International Journal of Advance Research in Science and Engineering
Volume No.10, Issue No. 03, March 2021
www.ijarse.com
IJARSE
ISSN 2319 - 8354

MODIFIED ADOMIAN DECOMPOSITION METHOD TO INITIAL VALUE PROBLEMS

ANJU DHIRAJ KUMAR NAIDU AND AMIT UJLAYAN

ABSTRACT.

In this paper we study a modified Adomian decomposition method to solve an initial value problem. This method may be applied to a singular or a nonsingular problem. In particular, we may apply this method to a Riccati differential equation. Finally ,acomparison is made between the standard and the modified Adomian decomposition method using some examples.

1. INTRODUCTION

In the last two decades, many papers have been devoted on the Adomian decomposition method and applied to a variety of the problems governed by certain differential equations, integral equations and integro-differential equations .On the performance of the Adomian method in scientific applications, there are several comparative studies made between the Adomian decomposition method and the other existing methods. All results of these investigations have shown that the Adomian decomposition method is quite effective and practical. Bellomo et al. [3]has done a comparison between the Adomian decomposition method and the perturbationtechniques to show that the Adomian decomposition method provides improvement over the perturbation techniques and minimizes the volume of the computational work. In Bellomo [3], a study was conducted by the analytical discussion supported by some numerical examples to demonstrate the power of the method. Rach [4] conducted a comparative study, using some examples, between the Adomian decomposition method and Picard's iteration method. It has been shown by Rach[4], that the similarity between the Adomian decomposition method and Picard's method is purely superficial. Wazwaz[5],[6] has carried out a comparison between theAdomian decomposition method with the Taylor's series solution method and have shown the advantages of the Adomian decomposition method concerning the calculation size.

Before closing these remarks, we point out that the Adomian decomposition method has been used in various classes of differential, integral and integro-differential equations. The main advantage of the Adomian decomposition method is to provide analytical approximation to a variety of linear and nonlinear equations without going into the linearization, perturbation, or discretization methods which may bring in various kinds of errors in the calculation of an approximate solution.

1991 Mathematics Subject Classification. 49M27, 34A34, 35F15, 35F25, 35F20.

Key words and phrases. Initial value problem, Adomian decomposition method, modified Adomian decomposition method , Riccati differential equation.

In this paper we study an initial value problem of the following type:

$$y'(x) + p(x)y(x) + f\big(x, y(x)\big) = g(x), \quad \forall x \geq 0, \quad y(0) = a, \tag{1.1}$$

where a is a constant, $p : \mathbb{R} \to \mathbb{R}$, $p(x) \neq 0\ for\ all\ x \geq 0$, $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}\ and\ g : \mathbb{R} \to \mathbb{R}$ is a polynomial function. In the present study we apply a modified Adomian decomposition method to analyse (1.1). This method is equally applicable to a singular problem or a non-singular problem. Our work is motivated by the study of Wazwaz [1] and reference cited therein. Author in [1] has been applied the modified decomposition method to solve a second order initial value problem. The organization of the present study is as follows. In the next section we give basics about the modified Adomian method. At the last a comparative study is made between the standard Adomian method [2] and modified Adomian decomposition method with the help of some examples.

2. Modified Adomian decomposition method

Let us consider the initial value problem (1.1) , we define an operator on the space of continuously differentiable functions, as

$$L(\,.\,) = e^{-\int p(x)dx}\ \frac{d}{dx}\Big(e^{-\int p(x)dx}\ (.)\Big) \tag{2.1}$$

Using this operator we can rewrite (1.1) as

$$L(y) = g(x) - f(x, y) \tag{2.2}$$

Now we define a operator known as the inverse operator (cf. ch. 2, [7]) for the operator $L(\,.\,)$ given in (2.1)

$$L^{-1}(\,.\,) = e^{-\int p(x)dx} \int_0^x e^{-\int p(x)dx}\ (\,.\,)\, dx\,. \quad (2.3)$$

