Stability of an SIQS Epidemic Model with Behavioural Changes

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ABSTRACT

This paper is aim to study the effect of combining the method of quarantining an infective with behavioural changes of the infective and the susceptible as a result of the disease to the transmission of the disease. Therefore, this paper considers an SIQS model with constant recruitment rate and parameters that measure behavioural changes of the infective and susceptible. We obtained a threshold R_0 *which we use to analysis the stabilities of the disease. It is established that threshold value is not 1 but* $1+\frac{d}{\alpha b}$*. That is if* R_0 *is less than* $1+\frac{d}{\alpha b}$*, then the disease-free-equilibrium exists and is stable. And if* R_0 *is greater than* $1+\frac{d}{\alpha b}$*, then there exists a unique positive equilibrium and is stable.*

Keywords: *Endemic, Inhibition, Quarantine, Reproduction number, Stability.*

1. INTRODUCTION

The rapid spread of infectious *disease* to a large number of people in a given population within a short period of time has been a great concern to humanity in many ways. One of the negative effects of infectious disease is the high number of people it can kill within a short period of time. For instance, out of the 66 recorded cases of Ebola virus in Zaire of Democratic Republic of Congo (DRC) in 2014, 49 (74%) people were reported dead [1]. In 2008 Zimbabwean cholera outbreak that lasted for six months, 4,369 out 98,596 reported cases were killed by the disease [2]. Apart from this direct killing of the patients, the infectious diseases still did not leave the survivals without an untold hardship. Huge part of the government's budget that is supposed to be used to better the life of the citizenry will be used to control it and economic activities of the affected areas are usually paralyzed. All these among contribute to the low standard of living of the survivals.

With respect to the aforementioned reasons among others, the quick response to the control of infectious diseases cannot be overemphasized. The measures taken by the health workers are vaccination, quarantine and treatment. Among the adopted measures, quarantine is the quickest measure that can be used to prevent the spread of infectious disease. Quarantine is the process of separating an individual who is exposed to a communicable disease, and has high chances of spreading the disease because of his exposure. A lot of researchers have modeled and worked on quarantining with rest to epidemic. Feng et al [3, 4, and 5], Adebimpe, O. et al [6] and Hethcote et al [7] formulated and worked on models with different reaction incidence rate and different endemic models with a view of studying the effects of quarantine.

The spread of disease depend majorly on the interaction between the susceptible and the infective. And because of the knowledge of the disease among other reasons, the models that consider free interaction-simple mass action cannot adequately describe the transmission of disease [8, 9, and 10]. The chances are high that due to the knowledge of the disease, the susceptible and infective will be cautious on how they interact with one another [11]. The consciousness of the disease, apart from the quarantine method measured above, if well managed can help to control the transmission of the disease.

In this paper, we consider an SIQS model that combined a quarantine method and incidence rate $\frac{kS(t)I(t)}{1+\alpha S(t)+\beta(t)}$,proposed by Pathak S et al [11], that consider the behavioural changes in both the susceptible and infective. Thereafter, we obtained the disease-free and the endemic equilibria and then discuss the local stability at these equilibria. In conclusion, we carried out the numerical simulation of the model to compare the effectiveness of the quarantine and the behavioural changes on the transmission of the infectious disease.

2. The Mathematical Model

The SIQS model is given by the following non-linear ordinary differential equation:

$$S' = b + \gamma I + \mu Q - dS - \frac{kSI}{1+\alpha S+\beta I} \quad (1)$$

$$I' = \frac{kSI}{1+\alpha S+\beta I} - (\gamma + d + \rho + \sigma)I \quad (2)$$

$$Q' = \sigma I - (d + \varphi + \mu)Q \quad (3)$$

In the *SIQS* model, infectious do not confer immunity, some of the susceptible becomes infected and some infective remains in the infectious class *I* for their whole infectious period before they return to the susceptible class *S*, while some remain in the infective class until they die, while other infective individuals are transferred into the quarantine class *Q*. The parameters $b, \gamma, \mu, d, k, \alpha, \beta, \rho, \sigma, and\ \varphi$ are positive constants. The constant *b* is the recruitment rate of the susceptible, d is the natural death rate, γ is the rate at which infectious individuals recover into the class S, μ is the rate at which quarantine individuals recover into class S, ρ represents death rate of the infectious of people in the I class, φ represents death rate as a result of the infection on the people in Q class, σ is the rate at which infectious people are isolated from the I class while k is the infectious force constant of the disease. And α and β are parameters that measure the behavioural changes of the Susceptible and Infective respectively.

