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Existence of Solution and Iterative Algorithm involving Maximal η-Monotone Operator for a System of Variational Inclusions

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Abstract: In this paper, we consider the system of generalized variational inclusions in Hilbert spaces, which is an extension of variational inclusion studied by Hassouni and Moudafi. Using proximal operator technique, we construct an iterative algorithm for solving the system of generalized variational inclusions. Further, we prove the existence of solution and discuss convergence criteria for the approximate solution of the system of generalized variational inclusions. Our suggested iterative algorithm and its convergence results are new and the theorems presented in this paper improve and unify many known results in the literature as well.

Keywords: System of generalized variational inclusions; Maximal η-monotone; η-proximal mapping; Iterative algorithm; Convergence criteria

1. Introduction

One of the most significant and important problems in the variational inequality theory is the development of efficient iterative algorithms to compute approximate solutions. Although one of the most effective numerical technique for solving variational inequalities is the projection method and its variant forms. For further generalizations of variational and quasi-variational inequalities/inclusions see for example [4-9,11].

Motivated by recent research work going on variational inequalities, we consider the system of generalized variational inclusions in Hilbert spaces and suggest an iterative algorithm. Further, we prove the existence of solution of the system of generalized nonlinear variational-like inclusions and discuss the convergence criteria for the iterative algorithm. The suggested iterative algorithm include as special cases the algorithm developed by Kazmi and Bhat [4]. The results presented in this paper improve and extend some known results in the literature.

2. Preliminaries and Basic Results

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. The following concepts and results are needed in the sequel:

Definition 2.1. Let $\eta : H \times H \to H$ be a single-valued mapping. Then a multivalued mapping $M : H \to 2^H$, where 2^H is the power set of H, is said to be

(i) *η-monotone*, if

$$\langle x - y, \eta(u, v) \rangle \geq 0, \quad \forall\, u, v \in H,\ \forall x \in M(u), y \in M(v);$$

(ii) *σ-strongly η-monotone*, if there exists a constant $\sigma > 0$ such that

$$\langle x - y, \eta(u, v) \rangle \geq \sigma \|u - v\|^2, \quad \forall\, u, v \in H, \quad \forall x \in M(u), y \in M(v);$$

(iv) *maximal η-monotone*, if M is η-monotone and $(I + \rho M)(H) = H$, for any $\rho > 0$, where I stands for an identity operator.

Definition 2.2 ([11]). Let $\eta : H \times H \to H$ be a single-valued mapping. A proper convex function $\phi : H \to I\!R \cup \{+\infty\}$ is said to be *η-subdifferentiable* at a point $u \in H$, if there exists a point $f^\star \in H$ such that

$$\phi(v) - \phi(u) \geq \langle f^\star, \eta(u, v) \rangle, \quad \forall v \in H, \tag{2.1}$$

where $f^\star$ is called an η-subdifferentiable of ϕ at u. The set of all η-subdifferentiable of ϕ at u is denoted by $\partial\phi(u)$. The mapping $\partial\phi : H \to 2^H$ defined by

$$\partial\phi(u) = \{f^\star \in H : \phi(v) - \phi(u) \geq \langle f^\star, \eta(u, v) \rangle, \ \ \forall v \in H\},$$

is said to be η-subdifferential of ϕ at u.

Definition 2.3 ([11]). Let $\phi : H \to I\!R \cup \{+\infty\}$ be a proper convex function. For any given $u \in H$ and $\rho > 0$, if there exists a mapping $\eta : H \times H \to H$ and a given unique point $w \in H$ such that

$$\langle \eta(v, w), w - u \rangle \geq \rho\phi(w) - \rho\phi(v), \quad \forall v \in H, \tag{2.2}$$

then the mapping $u \mapsto w$, denoted by $J_\rho^{\partial\phi}(u)$ is said to be η-proximal mapping of ϕ. By (2.1) and the definition of $J_\rho^{\partial\phi}(u)$, it follows that

$$J_\rho^{\partial\phi}(u) = (I + \rho\partial\phi)^{-1}(u), \quad \forall u \in H, \tag{2.3}$$

is called the *proximal (resolvent) mapping* of ϕ, where I stands for identity mapping on H.