Applying the inverse operator L^{-1} on both the sides of (2.2) , we have

IJARSE
ISSN 2319 - 8354

$$y(x) = \psi(x) + L^{-1}(g(x)) - L^{-1}f(x,y). \qquad (2.4)$$

Here ψ is a function satisfying $L\psi(x) = 0$ .A possible example for such function $\psi(x)$ is given as follows:

$$\psi(x) = \begin{cases} \alpha_0 & if\ L(\,.\,) = \frac{d}{dx} \\ \alpha_0 + \alpha_1 x & if\ L(.) = \frac{d^2}{dx^2} \\ \alpha_0 + \alpha_1 x + \alpha_2 \frac{x^2}{2} & if\ L(.) = \frac{d^3}{dx^3} \end{cases} \qquad (2.5)$$

where $\alpha_i, i = 0,1,2$ are given constant . To solve (1.1), we use the Adomian decomposition method , which introduces the solution of $y(x)$ by an infinite series of components

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \qquad (2.6)$$

and the nonlinear function $f(x,y)$ by an infinite series of polynomials

$$f(x,y) = \sum_{n=0}^{\infty} A_n \qquad (2.7)$$

Where the components $y_n(x)$ of the solution $y(x)$ will be determined by a recursive formula and A_n, the Adomian polynomials are determined as per the following schemes :

$$A_0 = f(u_0)\,,$$

$$A_1 = f^{'}(u_0)u_1\,,$$

$$A_2 = f^{'}(u_0)u_2 + \frac{1}{2}f^{''}(u_0)u_1^2\,,$$

$$A_3 = f^{'}(u_0)u_3 + \frac{1}{3}f^{'''}(u_0)u_1^3\,,$$

$$A_4 = f^{'}(u_0)u_4 + \left(\frac{u_2^2}{2!} + u_1u_3\right)f^{''}(u_0) + \left(\frac{u_1^2u_2}{2!}\right)f^{'''}(u_0) + \frac{1}{4!}u_1^4f^{''''}(u_0)\,,$$

$$\vdots \ = \quad \vdots \qquad (2.8)$$

Using (2.6) – (2.8) , we have

$$\sum_{n=0}^{\infty} y_n = \psi(x) + L^{-1} g(x) - L^{-1} \sum_{n=0}^{\infty} A_n .$$

The components $y_n(x)$ of the solution $y(x)$, can be easily determined by the following recurrence relations

$$\begin{cases} y_0 = \psi(x) + L^{-1} g(x), \\ y_{n+1} = -L^{-1} A_n , \qquad n \geq 0 \end{cases}$$

Or

$$y_0 = \psi(x) + L^{-1} g(x),$$

$$y_1 = -L^{-1} A_0 ,$$

$$y_2 = -L^{-1} A_1 ,$$

$$y_3 = -L^{-1} A_2 ,$$

$$\vdots \ = \quad \vdots$$

$$y_n = -L^{-1} A_{n-1} , \qquad (2.9)$$

Using (2.8) – (2.9) , we can easily find $y_n(x), n \geq 0$. Define n-terms approximation to the solution $y(x), by\ \psi_n(y) = \sum_{i=0}^{n-1} y_i$ and so the solution may be obtained as the limit of the n^{th} approximate solution, i.e.

$$\lim_{n\to\infty} \psi_n(y) = y .$$

***Remark*1:**We can apply our method to a homogeneous second order problem as a second order homogenous differential equation of the type $u'' + p(x)u' + q(x)u = 0$ may be easily rewritten as $v' + pv + v^2 + q = 0$, using a transformation of the type $u(x) = e^{\int v(x)dx}$.

***Remark*2:** We can use our study to solve the Riccati differential equation of the form

$$a(x)y' + b(x)y + c(x)y^2 + d(x) = g(x), \quad x > x_0, \qquad y(x_0) = y_0 ,$$

Where$a(x) \neq 0, b(x), c(x), d(x)$ are given functions and g(x) is a given polynomial function in *x*.