From the fact that $N = S + I + Q$, we obtain the equation

$$N' = b - dN - \rho I - \varphi Q \tag{4}$$

In the absence of disease, the population size N approaches the carrying capacity $\frac{b}{d}$. The differential equation (4) implies that the solution of (1-3) starting in $\mathcal{R}_+^3 = \{(S, I, Q) \in \mathcal{R}^3 | S \geq 0, I \geq 0, Q \geq 0\}$ either approaches, enters or remains in the subset of $\mathcal{R}_+^3$ defined by $\mathcal{D} = \left\{(S, I, R) \in \mathcal{R}^3 \middle| 0 < S + I + Q \leq \frac{b}{d}, S \geq 0, I \geq 0, Q \geq 0\right\}$.

For any parameters, the model (1-3) has disease-free equilibrium $E^0 = (S^0, I^0, Q^0) = (\frac{b}{d}, 0, 0)$ and unique endemic equilibrium (S^*, I^*, Q^*) where $S^* = \frac{R_0(\gamma - b\beta - \mu\beta Q) - \frac{k}{\alpha}}{k - R_0(k + \gamma\alpha + d\beta) + \gamma\alpha {R_0}^2}$, $I^* = \frac{\alpha S}{\beta}[R_0 - (\frac{1+\alpha S}{\alpha S})]$ and $Q^* = \frac{\sigma I^*}{d + \varphi + \mu}$.

Define the basic reproduction number as $R_0 = \frac{k}{\alpha(\gamma + d + \rho + \sigma)}$.

$$I' = \frac{kId(\frac{b}{d} - I - Q)}{d + \alpha b + Id(\beta - \alpha) - \alpha d Q} - (\gamma + d + \rho + \sigma)I \tag{5}$$

$$Q' = \sigma I - (d + \varphi + \mu)Q \tag{6}$$

We rescale (5 and 6) with $x = \frac{kI}{d + \varphi + \mu}$, $y = \frac{kQ}{d + \varphi + \mu}$, $\tau = (d + \varphi + \mu)t$

Then we obtain

$$\frac{dx}{d\tau} = \frac{\lambda x(\eta - x - y)}{1 + \pi x - \varpi y} - \partial x \tag{7}$$

$$\frac{dy}{d\tau} = px - y \tag{8}$$

Where $\lambda = \frac{d}{(d + \alpha b)}$, $\eta = \frac{kb}{d(d + \varphi + \mu)}$, $p = \frac{\sigma}{d + \varphi + \mu}$, $\pi = \frac{d(d + \varphi + \mu)(\beta - \alpha)}{k(d + \alpha b)}$, $\varpi = \frac{\alpha d(d + \varphi + \mu)}{k(d + \alpha b)}$, $\partial = \frac{\gamma + d + \rho + \sigma}{d + \varphi + \mu}$

The trivial equilibrium *(0,0)* of the system (7-8) is the disease free equilibrium of the E^0 of the model (1-3) and the unique positive equilibrium (x^*, y^*) of the system (7-8) is the endemic equilibrium E^* of the model (1-3) where $x^* = \frac{\lambda\eta - \partial}{\lambda(p+1)+\partial(\pi-\varpi p)}$ and $y^* = px^*$

3. The Local Stability of Disease-Free Steady State

Theorem 3.1: At E^0, the disease-free equilibrium of the model is locally asymptotically stable when $R_0 \le 1 + \frac{d}{\alpha b}$.

To establish the local stability of E^0, we use the Jacobian of the system (7-8) evaluated at E^0. The Jacobian of the system at E^0 is $\begin{pmatrix} \lambda\eta - \partial & 0 \\ p & -1 \end{pmatrix}$.

The two eigenvalues of the Jacobian matrix have negative real parts if and only if the coefficients are positive. And this will only happen if $R_0 \le 1 + \frac{d}{\alpha b}$.

4. The Local Stability of Endemic Steady State

Theorem 4.1: The system is locally asymptotically stable at E^* if $R_0 > 1 + \frac{d}{\alpha b}$, otherwise unstable.