Let $N_1, N_2, N_3 : H \times H \to H$, $g : H \to H$ be single-valued mappings and let $M : H \to 2^H$ be a maximal η-monotone mapping. Then the system of generalized nonlinear variational-like inclusions (in short, SGNVLI) is to find $u, v, w \in H$ such that

$$0 \in g(u) - g(v) + \rho_1\Big[N_1(u,v) + M(g(u))\Big], \ \rho_1 > 0, \tag{2.4}$$

$$0 \in g(v) - g(w) + \rho_2\Big[N_2(v,w) + M(g(v))\Big], \ \rho_2 > 0, \tag{2.5}$$

$$0 \in g(w) - g(u) + \rho_3\Big[N_3(w,u) + M(g(w))\Big], \ \rho_3 > 0. \tag{2.6}$$

We remark that if $u = v = w$ and $\rho_1 = \rho_2 = \rho_3$, SGNVLI (2.4)-(2.6) reduces to a variational inclusion of finding $u \in H$ such that

$$0 \in N_1(u,u) + M(g(u)). \tag{2.7}$$

Variational inclusion (2.7) is an important generalization of variational inclusion considered by Hassouni and Moudafi [1]. For applications of such variational inclusions, see [2,4,8,11].

Remark 2.4. For the suitable choices of the mappings N_1, N_2, N_3, g and M, SGNVLI (2.4)-(2.6) reduces to similar types of variational inclusions and variational inequalities considered by Yang *et al.* [3], Verma [9], Chang *et al.* [10], He and Gu [12].

Next, we give the following results, which are used in the sequel.

Lemma 2.5 ([2]). *Let $\eta : H \times H \longrightarrow H$ be a strictly monotone and let $M : H \longrightarrow 2^H$ be a maximal η-monotone mapping. Then the following conclusions hold:*
(a) $\langle x - y, \eta(u,v)\rangle \geq 0, \ \ \forall (y,v) \in Graph(M)$ implies $(x,u) \in Graph(M)$,
where $Graph(M) := \{(x,u) \in H \times H : x \in Mu\}$;
(b) the mapping $(I + \rho M)^{-1}$ is single-valued for any $\rho > 0$.

Lemma 2.6 ([2]). *Let $\eta : H \times H \longrightarrow H$ be δ-strongly monotone and τ-Lipschitz continuous mapping and let $M : H \longrightarrow 2^H$ be a maximal η-monotone mapping. Then the η-proximal mapping of M, $J_\rho^M := (I + \rho M)^{-1}$ is $\frac{\tau}{\delta}$-Lipcshitz continuous, i.e.,*

$$||J_\rho^M(u) - J_\rho^M(v)|| \leq \frac{\tau}{\delta}||u - v||, \ \forall u, v \in H. \tag{2.8}$$

where $\rho > 0$ is a constant.

3. Iterative Algorithms

In this section, an iterative algorithm for solving SGNVLI (2.4)-(2.6) is suggested and analyzed. First, we give the following lemma:

Lemma 3.1. *$u, v, w \in H$ is the solution of SGNVLI (2.4)-(2.6) if and only if it satisfies:*

$$g(u) = J_{\rho_1}^M\Big[g(v) - \rho_1 N_1(u, v)\Big]; \quad \rho_1 > 0, \tag{3.1}$$

where

$$g(v) = J_{\rho_2}^M\Big[g(w) - \rho_2 N_2(v, w)\Big]; \quad \rho_2 > 0, \tag{3.2}$$

and

$$g(w) = J_{\rho_3}^M\Big[g(u) - \rho_3 N_3(w, u)\Big]; \quad \rho_3 > 0. \tag{3.3}$$

Here $J_{\rho_i}^M := (I + \rho_i M)^{-1}$; $i = 1, 2, 3, ...$ is the proximal mapping, I stands for the Identity mapping on H.

The proof of this result follows from the definition of $J_{\rho_1}^M$, and hence is omitted.

