International Journal of Advance Research in Science and Engineering
Volume No.07, Issue No.04, April 2018
www.ijarse.com
IJARSE
ISSN: 2319-8354

VECTOR SPACES

P.Malathi[1], M.Praveena[2], P.Naveen[3]

Department of Mathematics, IIIT SRIKAKULAM, India

ABSTRACT

A vector space is a set with two operations defined upon it. They are generally called as vector addition and scalar multiplication. Here the different conditions for the set to be a vector space are defined and verified. The importance of a vector space lies in the fact that many mathematical questions can be re phrased as a question about vector spaces. The geometric interpretation for elements of $\mathbb{R}^2$ and $\mathbb{R}^3$ as points in the Euclidean plane and Euclidean space. For every given vector space there exists two subspaces which are called trivial subspaces . Here the properties of vector spaces are studied. The conditions for the existence of sums and direct sums are also verified . The finite and infinite dimensional vector spaces are studies using span . The linear dependence and independence of a set of vectors is verified. The terms Basis and Dimension of a vector space are discussed.

Key words: vector space, Euclidean space, Euclidean plane, subspace, span, Basis, dimension.

A vector space is a set V with two operations defined upon it ; addition of Vectors and Multiplication by Scalars. These operations must satisfy certain properties. These scalars are obtained from a field F, where F is either is either equals to $\mathbb{R}$ for the real numbers, equals to $\mathbb{C}$ for complex numbers. Obviously $\mathbb{R},\mathbb{C}$ are examples of Filed.

A vector space over F is a set V together with the two operations $V \times V \to V$ and scalar multiplication $F \times V \to V$ satisfying the following conditions.

1. Commutative Property : $u+v=v+u;\ u,v \in V$
2. Associative Property : $u+(v+w)=(u+v)+\mathrm{w};\ u,v,w \in V \ \ and\ \ (ab)v=a(bv); a,b \in F$
3. Additive Identity: There exists an element $0 \in V$ such that $0+v=0,\ \forall v \in V$
4. Additive Inverse: for every $v \in V$, there exists and element $w \in V$ such that $v+w=0$
5. Multiplicative Identity : $1(v)=v;\ \forall v \in V$
6. Distributive Property: $a(u+v)=au+av \ \ and\ \ (a+b)u=au+bu\ ;\ \forall a,b \in F,\ u,v \in V$

A vector space over $\mathbb{R}$ is called as a Real vector space. The elements $v \in V$ are called vectors.

Example: consider F^n , the set of all n-tuples . This is a vector space with addition and scalar multiplication.

$u=(u_1,u_2,u_3,............u_n)$ & $v=(v_1,v_2,v_3,............v_n)\in F^n$ & $a\in F.$

$we\ define\ \ u+v=(u_1+v_1,u_2+v_2,u_3+v_3,............u_n+v_n),\ au=(au_1,au_2,au_3,.......au_n)$

The geometric interpretation for elements of $\mathbb{R}^2$ and $\mathbb{R}^3$ as points in the Euclidean plane and Euclidean space.

The following are the basic properties of vector spaces.

1. In every vector space additive identity is unique
2. For every $u\in V$ has a unique additive inverse
3. $ov=0,\quad \forall v\in V$
4. $ov=0,\quad \forall v\in V$
5. $(-1)v=-v\ \ for\ every\ v\in V$

Subspace: Let V is a vector space over F, and let $U\subset V$ be a subset of V. Then we say that U is a subspace of V if U is a vector space over F under the same operations that make V into a vector space over F.

Let $U\subset V$ be a subset of a vector space V over F. Then U is a sub space of V if and only if the following conditions are hold good

1. Additive Identity : $0\in U$
2. Closure under addition : $u,v\in U\Rightarrow u+v\in U$
3. Closed under the Scalar multiplication: $a\in F,u\in U\ \Rightarrow au\in U$

In every vector space V, the subsets $\{0\},V$ are obviously forms a subspace. These are called as the trivial subspaces. Any other apart from these two are called as non trivial subspaces.

Sum of Subspaces : Let U_1,U_2 are the two sub spaces of V. Then the sum of the subspaces is denoted and defined as $U_1+U_2=\{u_1+u_2\,/\,u_1\in U_1,u_2\in U_2\}$.