Proof: At the endemic equilibriumE^*, the Jacobian matrix of system (7-8) is given by

$$J_{E^*} = \begin{pmatrix} \frac{(1+\pi x^* - \varpi y^*)(\lambda\eta - 2\lambda x^* - \lambda y^*) - (\lambda\eta x^* - \lambda x^{*2} - \lambda x^* y^*)\pi}{(1+\pi x^* - \varpi y^*)^2} - \partial & \frac{(1+\pi x^* - \varpi y^*)(-\lambda x^*) - (\lambda\eta x^* - \lambda x^{*2} - \lambda x^* y^*)(-\varpi)}{(1+\pi x^* - \varpi y^*)^2} \\ p & -1 \end{pmatrix}$$

The determinant,

$$\det(J_{E^*}) = -\left[\frac{(1+\pi x^* - \varpi y^*)(\lambda\eta - 2\lambda x^* - \lambda y^*) - (\lambda\eta x^* - \lambda x^{*2} - \lambda x^* y^*)\pi}{(1+\pi x^* - \varpi y^*)^2} - \partial\right] - p\left[\frac{(1+\pi x^* - \varpi y^*)(-\lambda x^*) - (\lambda\eta x^* - \lambda x^{*2} - \lambda x^* y^*)(-\varpi)}{(1+\pi x^* - \varpi y^*)^2}\right]$$

$$= \partial + \lambda(p+1)\left[\frac{x^{*2}(\pi - \varpi p) + 2x^*}{(1+\pi x^* - \varpi p x^*)^2}\right] - \frac{\lambda\eta}{(1+\pi x^* - \varpi p x^*)^2}$$

With any positive values for the parameters with $\beta > \alpha$ and after careful analysis, the sign of the determinant is determined by

$x^* = \frac{\frac{kb}{R_0}\left[R_0 - \left(1+\frac{d}{\alpha b}\right)\right]}{d(\sigma+d+\varphi+\mu)+\frac{d}{\alpha R_0}[(d+\varphi+\mu)(\beta-\alpha)-\alpha\sigma]}$. Hence $\det(J_{E^*}) > 0$ provided $R_0 > 1 + \frac{d}{\alpha b}$.

The trace is $tr(J_{E^*}) = -1 + \frac{(1+\pi x^* - \varpi y^*)(\lambda\eta - 2\lambda x^* - \lambda y^*) - (\lambda\eta x^* - \lambda x^{*2} - \lambda x^* y^*)\pi}{(1+\pi x^* - \varpi y^*)^2} - \partial$

$$= -1 - \partial + \frac{\lambda[x^{*2}(2\varpi p + \varpi p^2 - \pi) - x^*(2 + p + \varpi p\eta) + \eta]}{(1 + \pi x^* - \varpi px)^2}$$

For any positive value of the parameters and $R_0 > 1 + \frac{d}{\alpha b}$, we have $tr(J_{E^*}) < 0$. Therefore, E^* is locally asymptotically stable.

5. Numerical Simulations

We present computer simulation of some solutions of the system (1-3):

5.1 Disease-free Equilibrium: Choosing the parameters as :

$b = 6, \gamma = 0.3, \mu = 0.4, d = 0.1, k = 0.2, \alpha = 0.3, \beta = 4, \rho = 0.1, \sigma = 0.5, \varphi = 0.2, \big(S(0), I(0), Q(0)\big) = (50, 30, 10)$.

Then $E^0 = (60, 0, 0)$ and $R_0 < 1 + \frac{d}{\alpha b}$ and in this case $S(t)$ approaches to its steady state value while $I(t)$ and $Q(t)$ approaches to zero as time goes to infinity, the disease disappears and dies out as seen in (Figure 5.1).