The above lemma allows us to suggest the following iterative algorithm:

Iterative Algorithm 3.2. *For any arbitrary chosen $u_0 \in H$, compute $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ by the iterative schemes:*

$$u_{n+1} = u_n - g(u_n) + J_{\rho_1}^M\Big[g(v_n) - \rho_1 N_1(u_n, v_n)\Big]; \quad \rho_1 > 0$$

where

$$g(v_n) = J_{\rho_2}^M\Big[g(w_n) - \rho_2 N_2(v_n, w_n)\Big]; \quad \rho_2 > 0$$

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and

$$g(w_n) = J^M_{\rho_3}\Big[g(u_n) - \rho_3 N_3(w_n, u_n)\Big]; \quad \rho_3 > 0$$

$n = 0, 1, 2, \ldots$.

If $\rho_1 = \rho_2 = \rho_3$ and $u_n = v_n = w_n$ for all $n \geq 0$, then the above iterative algorithm reduces to the following iterative algorithm.

Iterative Algorithm 3.3. *For any arbitrary chosen $u_0 \in H$, compute $\{u_n\}$ by the iterative scheme*

$$u_{n+1} = u_n - g(u_n) + J^M_{\rho_1}\Big[g(u_n) - \rho_1 N_1(u_n, u_n)\Big]; \quad \rho_1 > 0$$

$n = 0, 1, 2, \ldots$.

We remark that Iterative Algorithm 3.3 gives the approximate solution to the variational inclusion (2.7).

4. Convergence Criteria

Now, we prove the following theorem, which ensures the existence of solution and the convergence criteria of Iterative Algorithm 3.2 for SGNVLI (2.4)-(2.6).

Theorem 4.1. *Let H be a real Hilbert space. Let $\eta : H \times H \longrightarrow H$ be δ-strongly monotone and τ-Lipschitz continuous mapping, $M : H \longrightarrow 2^H$ be a maximal η-monotone mapping, $N_1 : H \times H \longrightarrow H$ be α_1-strongly monotone with respect to second argument and (β_1, β_2)-Lipschitz continuous with respect to first and second argument, respectively, $N_2 : H \times H \longrightarrow H$ be α_2-strongly monotone with respect to second argument and (β_3, β_4)-Lipschitz continuous with respect to first and second argument, respectively, $N_3 : H \times H \longrightarrow H$ be α_3-strongly monotone with respect to second argument and (β_5, β_6)-Lipschitz continuous with respect to first and second argument, respectively, $g : H \longrightarrow H$ be σ-strongly monotone and ζ-Lipschitz continuous. If there exist constants $\rho_1 > 0$, $\rho_2 > 0$, $\rho_3 > 0$ such that*

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$$\left|\rho_1 - \frac{\alpha_1 + \beta_1\theta_1}{\beta_2^2 - \beta_1^2}\right| < \frac{\sqrt{\tau_1^2\left[\alpha_1^2 - (1-\theta_1^2)\beta_2^2\right] + \beta_1\tau_1^2(\beta_1 + 2\alpha\theta_1)}}{(\beta_2^2 - \beta_1^2)\tau_1}, \tag{4.1}$$

$$\alpha_1 > \beta_2\sqrt{1-\theta_1^2}, \qquad \theta_1 < 1,$$

$$\left|\rho_2 - \frac{\alpha_2\tau_2 - (\sigma\delta_2 - \theta_1\tau_2)\beta_3}{(\beta_4^2 - \beta_3^2)\tau_2}\right|$$

$$< \frac{\sqrt{\left[\alpha_2\tau_2 - (\sigma\delta_2 - \theta_1\tau_2)\beta_3\right]^2 - (\beta_4^2 - \beta_3^2)\{\sigma\delta_2(2\theta_1\tau_2 - \sigma\delta_2) + \tau_2^2(1-\theta_1^2)\}}}{(\beta_4^2 - \beta_3^2)\tau_2}, \tag{4.2}$$

$$\alpha_2\tau_2 - (\sigma\delta_2 - \theta_1\tau_2)\beta_3 \;>\; \sqrt{(\beta_4^2 - \beta_3^2)\{\sigma\delta_2(2\theta_1\tau_2 - \sigma\delta_2) + \tau_2^2(1-\theta_1^2)\}},$$

$$\left|\rho_3 - \frac{\alpha_3\tau_3 - (\sigma\delta_3 - \theta_1\tau_3)\beta_5}{(\beta_6^2 - \beta_5^2)\tau_3}\right|$$