Also we can apply the present study to find a solution to a Bernoulli type nonlinear differential equation.

3. NUMERICAL EXAMPLES:

In this section we present a comparison between the Adomian method and the modified Adomian method through some examples.

Example 1: Consider the regular initial value problem,

$$\begin{cases} y' + xy = x \quad , x > 0 \\ y(0) \quad = 1 . \end{cases} \tag{3.1}$$

We have $y(x) \equiv 1$ as a exact solution of the problem.

Standard Adomian decomposition method: Define a differential operator for (3.1) as

$$L(\,.\,) = \frac{d}{dx}(\,.\,) \tag{3.2}$$

Then using (3.2), (3.1) can be written as

$$L(y) = -xy + x \tag{3.3}$$

Now define the inverse operator for $L(\,.\,)$ as

$$L^{-1}(\,.\,) = \int_0^x (\,.\,)\,dx \tag{3.4}$$

Operating L^{-1}on both sides of (3.3) , we get

$$y(x) = y(0) - L^{-1}(xy) + L^{-1}(x). \tag{3.5}$$

Finally, we use the following recursive relation to evaluate, the components $y_n(x)$ as

$$\begin{cases} y_0 = \; y(0) + L^{-1}(x) \\ y_{n+1} \; = -L^{-1}(xy_n) n \geq 0 \end{cases} \tag{3.6}$$

or

$$\begin{cases} y_0 = +\left(1 + \frac{x^2}{2}\right) , \\ y_1 = -\left(\frac{x^2}{2} + \frac{x^4}{8}\right) , \\ y_2 = +\left(\frac{x^4}{8} + \frac{x^6}{48}\right) , \\ \quad \vdots \; = \quad \vdots \end{cases} \tag{3.7}$$

International Journal of Advance Research in Science and Engineering
Volume No.10, Issue No. 03, March 2021
www.ijarse.com
IJARSE
ISSN 2319 - 8354

The partial sums of the components give rise to the exact solution $y(x) \equiv 1$.

Modified Adomian decomposition method : We use the concept given in (2.1) to define the operators L and L^{-1} as,

$$L(.) = e^{-\frac{x^2}{2}} \frac{d}{dx}\left(e^{\frac{x^2}{2}}(.)\right)$$

And$L^{-1}(.) = e^{-\frac{x^2}{2}} \int_0^x e^{\frac{x^2}{2}}(.)\, dx$

Rewriting (3.1) in the operator form as

$L(y) = x\,.$ (3,8)

Applying L^{-1} on both the sides of (3.8), we have

$L^{-1}(L(y)) = e^{-\frac{x^2}{2}} \int_0^x e^{\frac{x^2}{2}} x\ dx$ (3.9)

And it shows that,

$$y(x) = y(0) + 1 - 1 \ \Rightarrow y(x) = 1\,.$$

So, the exact solution is easily obtained by the modified Adomian decomposition method.

Example 2: Consider the singular initial value problem,

$$\begin{cases} y' + \frac{1}{x} y = x \\ y(0) \quad = 1\,. \end{cases} \qquad (3.10)$$

With the exact solution $y(x) \equiv \frac{x^2}{3}$.

Standard Adomian decomposition method: Define a differential operators as

$$L(.) = \frac{d}{dx}(.), \qquad \text{and} \quad L^{-1}(.) = \int_0^x (.)\, dx$$

Rewriting (3.10) in the operator form, we have

International Journal of Advance Research in Science and Engineering
Volume No.10, Issue No. 03, March 2021
www.ijarse.com
IJARSE
ISSN 2319 - 8354

$$L(y) = -\frac{1}{x}y = x \tag{3.11}$$

Operating L^{-1} on both the sides of (3.11) , we get

$$y(x) = y(0) - L^{-1}\left(\frac{1}{x}y\right) + L^{-1}(x). \tag{3.12}$$

Finally, use the following recursive relation to obtain the components for the solution to (3.10),

$$\begin{cases} y_0 = y(0) + L^{-1}(x), \\ y_{n+1} = -L^{-1}\left(\frac{1}{x}y_n\right), \qquad n \geq 0 \end{cases}$$