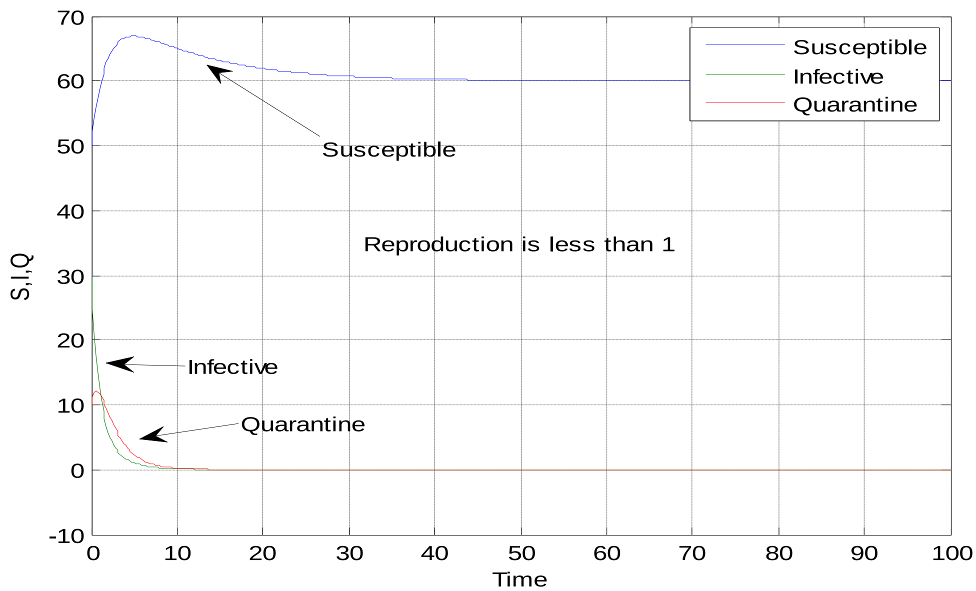

Figure 5.1

5.2 Endemic Equilibrium:

Choosing the parameters as:

$b = 6, \gamma = 0.1, \mu = 0.4, d = 0.1, k = 0.8, \alpha = 0.3, \beta = 4, \rho = 0.1, \sigma = 0.3, \varphi = 0.2, \big(S(0), I(0), Q(0)\big) = (50, 30, 10)$.

Then $E^*(S^*, I^*, R^*) = (32.8984, 8.2479, 3.5345)$. $S(t), I(t)$ and $R(t)$ get to their steady state and the disease become endemic as seen in Figure 5.2.

IJARSE
ISSN 2319 - 8354

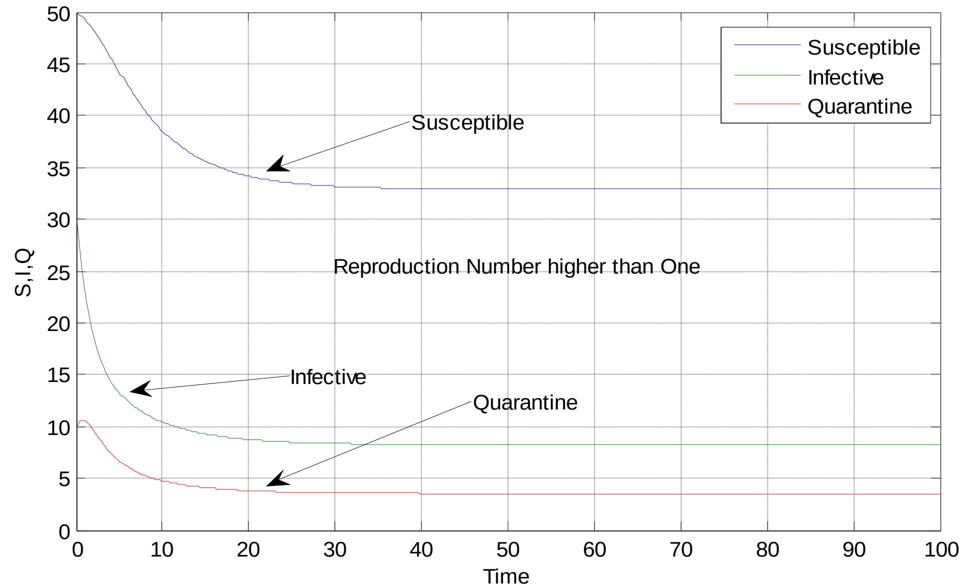

Figure 5.2

5.2.1. Modified Endemic Equilibrium I

If the σ, the infectious people isolated from the I class is reduced by 50% i.e $\sigma = 0.15$ and other parameters remain the same. Then $E^*(S^*, I^*, R^*) = (30.6924, 11.0894, 2.3765)$ $S(t), I(t)$ and $R(t)$ get to their steady state and the disease is still endemic as seen in Figure 5.3.

