

Farthest points in Abstract Spaces

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ABSTRACT. Consider a metric space (X, d) and its bounded subset K , an element $k_o \in K$ is called a farthest point to an $x \in X$ if $d(x, k_o) = \sup_{k \in K} d(x, k)$.

The set of all farthest points to $x \in X$ is denoted by $F_K(x)$. Set K is said to be remotal (uniquely remotal) if for every $x \in X$ $F_K(x)$ is nonempty (singleton). The set valued map F_K is called farthest point map. In this paper, we prove results related to maps concerning farthest points and show that for closed uniquely remotal set K the farthest point map F_K has closed graph. Well-posedness problem for farthest points is also discussed. The underlying spaces are metric spaces and convex metric spaces.

Keywords remotal set, uniquely remotal set, farthest point map, convex space.

1. Introduction

One of the most interesting and hitherto unsolved problems in the theory of farthest points, known as the **farthest point problem (f.p.p.)**, is: If every point of a normed linear space X admits a unique farthest point in a given bounded subset K , then must K be singleton? There are some partial affirmative answers to this problem and there are many special cases in which the answer is negative (see e.g. [4], [5], [7], [8] and references cited therein). The problem is not solved in general even in Hilbert spaces. Most natural examples of uniquely remotal sets in normed linear spaces are singletons.

Farthest points have applications in the study of extremal structure of sets, characterization of weakly compact convex sets and finding deviation of two sets. They are important building blocks of convex sets which are extensively applied in programming (see e.g. [8]). For most of the available literature in the theory of farthest points, the underlying spaces are Hilbert spaces and normed linear spaces (see e.g. [1], [3], [6], [8], [9], [10], [11], [12], [13] and the references cited therein). This problem is so closely related to the problem of convexity of Chebyshev sets



in a Hilbert space (which too is an open problem in the theory of nearest points) that a solution of one will lead to a solution of the other (Ficken's Theorem - see [7]). The construction of farthest point theory in spaces more general than normed linear spaces is a challenging one. Some attempts have already been made in this direction and present paper is also a step in this direction. In this paper, we prove results related to maps concerning farthest points and show that for closed uniquely remotal set K the farthest point map F_K has closed graph. Well posedness problem in best approximation was discussed in [2], in this paper, we discuss parallel problem in farthest point theory.

2. Notations and Definitions

In this section we give some notations and recall few definitions to be used in the sequel.

DEFINITION 2.1. Let (X, d) be a metric space and K a non-empty bounded subset of X . The set K is said to be **remotal (uniquely remotal)** if for each $x \in X$ there exists at least one (exactly one) $k \in K$ such that $d(x, k) = \sup\{d(x, y) : y \in K\} \equiv \delta(x, K)$. Such a point k is called a **farthest point** or **remotal point** of x in K .

The number $\delta(x, K)$ is called deviation of K from x . The map $r_K : x \rightarrow \delta(x, K)$ is called **radial function** for K .

The set-valued map $F_K : X \rightarrow 2^K$ the collection of all subsets of K , defined by

$$F_K(x) = \{k \in K : d(x, k) = \sup_{y \in K} d(x, y)\}$$

is called the **farthest point map (f.p.m.)**.

DEFINITION 2.2. In a metric space (X, d) , a sequence $\langle k_n \rangle$ in K with

$$\lim_{n \rightarrow \infty} d(x, k_n) = \delta(x, K)$$

is called a **maximizing sequence** for $x \in X$ in K .

DEFINITION 2.3. Let (X, d) be a metric space. A mapping $W : X \times X \times [0, 1] \rightarrow X$ is said to be a **convex structure** [14] on X if for all $x, y \in X$ and $\lambda \in [0, 1]$

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y)$$

holds for all $u \in X$. A metric space (X, d) together with a convex structure is called a **convex metric space** and is denoted by (X, d, W) .

The following properties (see [14]) are direct consequences of above inequality :

$$\begin{aligned} W(x, y, 1) &= x, \quad W(x, y, 0) = y, \quad d(W(x, y, \lambda), y) = \lambda d(x, y), \\ d(W(x, y, \lambda), x) &= (1 - \lambda) d(x, y), \quad d(x, y) = d(x, W(x, y, \lambda)) + d(W(x, y, \lambda), y). \end{aligned}$$

3. Farthest Point and Related Map

PROPOSITION 3.1. Let K be a non-empty bounded subset of convex metric space (X, d, W) and $x_o \in X$. Then $y_o \in K$ is a farthest point for x_o if and only if y_o is a farthest point for x_o in $\{W(y_o, y, \lambda) \mid 0 \leq \lambda \leq 1\}$ for each $y \in K$.

PROOF. Let $t, 0 \leq t \leq 1$ be arbitrary. Consider

$$\begin{aligned} d(x_o, W(y_o, y, t)) &\leq td(x_o, y_o) + (1 - t)d(x_o, y) \\ &\leq td(x_o, y_o) + (1 - t)d(x_o, y_o) \\ &= d(x_o, y_o). \end{aligned}$$

Therefore y_o is farthest point for x_o in $\{W(y_o, y, \lambda), 0 \leq \lambda \leq 1\}$ for each $y \in K$. The converse implication is obvious as $W(y_o, y, 0) = y$.

PROPOSITION 3.2. Let K be a remotal subset of a metric space (X, d) then the farthest distance function $r_K : X \rightarrow R$ defined by

$$r_K(x) = \delta(x, K) = \sup\{d(x, k) : k \in K\}$$

is non expansive and hence uniformly continuous.