$$< \frac{\sqrt{\left[\alpha_3\tau_3 - (\sigma\delta_3 - \theta_1\tau_3)\beta_5\right]^2 - (\beta_6^2 - \beta_5^2)\{\sigma\delta_3(2\theta_1\tau_3 - \sigma\delta_3) + \tau_3^2(1-\theta_1^2)\}}}{(\beta_6^2 - \beta_5^2)\tau_3}, \tag{4.3}$$

$$\alpha_3\tau_3 - (\sigma\delta_3 - \theta_1\tau_3)\beta_5 \;>\; \sqrt{(\beta_6^2 - \beta_5^2)\{\sigma\delta_3(2\theta_1\tau_3 - \sigma\delta_3) + \tau_3^2(1-\theta_1^2)\}},$$

where $\theta_1 = \sqrt{1 - 2\sigma + \zeta^2}$, *then the iterative sequences* $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ *generated by Iterative Algorithm* 3.2 *strongly converge to* u, v, w, *respectively, in* H *and* $u, v, w \in H$ *is the solution of SGNVLI* (2.4) − (2.6).

Proof. From Iterative Algorithm 3.2, Lemma 3.1 and (3.1), we have

$$\begin{aligned}
||u_{n+2} - u_{n+1}|| \\
&\le \left\|u_{n+1} - u_n - \left(g(u_{n+1}) - g(u_n)\right)\right\| \\
&\quad + \left\|J_{\rho_1}^M\left[g(v_{n+1}) - \rho_1 N_1(u_{n+1}, v_{n+1})\right] - J_{\rho_1}^M\left[g(v_n) - \rho_1 N_1(u_n, v_n)\right]\right\| \\
&\le \left\|u_{n+1} - u_n - \left(g(u_{n+1}) - g(u_n)\right)\right\| \\
&\quad + \frac{\tau_1}{\delta_1}\left\|g(v_{n+1}) - g(v_n) - \rho_1\left[N_1(u_{n+1}, v_{n+1}) - N_1(u_n, v_n)\right]\right\| \\
&\le \left\|u_{n+1} - u_n - \left(g(u_{n+1}) - g(u_n)\right)\right\| \\
&\quad + \frac{\tau_1}{\delta_1}\left\|g(v_{n+1}) - g(v_n) - \rho_1\left[N_1(u_{n+1}, v_{n+1}) - N_1(u_{n+1}, v_n)\right.\right. \\
&\quad \left.\left. + N_1(u_{n+1}, v_n) - N_1(u_n, v_n)\right]\right\| \\
&\quad + \frac{\tau_1}{\delta_1}\left\|v_{n+1} - v_n - \rho_1\left[N_1(u_{n+1}, v_{n+1}) - N_1(u_{n+1}, v_n)\right]\right\|
\end{aligned}$$

$$+\frac{\tau_1\rho_1}{\delta_1}\Big\|\Big[N_1(u_{n+1},v_n)-N_1(u_n,v_n)\Big]\Big\|. \tag{4.4}$$

Since g is σ-strongly monotone and ζ-Lipschitz continuous, we have the following estimate:

$$\begin{aligned}||g(v_{n+1})-g(v_n)-(v_{n+1}-v_n)||^2 \\ &= ||g(v_{n+1})-g(v_n)||^2-2\Big\langle g(v_{n+1})-g(v_n),v_{n+1}-v_n\Big\rangle \\ &\quad +||v_{n+1}-v_n||^2 \\ &\leq \zeta^2||v_{n+1}-v_n||^2-2\sigma||v_{n+1}-v_n||^2+||v_{n+1}-v_n||^2 \\ &\leq (1-2\sigma+\zeta^2)||v_{n+1}-v_n||^2.\end{aligned}$$

Hence,

$$||g(v_{n+1})-g(v_n)-(v_{n+1}-v_n)||\leq\sqrt{1-2\sigma+\zeta^2}\;\;||v_{n+1}-v_n||. \tag{4.5}$$

Similarly, we have

$$||g(u_{n+1})-g(u_n)-\Big(u_{n+1}-u_n\Big)||\leq\sqrt{1-2\sigma+\zeta^2}\;\;||u_{n+1}-u_n||. \tag{4.6}$$