Or

$$\begin{cases} y_0 = +\left(\frac{x^2}{2}\right), \\ y_1 = -\left(\frac{x^4}{4}\right), \\ y_2 = +\left(\frac{x^2}{8}\right), \\ y_3 = -\left(\frac{x^2}{16}\right), \\ \vdots = \vdots \end{cases} \tag{3.13}$$

We can easily see that the sum of the above expressions can not give the exact solution to the problem (3.10) , i.e. in this case the Adomian decomposition method diverges.

Modified Adomian decomposition method: We define the operators for the problem (3.10) , as

$$L(.) = \frac{1}{x}\frac{d}{dx}(x(.)), \qquad \text{and} \quad L^{-1}(.) = \frac{1}{x}\int_0^x (.)\, dx \tag{3.14}$$

Using the definition of the operators, we rewrite (3.10) as

$$L(y) = x \tag{3.15}$$

Applying L^{-1} on bothsides of (3.15) , we have

$$L^{-1}(L(y)) = \frac{1}{x}\int_0^x x\,.\,x\, dx$$

Which gives

$$y(x) = y(0) + \frac{x^2}{3} \Longrightarrow y(x) = \frac{x^2}{3}$$

So, the exact solution is easily obtained by the modified Adomian decomposition method.

Example 3: Consider the following nonlinear initial value problem,

$$\begin{cases} y' + y + x^2y^3 = 1 + x + x^5 , \\ y(0) = 0 . \end{cases} \quad (3.17)$$

with the exact solution $y(x) = x$.

Standard Adomian decomposition method: Define the operators as

$L(.) = \frac{d}{dx}(.)$, and $L^{-1}(.) = \int_0^x (.)\, dx$ (3.18)

We can rewrite (3.17) in the operator form as

$$L(y) = -y - x^2y^3 + (1 + x + x^5) \quad (3.19)$$

Applying the inverse operator L^{-1} on both the sides of (3.19) , we have

$$\begin{cases} y_0 = y(0) + L^{-1}\, g(x), \\ y_{n+1} = -L^{-1}\,(y_n) - L^{-1}\,(A_n) , \quad n \geq 0 \end{cases} \quad (3.20)$$

Where A_n, the Adomian polynomials of the nonlinear term x^2y^2 , are given by

$$\begin{cases} A_0 = x^2y_0^2 , \\ A_1 = x^2(2y_0y_1) , \\ A_2 = x^2(2y_0y_2 + y_1^2) , \\ A_3 = x^2(2y_0y_3 + (2y_2y_1) \\ A_4 = x^2(2y_0y_4 + 2(\frac{y^2}{2!} + y_1y_3) \\ \vdots = \vdots \end{cases} \quad (3.13)$$

Which yields

$$A_0 = x^5 + \frac{3x^6}{2} + \frac{3x^7}{4} + \frac{x^8}{8} + \frac{3x^6}{2} + \frac{x^{10}}{2} + \frac{x^{11}}{2} + \frac{x^{12}}{8} + \frac{x^{15}}{12} + \frac{x^{16}}{24} + \frac{x^{20}}{216} + O[x]^{21} ,$$

$$A_1 = -\frac{x^6}{2} - 2x^7 - \frac{7x^8}{8} - \frac{x^9}{8} - \frac{x^{10}}{2} - \frac{12x^{11}}{7} - \frac{1033x^{12}}{672} - \frac{131x^{13}}{224} - \frac{43\,x^{14}}{384} - \frac{331x^{15}}{1056} - \frac{557\,x^{16}}{924} - \frac{119531\,x^{17}}{288288} - \frac{905x^{18}}{7488} - \frac{53\,x^{19}}{3744} - \frac{475\,x^{20}}{6336} - \frac{80837\,x^{21}}{753984} - \frac{8405577\,x^{22}}{169728} - \frac{745\,x^{23}}{95472} - \frac{1285\,x^{24}}{133056} + O[x]^{26},$$