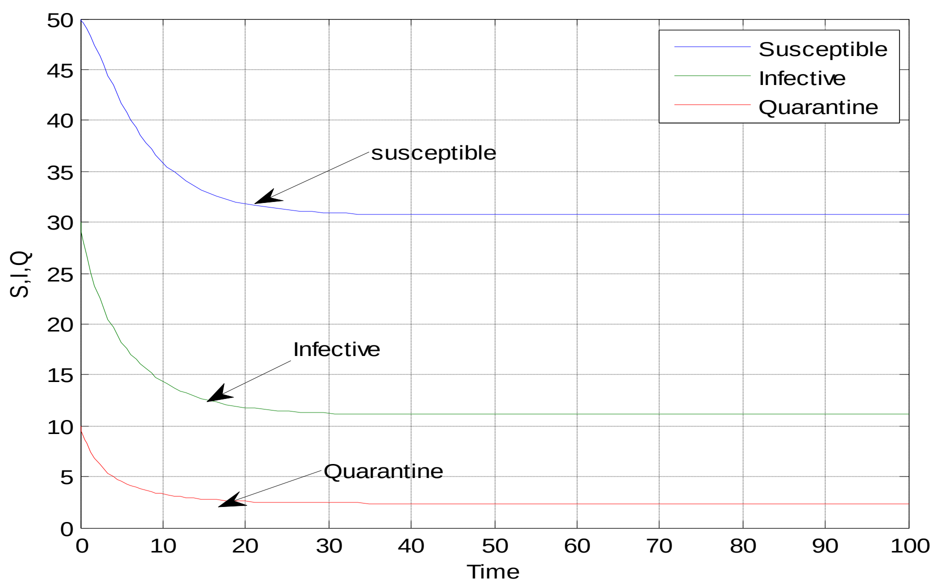

Figure 5.3

5.2.2Modified Equilibrium II

When α, parameters that measure the behavioural changes of the Susceptible is reduced by 50% i.e $\alpha = 0.15$ while other remain constant. $E^*(S^*, I^*, R^*) = (30.8432, 8.8741, 3.8017)$ $S(t), I(t)$ and $R(t)$ get to their steady state and the disease is still endemic as seen in Figure 5.4

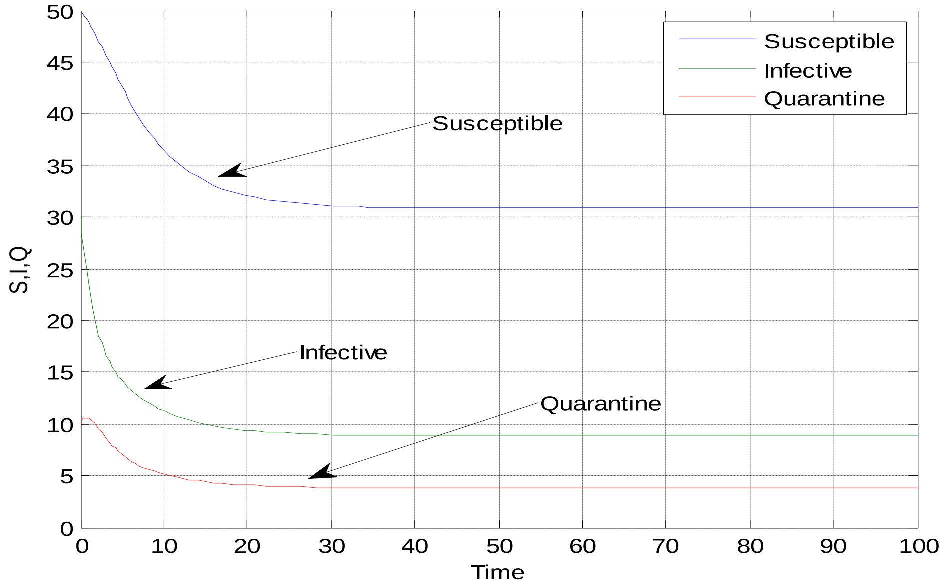

Figure 5.4

And β, parameter that measures the behavioural changes of the infective is reduced by 50% i.e $\beta = 2$ while other parameters remain constant. $E^*(S^*, I^*, R^*) = (22.8508, 11.3061, 4.8457)$ $S(t), I(t)$ and $R(t)$ get to their steady state and the disease is still endemic as seen in Figure 5.5.

