omega, f\omega, \frac{t}{k}\right)\right]}{\left[1 + M'\left(\omega, \upsilon, \frac{t}{k}\right)\right]}, M'\left(\omega, \upsilon, \frac{t}{k}\right)\right\}$$

$$\lambda\left(\omega, \upsilon, \frac{t}{k}\right) = \min\left\{\frac{M'\left(\upsilon, \upsilon, \frac{t}{k}\right)\left[1 + M'\left(\omega, \omega, \frac{t}{k}\right)\right]}{\left[1 + M'\left(\omega, \upsilon, \frac{t}{k}\right)\right]}, M'\left(\omega, \upsilon, \frac{t}{k}\right)\right\}$$

$$\lambda\left(\omega, \upsilon, \frac{t}{k}\right) = \min\left\{\frac{2}{\left[1 + M'\left(\omega, \upsilon, \frac{t}{k}\right)\right]}, M'\left(\omega, \upsilon, \frac{t}{k}\right)\right\}$$

$$= \min\{1, 1\}$$

$$\Rightarrow \lambda\left(\omega, \upsilon, \frac{t}{k}\right) = 1 \tag{1.10}$$

Use it in (1.9), we get $\omega = \upsilon$. Thus υ is unique fixed point of f. This completes the proof of Theorem 1.

Let us define $\Phi = \{\phi/\phi : [0, 1] \to [0, 1]\}$ is a continuous function such that $\phi(1) = 1$, $\phi(0) = 0$, $\phi(\alpha) > \alpha$ for each $0 < \alpha < 1$.

Example 1: Consider the complete fuzzy metric space $(X', M', *)$ where $X' = \{0, 1\}$ and the fuzzy metric M' is given, for each $t \in [0, \infty]$ by

$$M'(x,y,t) = \begin{cases} x*y & if \quad x \neq y \\ 1 & if \quad x = y \end{cases}$$

Consider the function $\delta \in \Delta$ given by

$\delta(\in) = \in/2$ for $\in \in [0, 1]$

Next, define the self-mapping $T_1 : X' \to X'$ by $T_1(0) = 1$ & $T_1(1) = 0$. It is a simple matter to check that T_1 is a fuzzy Meir-Keeler Contractive mapping with respect to δ. Nowever T_1 does not have fixed points. Observe that $*t > 0$

$M'(x, T(x), t) = 0 \qquad$ for all $x \in X'$

Now, define the self-mapping $T_2(1) = 1$. It is clear that T_2 is also a fuzzy Meir-Keeler Contractive mapping with respect to δ clearly T_2 has two fixed points.

Theorem 2: Let $(X', M', *)$ be a complete fuzzy metric space and $f : X' \to X'$ be a mapping satisfying

$$M'(x, y, t) = 1 \tag{1.11}$$

and

$$M'(fx, fy, kt) \geq \phi\{\lambda(x, y, t)\} \tag{1.12}$$

where

$$\lambda(x,y,t) = \min\left\{\frac{M'(y, fy, t)[1 + M'(x, fx, t)]}{[1 + M'(x, y, t)]}, M'(x, y, t)\right\} \tag{1.13}$$

for all $x, y \in X'$, $k \in (0, 1)$, $\phi \in \Phi$. Then f has a unique fixed point.

Proof: Since $\phi \in \Phi$. This implies that $\phi(\alpha) > \alpha$ for each $0 < \alpha < 1$. Thus from (1.12)

$M'(fx, fy, kt) \geq \phi\{\lambda(x, y, t)\} \geq \lambda(x, y, t)$

Now, applying Theorem 1, we obtain the desired result.

4. Applications

In this section, we gives some application related to our results. Let us define $\Psi: [0,\infty] \to [0,\infty]$, as $\Psi(t) = \int_0^t \varphi(t)dt \ \forall t > 0$, be a non-decreasing and continuous function. Moreover, for each $\varepsilon > 0$, $\varphi(\varepsilon) > 0$. Also implies that $\varphi(t) = 0$ iff $t = 0$.