Similarly , we can find the values of $A_2, A_3, A_4, A_5, \ldots \ldots$. Using the values of Adomian polynomials we can find the components of $y(x)$ as follows

$$y_0 = x + \frac{x^2}{2} + \frac{x^6}{6},$$

$$y_0 + y_1 = x - \frac{x^3}{6} - \frac{5x^7}{21} - \frac{3x^8}{32} - \frac{x^9}{72} - \frac{x^{11}}{22} - \frac{x^{12}}{24} - \frac{x^{13}}{104} - \frac{x^{16}}{192} - \frac{x^{17}}{408} - \frac{x^{21}}{4536} + O[x]^{22},$$

$$y_0 + y_1 + y_2 = x + \frac{X^{24}}{24} + \frac{125X^8}{672} + \frac{3X^9}{32} + \frac{X^{10}}{72} + \frac{97X^{12}}{924} + \frac{977X^{13}}{8736} + \frac{1731X^{14}}{40768} + \frac{43X^{15}}{5760} + \frac{81X^{16}}{5632} + \frac{8693X^{17}}{251328} + \frac{2944039X^{18}}{88216128} + \frac{905X^{19}}{142272} + \frac{53X^{20}}{74880} + \frac{191X^{21}}{57024} + O[x]^{22}$$

Modified Adomian decomposition method : We define the operators for the problem (3.17), as

$L(.) = \frac{1}{x}\frac{d}{dx}(x(.)),$ and $L^{-1}(.) = \frac{1}{x}\int_0^x (.)\,dx$ (3.22)

We may rewrite (3.17) in the operator form as

$$L(y(x)) = -x^2y^3 + (1 + x + x^2) \qquad (3.23)$$

Applying the inverse operator L^{-1} on both the sides of (3.23) , we have

$$y(x) = y(0) - L^{-1}(x^2y^3) + L^{-1}(1 + x + x^2) \qquad (3.24)$$

$$\begin{cases} y_0 = \psi(x) + L^{-1}g(x) \\ y_{n+1} = -L^{-1}A_{n-1,} n \geq 0 \end{cases} \qquad (3.25)$$

Where, $g(x)=(1 + x + x^2)$. In this case the Adomian polynomials A_n are given by

$$A_0 = x^5 + \frac{x^{10}}{2} - \frac{x^{11}}{14} + \frac{x^{12}}{112} - \frac{x^{13}}{1008} + \frac{x^{14}}{10080} - \frac{9239x^{15}}{110880} + \frac{31679x^{16}}{1330560} - \frac{5662273x^{17}}{121080960} + \frac{61013x^{18}}{80720640} - \frac{97061x^{19}}{908107200} + \frac{26827632x^{20}}{58118860800} - \frac{839476709x^{21}}{423437414400} + \frac{726817477x^{22}}{1368028569600} - \frac{52935235883x^{23}}{473064279367680} + \frac{45319397867x^{24}}{2252687044608000} + O[x]^{25},$$

$$A_1 = \frac{-x^{10}}{2} + \frac{x^{11}}{14} - \frac{x^{12}}{112} + \frac{x^{13}}{1008} - \frac{x^{14}}{10080} + \frac{3599x^{15}}{110880} - \frac{102239x^{16}}{1330560} + \frac{1654193x^{17}}{121080960} - \frac{70297x^{18}}{34594560} + \frac{19889x^{19}}{74131200} - \frac{4355198689x^{20}}{58118860800} + \frac{80928718147x^{21}}{2964061900800} - \frac{1119915756495x^{22}}{1368028569600} + \frac{2729027366119x^{23}}{473064279367680} - \frac{8614347574919x^{24}}{2252687044608000} + O[x]^{25}$$