Direct Sum of Subspaces : Suppose that every $u\in U$ can be written as $u=u_1+u_2\ \ for\ u_1\in U_1$ and $u_2\in U_2$, then we write $U=U_1\oplus U_2$ and we call it as the direct sum of $U_1\ \&\ U_2$

Let $U_1\ \&\ U_2$ are the two sub spaces of V. Then $V=U_1\oplus U_2$ if and only if the following two conditions are holds good.

1. $V=U_1+U_2$
2. $if\ \ 0=u_1+u_2;\ u_1\in U_1\ \&\ u_2\in U_2,\ then\ \ u_1=u_2=0$

Similarly the following conditions are also sufficient enough to define the direct sum of sub spaces

Let U_1 & U_2 are the two sub spaces of V. Then $V = U_1 \oplus U_2$ if and only if the following two conditions are holds good.

1. $V = U_1 + U_2$
2. $\{0\} = U_1 \cap U_2$

The linear span of $(v_1, v_2, v_3, ..., v_m)$ is defined as the $span(v_1, v_2, v_3, ..., v_m) = \{a_1v_1 + a_2v_2 + ... + a_mv_m / a_1, a_2, ..., a_m \in F\}$

With the help of the above definition we can state the following lemma

Let V is a vector space and $v_1, v_2, v_3, ..., v_m \in V$, then the following holds good

1. $v_j \in span(v_1, v_2, v_3, ..., v_m)$
2. $span(v_1, v_2, v_3, ..., v_m)$ is a subspace of V.
3. If $U \subset V$ is a sub space such that $v_1, v_2, v_3, ..., v_m \in U$, then $span(v_1, v_2, v_3, ..., v_m) \subset U$

Using the definitions of a span we can say that a vector space is a finite dimensional or infinite dimensional using the following criteria

If $span(v_1, v_2, v_3, ..., v_m) = V$, then we can say that $(v_1, v_2, v_3, ..., v_m)$ spans V and we call V as **finite dimensional** . otherwise it is called **infinite dimensional.**

For example 1. the vectors $e_1 = (1,0,0,0,......0), e_2 = (0,1,0,0,......0), ..., e_n = (0,0,0,0,......1)$ spans F^n. Hence F^n is finite dimensional.

2. The vectors v_1 = (1, 1, 0) and v_2 = (1, -1, 0) span a subspace of R^3.

A list of vectors $(v_1, v_2, v_3, ..., v_m)$ is called **linearly independent** if the only solution for $a1, ...am$ $\in$ F to the equation $a_1v_1 + a_2v_2 + ... + a_mv_m = 0$ $a_1 = a_2 = ... = a_m = 0$.

In other words, the zero vector can only trivially be written as a linear combination of $(v_1, v_2, v_3, ..., v_m)$

A list of vectors $(v_1, v_2, v_3, ..., v_m)$ is called **linearly dependent** if it is not linearly independent. That is, $(v_1, v_2, v_3, ..., v_m)$ is linear dependent if there exist $a1, ...am$ $\in$ F, not all zero, such that $a_1v_1 + a_2v_2 + \dots\dots + a_m v_m = 0$

Basis: A list of vectors $(v_1, v_2, v_3, \dots, v_m)$ is a **basis** for the finite-dimensional vector space *V* if $(v_1, v_2, v_3, \dots, v_m)$ is linearly independent and *V* = span $(v_1, v_2, v_3, \dots, v_m)$

(**Basis Reduction Theorem**). *If V* = span $(v_1, v_2, v_3, \dots, v_m)$, *then either* $(v_1, v_2, v_3, \dots, v_m)$ *is a basis of V or some* v_i *can be removed to obtain a basis of V*

And here we can observe that every finite dimensional vector space has a basis.

And we can prove some additional content related to the basis. One of the main concept is Basis extension theorem.

It states that Every linearly independent list of vectors in a finite-dimensional vector space V can be extended to a basis of V .

Dimension:

The length of any basis of a vector is said to be the dimension of that vector space. It is generally denoted by $\dim(V)$.

An important observation regarding the dimension of a vector space is that any two bases of a given vector space are having same dimension.

REFERENCES

[1]. Linear Algebra, An Introduction to Abstract Mathematics : by Isaiah Lankham, Bruno Nachtergaele, Anne Schilling

[2]. Linear Algebra and its Applications by Gilbert Strang fourth edition

[3]. Linear Algebra and its Applications by DavidClay, fourth edition

[4]. Linear Algebra by Jim Hefferon, Third edition