Also, since N_1 is α_1-strongly monotone with respect to second argument and (β_1,β_2)-Lipschitz continuous with respect to first and second arguments, respectively, we have the following estimates:

$$||N_1(u_{n+1},v_n)-N_1(u_n,v_n)||\leq\beta_1||u_{n+1}-u_n||,$$

and

$$\begin{aligned}\Big\|v_{n+1}-v_n-\rho_1\Big[N_1(u_{n+1},v_{n+1})-N_1(u_{n+1},v_n)\Big]\Big\|^2 \\ &\leq ||v_{n+1}-v_n||^2-2\rho_1\Big\langle N_1(u_{n+1},v_{n+1})-N_1(u_{n+1},v_n),v_{n+1}-v_n\Big\rangle \\ &\quad +\rho_1^2||N_1(u_{n+1},v_{n+1})-N_1(u_{n+1},v_n)||^2 \\ &\leq ||v_{n+1}-v_n||^2-2\rho_1\alpha_1||v_{n+1}-v_n||^2+\rho_1^2\beta_2^2||v_{n+1}-v_n||^2 \\ &= \Big(1-2\rho_1\alpha_1+\rho_1^2\beta_2^2\Big)||v_{n+1}-v_n||^2.\end{aligned}$$

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Hence,

$$\Big\|v_{n+1}-v_n-\rho_1\big[N_1(u_{n+1},v_{n+1})-N_1(u_{n+1},v_n)\big]\Big\| \leq \sqrt{1-2\rho_1\alpha_1+\rho_1^2\beta_2^2}\ ||v_{n+1}-v_n||. \quad (4.7)$$

Now, we have

$$\begin{aligned} ||g(v_{n+1})-g(v_n)||\ ||v_{n+1}-v_n|| &\geq \Big\langle g(v_{n+1})-g(v_n), v_{n+1}-v_n\Big\rangle \\ &\geq \sigma||v_{n+1}-v_n||^2, \end{aligned}$$

which implies

$$\begin{aligned} ||v_{n+1}-v_n|| &\leq \frac{1}{\sigma}||g(v_{n+1})-g(v_n)|| \\ &\leq \frac{1}{\sigma}\Big\|J_{\rho_2}^M\big[g(w_{n+1})-\rho_2N_2(v_{n+1},w_{n+1})\big]-J_{\rho_2}^M\big[g(w_n)-\rho_2N_2(v_n,w_n)\big]\Big\| \\ &\leq \frac{\tau_2}{\sigma\delta_2}\Big\|g(w_{n+1})-g(w_n)-\rho_2\big[N_2(v_{n+1},w_{n+1})-N_2(v_n,w_n)\big]\Big\| \\ &\leq \frac{\tau_2}{\sigma\delta_2}\Big\|g(w_{n+1})-g(w_n)-(w_{n+1}-w_n)\Big\| \\ &\quad +\frac{\tau_2}{\sigma\delta_2}\Big\|w_{n+1}-w_n-\rho_2\big[N_2(v_{n+1},w_{n+1})-N_2(v_{n+1},w_n)\big]\Big\| \\ &\quad +\frac{\tau_2\rho_2}{\sigma\delta_2}\Big\|N_2(v_{n+1},w_n)-N_2(v_n,w_n)\Big\|. \quad (4.8) \end{aligned}$$

Since N_2 is α_2-strongly monotone with respect to second argument and (β_3, β_4)-Lipschitz continuous with respect to first and second arguments, respectively, we have the following estimates:

$$||N_2(v_{n+1},w_n)-N_2(v_n,w_n)|| \leq \beta_3||v_{n+1}-v_n||\ ,$$

and

$$\begin{aligned} &\Big\|w_{n+1}-w_n-\rho_2\big[N_2(v_{n+1},w_{n+1})-N_2(v_{n+1},w_n)\big]\Big\| \\ &\qquad \leq \sqrt{1-2\rho_2\alpha_2+\rho_2^2\beta_4^2}\ ||w_{n+1}-w_n||. \quad (4.9) \end{aligned}$$

Similarly, we have

$$||g(w_{n+1})-g(w_n)-(w_{n+1}-w_n)|| \leq \sqrt{1-2\sigma+\zeta^2}\ ||w_{n+1}-w_n||. \quad (4.10)$$