Similarly, we can find the values of $A_2, A_3, A_4, A_5, \dots \dots$. Using the values of Adomian polynomials we can find the components of $y(x)$ as follows

$$y_0 = x + \frac{x^6}{6} - \frac{x^7}{42} + \frac{x^8}{336} - \frac{x^9}{3024} + \frac{x^{10}}{30240} - \frac{x^{11}}{332640} + \frac{x^{12}}{3991680} - \frac{x^{13}}{51891840} + \frac{x^{14}}{726485760} - \frac{x^{15}}{10897286400} + \frac{x^{16}}{174356582400} - \frac{x^{17}}{29664061900800} + \frac{x^{18}}{53353114214400} - \frac{x^{19}}{1013709170073600} + \frac{x^{20}}{20274183401472000} + O[x]^{21}$$

$$y_0 + y_1 = x - \frac{x^{11}}{22} + \frac{3x^{12}}{308} - \frac{23x^{13}}{16016} + \frac{25x^{14}}{144144} - \frac{131x^{15}}{7207200} + \frac{2144x^{16}}{4118400} - \frac{73807x^{17}}{43243200} + \frac{39037x^{18}}{110073600} - \frac{96043x^{19}}{1643241600} + \frac{5705221x^{20}}{690161472000} + O[x]^{21}$$

$$y_0 + y_1 = x - \frac{29x^{16}}{2112} - \frac{47x^{17}}{11968} + \frac{295313x^{18}}{411675264} - \frac{820681x^{19}}{7821830016} + \frac{14880121x^{20}}{1117404288000} + O[x]^{21}$$

In the above discussion it is shown that by using the modified Adomian decomposition method we may achieve the higher rate of convergence than the classical one, i.e., .Adomian decomposition method.

Remark: However, the modified Adomian decomposition method may not work if we assume $g(x)$ to be other than polynomial function . For example, in the following differential equations

$$y' + xy + x^2y^2 = e^x(1+x) + x^2e^{3x} \qquad (3.26)$$

$$y' - 2xy + x^2y^2 = x^2e^{2x^2}, \qquad (3.27)$$

With exact solution $y(x) = e^x$ and $y(x) = e^{x^2}$ respectively. The modified Adomian decomposition method fails, for solution of this problem further investigation are required .

REFERENCES

[1] A. M. Wazwaz, A reliable modification of Adomian decomposition method, Appl. Math. Comput., 102 (1999), 77-86.

[2] G. Adomian, A review of the decomposition method in applied mathematics, J. Math. Anal. Applic, 135 (1988), 501-544.

[3] N. Bellomo and R. Monaco, A comparison between Adomian decomposion method and perturbation techniques for nonlinear random differntial equations, J. Math. Anal. Applic. 110 (1985), 495-502.

[4] R. Rach, On the Adomian decomposition method and comparison with Picard's method, J. Math. Anal. Applic. 128 (1987), 480-483.

[5] A. M. Wazwaz, A first course in Integral Equations, World Scientific, Singapore 1997.

[6] A. M. Wazwaz, A comparison between Adomian decomposion method taylor's series method in series solution, Appl. Math. Comput. 79 (1998), 37-44.

[7] G. Adomian, Nonlinear Stochastic Operator Equation, Academic. Press. San Diego, CA, 1986.

[8] G. Adomian, Solving Frontier Problem in Physics: The Decomposition Method, Kluwer, Boston. MA, 1994.

Department of Mathematics, G.H Raisoni University, Amravati, India. Email address: adnaidu8@gmail.com,anju.naidu@ghru.edu.in

Department of Mathematics, Gautam Buddha University, Gr Noida- 208 013, India. Email address: iitk.amit@gmail.comac.in