PROOF. Let $x, y \in X$ and $k \in K$ then

$$d(x, k) \leq d(x, y) + d(y, k)$$

$$\text{This implies } \sup\{d(x, k) : k \in K\} \leq d(x, y) + \sup\{d(y, k) : k \in K\}$$

$$\text{i.e. } \delta(x, K) \leq d(x, y) + \delta(y, K).$$

$$\text{and so } \delta(x, K) - \delta(y, K) \leq d(x, y) \tag{1}$$

for all $x, y \in X$. Interchanging x and y , we obtain

$$\delta(y, K) - \delta(x, K) \leq d(x, y) \quad (2)$$

and hence $|\delta(x, K) - \delta(y, K)| \leq d(x, y)$ for all $x, y \in X$ i.e. $|r_K(x) - r_K(y)| \leq d(x, y)$ for all $x, y \in X$.

Consequently, r_K is non expansive and hence uniformly continuous. $\square$

PROPOSITION 3.3. Let K be a non-empty bounded closed subset of a metric space (X, d) If $\langle (x_n, y_n) \rangle$ is a sequence in $X \times X$ with $y_n \in F_K(x_n)$ for all $n \in N$ and $\lim_{n \rightarrow \infty} \langle (x_n, y_n) \rangle = (x, y)$ then $y \in F_K(x)$.

PROOF. Since $\langle (x_n, y_n) \rangle \rightarrow (x, y)$, $\langle y_n \rangle \rightarrow y$. Since $\langle y_n \rangle \in K$ and K is closed, $y \in K$. Since $y_n \in F_K(x_n)$, $d(x_n, y_n) = \delta(x_n, K) \equiv \sup\{d(x_n, k) : k \in K\}$. This implies $d(x, y) = \lim_{n \rightarrow \infty} d(x_n, y_n) = \lim_{n \rightarrow \infty} \delta(x_n, K) = \delta(x, K)$ as radial function is continuous by Proposition 3.2. So $y \in F_K(x)$. $\square$

COROLLARY 3.1. If K is a closed uniquely remotal subset of a metric space (X, d) then the farthest point map for K has a closed graph.

PROOF. $G(F_K) = \{(u, F_K(u)) : u \in X\}$. Let (x, y) be a limit point of $G(F_K)$. Then there exists a sequence $\langle (x_n, y_n) \rangle$ in $G(F_K)$ such that $\langle (x_n, y_n) \rangle \rightarrow (x, y)$ where $y_n \in F_K(x_n)$. So by the above proposition $y \in F_K(x)$. Since K is uniquely remotal $y = F_K(x)$ i.e. $(x, y) = (x, F_K(x)) \in G(F_K)$. Hence $G(F_K)$ is closed.

4. Wellposedness and Farthest Point Problem in Metric Spaces

For a bounded subset K of a metric space (X, d) and $x \in X$, we denote by $Max(x, K)$ the problem of worst approximation of x by the elements of K i.e. the problem to find $k_o \in K$ such that $d(x, k_o) = \delta(x, K) = \sup\{d(x, k) : k \in K\}$. We say that the problem $Max(x, K)$ is well posed if it has a unique solution $k_o \in K$ and every maximizing sequence converges to k_o .

Concerning the wellposedness of this problem, we have

THEOREM 4.1. Let K be a non-empty complete bounded subset of a metric space (X, d) and $x \in X$ then the problem $Max(x, K)$ is well posed if and only if

$$\lim_{\varepsilon \rightarrow 0} \text{diam} Q_K(x, \varepsilon) = 0$$

where $Q_K(x, \varepsilon) = \{k \in K : d(x, k) \geq \delta(x, K) - \varepsilon\}$

PROOF. Suppose that the problem $Max(x, K)$ is well posed i.e. it has a unique solution $k_o \in Q_K(x, \varepsilon)$ for every $\varepsilon > 0$ and every maximizing sequence converges to k_o . If $y_n, y'_n \in Q_K(x, \frac{1}{n})$ then the sequences $\langle y_n \rangle$ and $\langle y'_n \rangle$ are both maximizing and converge to k_o -the solution of the problem. So $d(y_n, y'_n) \rightarrow 0$. It follows that $diam Q_K(x, \frac{1}{n}) \rightarrow 0$ as $n \rightarrow \infty$ and so $\lim_{\varepsilon \rightarrow 0} diam Q_K(x, \varepsilon) = 0$.

Now suppose $diam Q_K(x, \varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Let $\langle x_n \rangle$ be a maximizing sequence. Then for any $\varepsilon > 0$ there exists n_ε such that $x_n \in Q_K(x, \varepsilon)$ for all $n \geq n_\varepsilon$. Therefore $d(x_n, x_m) \leq diam Q_K(x, \varepsilon)$ for all $n, m \geq n_\varepsilon$ and so $\langle x_n \rangle$ is a Cauchy sequence in K . Since K is complete $\langle x_n \rangle \rightarrow k_o \in K$ and $d(x, k_o) = \delta(x, K)$. If x has two distinct elements of worst approximation k_o and k'_o in K then the sequence $k_o, k'_o, k_o, k'_o, \dots$ is a maximizing sequence which is not Cauchy. But this is not true as shown above that every maximizing sequence must be Cauchy. Hence the problem $Max(x, K)$ is well posed. $\square$

REMARK 4.1. The problem $Max(x, K)$ is well posed if K is a non-empty closed bounded subset of complete metric space (X, d) and $\lim_{\varepsilon \rightarrow 0} diam Q_K(x, \varepsilon) = 0$

NOTE 4.1. For normed linear spaces these results were proved in [3].

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