Theorem 3: Let $(X', M', *)$ be a complete fuzzy metric space and $f : X' \to X'$ be a mapping satisfying

$M'(x, y, t) = 1$

$$\int_0^{M'(fx,fy,kt)} \varphi(t)dt \geq \int_0^{\lambda(x,y,t)} \varphi(t)dt$$

where

$$\lambda(x,y,t) = \min\left\{\frac{M'(y,fy,t)[1+M'(x,fx,t)]}{[1+M'(x,y,t)]}, M'(x,y,t)\right\}$$

for all x, y ∈ X′, φ ∈ Ψ and k ∈ (0, 1). Then f has a unique fixed point.

Proof: By taking φ(t) = 1 and applying Theorem 1, we obtain the result.

Theorem 4: Let (X′, M′, ∗) be a complete fuzzy metric space and f : X′ → X′ be a mapping satisfying

M′(x, y, t) = 1

$$\int_0^{M'(fx,fy,kt)} \varphi(t)dt \geq \phi\left\{\int_0^{\lambda(x,y,t)} \varphi(t)dt\right\}$$

where

$$\lambda(x,y,t) = \min\left\{\frac{M'(y,fy,t)[1+M'(x,fx,t)]}{[1+M'(x,y,t)]}, M'(x,y,t)\right\}$$

for all x, y ∈ X′, φ ∈ Ψ, k ∈ (0, 1) and ϕ ∈ Φ. Then f has a unique fixed point.

Proof: Since ϕ(α) > α for each 0 < α < 1, therefore result follows immediately from Theorem 3.

Remark 1: Our paper extends and generalizes the result of Mariusz Grabeic [4] and also many other authors existing in the literature.

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References

[1] A. George and P. Veermani, On some results in fuzzy metric spaces, *Fuzzy Sets and Systems,* 64 (1994), 395-399.

[2] B. Schweizer and A. Sklar, Probabilistic Metric Spaces, North-Holland Series in Probability and Applied Mathematics, North-Holland Publishing Co., New York (1983), ISBN: 0-444-00666-4 MR0790314 (86g:54045).

[3] L. A. Zadeh, Fuzzy sets, *Inform. Control,* 8 (1965), 338-353.

[4] M. Grabeic, Fixed points in fuzzy metric spaces, *Fuzzy Sets and Systems,* 27 (1988), 385-389.

[5] O. Kramosil and J. Michalek, Fuzzy metric and statistical metric spaces, *Kybornetica,* 11 (1975), 326-334.

[6] P. Balasubramaniam and S. Murlisankar and R. P. Pant, Commong fixed points of four mappings in a fuzzy metric Common fixed points of four mappings in a fuzzy metric space, *J. Fuzzy Math.,* 10(2) (2002), 379-384.

[7] R. Vasuki, A common fixed point theorem in a fuzzy metric space. *Fuzzy Sets and Systems,* 97 (1998), 395-397.

[8] R. Vasuki, Common fixed point for R-weakly commuting maps in fuzzy metric space, *Indian J. Pure Appl. Math.,* 30 (1999), 419-423.

[9] R. K. Saini and V. Gupta, Fuzzy Version of Some Fixed Points Theorems On Expansion Type Maps in Fuzzy Metric Space, *Thai Journal of Mathematics, Mathematical Assoc. of Thailand,* 5, No. 2 (2007), 245-252, ISSN 1686-0209.

[10] R. K. Saini and V. Gupta, Common coincidence Points of R-Weakly Commuting Fuzzy Maps, *Thai Journal of Mathematics, Mathematical Assoc. of Thailand,* 5, No. 2 (2007), 109-115, ISSN 1686-0209.

[11] S. N. Mishra, S.N. Sharma and S. L. Singh, Common fixed point of maps of fuzzy metric spaces, *Internat. J. Math. Sci.* 17 (1994), 253-58.

[12] Satish Shukla, Shweta Rai and Juan-Jose Minana, On graphical fuzzy metric spaces and related fixed point theorems, Fixed Point Theory, Volume 25, No.2, 2024, 723-746.

[13] SEONG. HOON. CHOET. AL., On common fixed point theorems in fuzzy metric spaces, *Int. Mathematical Forum,* 1 (No. 9-12) (2006), 471-479.

[14] Vatentin. Gregori and Almanzor, Sapena, On fixed point theorems in fuzzy metric spaces, *Fuzzy Sets and Systems,* 125 (2002), 245-252.

[15] Y. J. Cho and S. Sedghi and N. Shobe, Generalized fixed point theorems for Compatible mappings with some types in fuzzy metric spaces, Chaos, Solitons and Fractals, 39 (2009), 2233-2244.