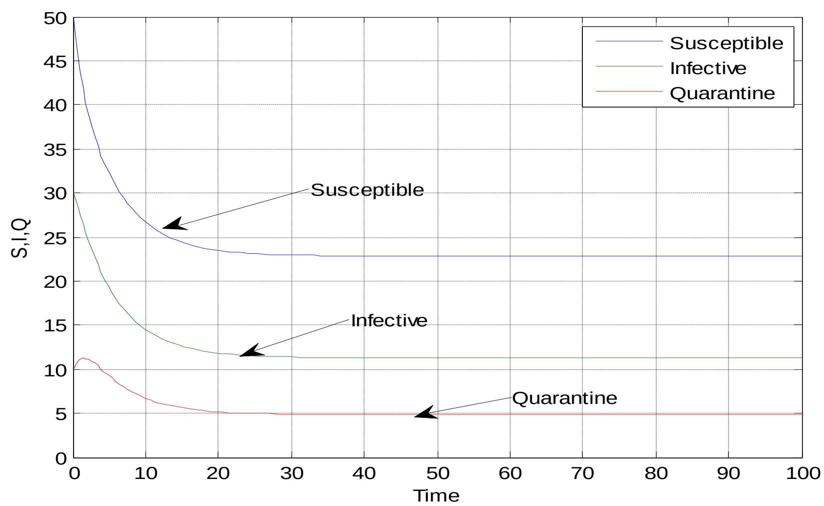

Figure 5.5

6. Conclusion

This paper has considered the $SIQS$ model with constant recruitment rate with quarantine and incidence rate that consider the behavioural changes of the population. It is established that in the model, the threshold value (reproduction number R_0) is not 1 but $1+\frac{d}{\alpha b}$. That is, $R_0 \leq 1+\frac{d}{\alpha b}$ for the disease to die out and $R_0 > 1+\frac{d}{\alpha b}$ for the disease to be endemic.

We also compare the effect of quarantine and behavioural change on the transmission of the disease. It is established that β that measures the rate of behavioural changes in infective and σ that measure the rate of quarantine play vital role in the propagation of the disease. As seen in Figure 5.3 that 50% reduction in σ leads to 34.45% increment in the infective and 50% reduction β leads to 37.01% increment in the infective while 50% reduction in α, parameter that measures the behavioural changes in susceptible, leads to about 7.59% in the infective. Therefore, to get quick result in the fight against epidemic, more attention should be given to the infective. This attention could be in the form of quarantine or public enlightenment of the infective class.

Acknowledgement

The authors would like to thank whose publications were used in this work.

REFERENCES

[1] www.who.int/news-room/fact-sheets detail/ebola-virus-disease.

[2] www.en.wikipedia.org/wiki/2008_Zimbabwean_cholera_outbreak

[3] Feng, Z and Thieme, H.R., Endemic models with arbitrarily distributed periods of infection,I: Fundamental properties of the model, SIAM J, Appl.Math.61 (2000) 803-833.

[4] Feng, Z., and Thieme, H.R., Recurrent outbreaks of childhood disease revisited: The impact of Isolation, math.Biosci. 128(1995)93-130.

[5] Feng, Z., and Thieme, H.R, Endemic models with arbitrarily distributed periods of infection, II: Fast disease dynamics and permanent recovery, SIAM J. Appl. Math. 61(2000), 983-1012.

[6] Adebimpe, O., Erinle-Ibrahim, L.M. and Adebisi, A.F. (2016), Stability Analysis of SIQS Epidemic Model with Saturated Incidence, Applied Mathematics, 7, 1082-1086.

[7] Hethcote, H.W.,Zhien Ma and Liao S., Effects of Quarantine in six endemic models for the infectious diseases,Math.Biosci.,180 (2002) 141-160.

[8] H. McCallum, N. Barlow, J. Hone, How should pathogen transmission be modeled?., Trends Ecol. Evol., 16, pp. 295-300, 2001.

[9] W.M. Liu, S.A. Levin, Y.Iwasa, Influence of non-linear incidence rates upon the behavior of SIRS epidemiological models,J.Math.Biol.,23,pp 187-204, 1986.

[10] H.W. Hethcote, P.van den Driessche, Some epidemiological model with nonlinear incidence J.Math.Biol., 29, pp 271-287, 1991.

[11] S. Pathak, A.Maiti, G.P. Samanta, Rich Dynamics of SIR epidemic model, Nonlinear Analysis: Modeling and control, 2010, vol.15, No.1, 71-81.