Now, we have

$$||g(w_{n+1})-g(w_n)||\;||w_{n+1}-w_n|| \;\geq\; \Big\langle g(w_{n+1})-g(w_n), w_{n+1}-w_n\Big\rangle \;\geq\; \sigma||w_{n+1}-w_n||^2 ,$$

which implies

$$||w_{n+1}-w_n|| \leq \frac{1}{\sigma}||g(w_{n+1})-g(w_n)||$$

$$\begin{aligned}
&\leq \frac{1}{\sigma}\Big\|J^M_{\rho_3}\Big[g(u_{n+1})-\rho_3 N_3(w_{n+1},u_{n+1})\Big]-J^M_{\rho_3}\Big[g(u_n)-\rho_3 N_3(w_n,u_n)\Big]\Big\| \\
&\leq \frac{\tau_3}{\sigma\delta_3}\Big\|g(u_{n+1})-g(u_n)-\rho_3\Big[N_3(w_{n+1},u_{n+1})-N_3(w_n,u_n)\Big]\Big\| \\
&\leq \frac{\tau_3}{\sigma\delta_3}\Big\|g(u_{n+1})-g(u_n)-(u_{n+1}-u_n)\Big\| \\
&\quad+\frac{\tau_3}{\sigma\delta_3}\Big\|u_{n+1}-u_n-\rho_3\Big[N_3(w_{n+1},u_{n+1})-N_3(w_{n+1},u_n)\Big]\Big\| \\
&\quad+\frac{\tau_3\rho_3}{\sigma\delta_3}\Big\|N_3(w_{n+1},u_n)-N_3(w_n,u_n)\Big\|. \qquad (4.11)
\end{aligned}$$

Since N_3 is α_3-strongly monotone with respect to second argument and (β_5,β_6)-Lipschitz continuous with respect to first and second arguments, respectively, we have the following estimates:

$$||N_3(w_{n+1},u_n)-N_3(w_n,u_n)|| \leq \beta_5||w_{n+1}-w_n||,$$

and

$$\Big\|u_{n+1}-u_n-\rho_3\Big[N_3(w_{n+1},u_{n+1})-N_3(w_{n+1},u_n)\Big]\Big\| \;\leq\; \sqrt{1-2\rho_3\alpha_3+\rho_3^2\beta_6^2}\;||u_{n+1}-u_n||.$$

Thus from (4.11), we have

$$||w_{n+1}-w_n|| \;\leq\; \frac{\tau_3(\theta_1+\theta_2)}{\sigma\delta_3-\tau_3\rho_3\beta_5}\;||u_{n+1}-u_n||, \qquad (4.12)$$

where $\theta_1=\sqrt{1-2\sigma+\zeta^2}$; $\theta_2=\sqrt{1-2\rho_3\alpha_3+\rho_3^2\beta_6^2}$.

Now from (4.8), we have

$$||v_{n+1}-v_n|| \leq \frac{\tau_2(\theta_1+\theta_3)}{\sigma\delta_2-\tau_2\rho_2\beta_3}\;||w_{n+1}-w_n||, \qquad (4.13)$$

where $\theta_1=\sqrt{1-2\sigma+\zeta^2}$; $\theta_3=\sqrt{1-2\rho_2\alpha_2+\rho_2^2\beta_4^2}$.

Now from (4.4), we have

$$||u_{n+2} - u_{n+1}||$$
$$\leq \left(\theta_1 + \frac{\tau_1\rho_1\beta_1}{\delta_1}\right) ||u_{n+1} - u_n|| + \frac{\tau_1\tau_2(\theta_1+\theta_3)(\theta_1+\theta_4)}{\delta_1(\sigma\delta_2 - \tau_2\rho_2\beta_3)} ||w_{n+1} - w_n||$$

$$= \left\{ \theta_1 + \frac{\tau_1\rho_1\beta_1}{\delta_1} + \frac{\tau_1\tau_2\tau_3(\theta_1+\theta_2)(\theta_1+\theta_3)(\theta_1+\theta_4)}{\delta_1(\sigma\delta_2 - \tau_2\rho_2\beta_3)(\sigma\delta_3 - \tau_3\rho_3\beta_5)} \right\} ||u_{n+1} - u_n||,$$

where

$$\theta_1 = \sqrt{1 - 2\sigma + \zeta^2}\ ; \qquad \theta_2 = \sqrt{1 - 2\rho_3\alpha_3 + \rho_3^2\beta_6^2}\ ;$$
$$\theta_3 = \sqrt{1 - 2\rho_2\alpha_2 + \rho_2^2\beta_4^2}\ ; \qquad \theta_4 = \sqrt{1 - 2\rho_1\alpha_1 + \rho_1^2\beta_2^2}\ .$$

Hence, we have

$$||u_{n+2} - u_{n+1}|| \leq \theta\ ||u_{n+1} - u_n||, \tag{4.14}$$

where

$$\theta := \theta_1 + \frac{\tau_1\rho_1\beta_1}{\delta_1} + \frac{\tau_1\tau_2\tau_3(\theta_1+\theta_2)(\theta_1+\theta_3)(\theta_1+\theta_4)}{\delta_1(\sigma\delta_2 - \tau_2\rho_2\beta_3)(\sigma\delta_3 - \tau_3\rho_3\beta_5)}$$
$$< \theta_1 + \frac{\tau_1\rho_1\beta_1}{\delta_1} + \frac{\tau_1(\theta_1+\theta_4)}{\delta_1}.$$

Since $\frac{\tau_3(\theta_1+\theta_2)}{(\sigma\delta_3 - \tau_3\rho_3\beta_5)} < 1$, $\frac{\tau_2(\theta_1+\theta_3)}{(\sigma\delta_2 - \tau_2\rho_2\beta_3)} < 1$ by conditions (4.2) and (4.3).

Also condition (4.1) ensures that $\theta_1 + \frac{\tau_1\rho_1\beta_1}{\delta_1} + \frac{\tau_1(\theta_1+\theta_4)}{\delta_1} < 1$.

Thus $0 < \theta < 1$. Now (4.14) implies that $\{u_n\}$ is a Cauchy sequence in H. Also, (4.12) and (4.13) implies that $\{v_n\}$, $\{w_n\}$ are cauchy sequences in H. Hence, there exist $u, v, w \in H$ such that $u_n \to u$, $v_n \to v$ and $w_n \to w$. Since N_1,N_2,N_3,g, $J^M_{\rho_1}$,$J^M_{\rho_2}$, $J^M_{\rho_3}$ are continuous, then it follows from Iterative Algorithm 3.2 that $u, v, w \in H$ satisfy (3.1),(3.2),(3.3), and thus, by Lemma 3.1, it follows that $u, v, w \in H$ is a solution of SGNVLI (2.4)-(2.6). This completes the proof.

If $\rho_1 = \rho_2 = \rho_3$ and $u = v = w$, Theorem 4.1 reduces to the following theorem which ensures the existence of a solution and the convergence criteria of Iterative Algorithm 3.3 for variational inclusion (2.7).

Theorem 4.2. *Let η, M, N_1 and g be same as in Theorem 4.1. If there exists a constant $\rho_1 > 0$ such that*

$$\left|\rho_1 - \frac{\tau_1\alpha_1 - \beta_1[\delta_1(1-\theta_1) - \tau_1\theta_1]}{\tau_1(\beta_2^2 - \beta_1^2)}\right|$$

$$< \frac{\sqrt{\left\{\tau_1\alpha_1 - \beta_1[\delta_1(1-\theta_1) - \tau_1\theta_1\right\}^2 - (\beta_2^2 - \beta_1^2)\left\{\tau_1^2 - \left(\delta_1(1-\theta_1) - \tau_1\theta_1\right)^2\right\}}}{(\beta_2^2 - \beta_1^2)\tau_1},$$

$$\tau_1\alpha_1 - \beta_1[\delta_1(1-\theta_1) - \tau_1\theta_1 \; > \; \sqrt{(\beta_2^2 - \beta_1^2)\left\{\tau_1^2 - \left(\delta_1(1-\theta_1) - \tau_1\theta_1\right)^2\right\}}, \qquad (4.15)$$

where $\theta_1 ;= \sqrt{1 - 2\sigma + \zeta^2}$, then the iterative sequence $\{u_n\}$ generated by Iterative Algorithm 3.3 strongly converges to $u \in H$ is the solution of variational inclusion (2.7).

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