International Journal of Advance Research in Science and Engineering
Volume No.07, Issue No.04, April 2018
www.ijarse.com
IJARSE
ISSN: 2319-8354

Fully discrete Finite Element Approximations of Semilinear Parabolic Equations in a Nonconvex Polygon

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ABSTRACT

In this paper, we consider the semilinear parabolic problems with homogeneous Dirichlet boundary conditions in a two-dimensional nonconvex polygon. We study the fully discrete error analysis for backward Euler method which is based on an error splitting technique. Previously, in [1], an effort has been made for problems in nonconvex polygons mainly focused on linear models. Also in [2], Thomée has discussed the error analysis for semilinear parabolic problems for a convex polygonal domain. A special feature in a nonconvex polygon is the presence of singularities in the solutions generated by the corners. Due to the nonlinearity in the forcing term and the non-smoothness of the solution in a nonconvex polygon, the analysis is not straightforward. We establish the convergence in $L^\infty(L^2)$ *for the semidiscrete finite element solution.*

Key words. *Semilinear parabolic problem, nonconvex polygon, singularity, fully discrete, error estimates*

AMS subject classifications. 65M60, 65N15

1. INTRODUCTION

The purpose of this paper is to study certain error estimates for piecewise linear finite element approximations to solutions of the semilinear parabolic equations in a nonconvex polygonal domain. We consider the discretization in both time and space, where the discretization with respect to space considered with piecewise linear finite elements and in time we apply the backward Euler method.

Let Ω be a bounded nonconvex polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega$. We restrict our attention to spatially semidiscrete approximate solutions of the semilinear initial-boundary value problem, for $u = u(x,t)$,

$$\begin{aligned} &u_t - \Delta u = f(u) \text{ in } \Omega,\ t \in J, \\ &u = 0 \quad \text{on } \partial\Omega,\ t \in J, \\ &\text{with } \ u = 0 \quad \text{in } \Omega, \end{aligned} \tag{1.1}$$

where u_t denotes $\partial u/\partial t$, the Laplacian denoted by $\Delta = \sum_{j=1}^{2} \partial^2/\partial x_j^2$ and $J = (0,T], T > 0$, be a finite interval in time. We assume the smooth function f on $\mathbb{R}$ such that

$$|f'(u)| \le B \quad \text{for } \ u \in \mathbb{R}. \tag{1.2}$$

The solution of parabolic partial differential equations in nonconvex polygonal domains is involved in many physical applications such as heat conduction in chip design, environmental modeling, porous media flow and modeling of complex technical engines (cf. [3]). The analysis for such PDEs and for the corresponding numerical methods is always been a challenging research area due to the non-smoothness of the solution around the reentrant corner of the domain. In the recent years Chatzipantelidiset. al. [1, 4] has studied the error analysis for linear parabolic models in a nonconvex polygon. To the best of author's knowledge the error estimates for the semilinear parabolic models in a nonconvex polygonal domain is introducing for the first time in the literature.

For simplicity, we assume that ω is exactly one interior angle $\omega$is reentrant,i.e., such that $\pi < \omega < 2\pi$. Setting$\beta = \pi/\omega$, we have$1/2 < \beta < 1$. For the case of L-shaped domain, $\omega = 3\pi/2$and$\beta = 2/3$.The regularity of the solutions of a simple elliptic problem

$$-\Delta u = f \quad in \ \Omega \quad with \quad u = 0 \quad on \ \partial\Omega, \tag{1.3}$$

for the nonconvex domain has been extensively studied, see Grisvard [5, 6]. In [7], Kellogg have shown the regularity shift-theorem for the solution of the problem (1.3) as

$$||u||_{H^{1+s}} \leq C||f||_{H^{-1+s}} = C||\Delta u||_{H^{-1+s}} \text{for} 0 \leq s < \beta, \tag{1.4}$$

where$H^s = H^s(\Omega)$ are fractional order Sobolev spaces, see Section 2. But we can not expect such estimate (1.4) for $s \geq \beta$ due to the singularity in the solution. A more precise analysis for the case $s = \beta$ was presented by Bacutaet. al. [8] in the framework of Besov spaces. In this paper we concentrate on the fully discrete finite element approximations for the problem (1.1) and derive convergence properties in the $L^\infty(L^2)$ norm.

The rest of the paper is organized as follows. In Section 2 we have presented some preliminary notations which will be used throughout this paper. In this section we have stated our main result for the estimates of the error between the solutions of the continuous and the fully discrete problem. Section 3 devoted to the proof of the fully discrete error estimates. There is a reduction in the convergence rate with respect to space discretization from optimal order to $\mathcal{O}(h^{2\beta})$caused by the presence of singularity in the solution due to the reentrant corner in the domain. However a systematical mesh refinement near the corners have been introduced in this section which gives an improvement of the convergence rate to the optimal order. With respect to time we have obtained an optimal convergence of order$\mathcal{O}(k)$. Numerical results are presented in Section 4. Finally, some concluding remarks are presented in the last section.

2. NOTATIONSANDPRELIMINARIES.

In this section, we introduce some basic preliminary notations which will be used throughout the paper. We denote the standard Lebesgue spaces by $L^p(\Omega), 1 \leq p \leq \infty$, with the norm$|| \cdot ||_{L^p(\Omega)}$. In particular, for $p = 2, L^2(\Omega)$is a Hilbert space with the norm $|| \cdot || = || \cdot ||_{L^2(\Omega)}$ induced by the inner product $(u,v) = \int_\Omega u(x)v(x)dx$. For an integer $m > 0$and $1 \leq p < \infty$, $W^{m,p}(\Omega)$ denotes the standard Sobolev space.In particular, for $p = 2$, we denote the Hilbert space $W^{m,2}(\Omega)$ by $H^m(\Omega)$ with the norm $|| \cdot ||_{H^m(\Omega)}$ (cf. [9, 10]). For an integer $m \geq 0$, set $s = m + \sigma, 0 < \sigma < 1$, and then $H^s = H^s(\Omega)$ denote the sobolev spaces of fractional order with the norm defined by

$$||u||_{H^s} = \left(||u||_{H^m}^2 + \sum_{|\alpha|=m} \int\int_{\Omega\times\Omega} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{2+2\sigma}} dxdy\right)^{1/2}.$$

For a given Banach space $\mathbf{B}$ and for $1 \leq p < \infty$, we define

$$L^p(0,T;\mathbf{B}) = \{v : [0,T] \to \mathbf{B} \mid v(t) \in \mathbf{B}\text{foralmostall } t \in [0,T]\text{and} \int_0^T ||v(t)||_{\mathbf{B}}^p\, dt < \infty\}$$

equipped with the norm

$$||v||_{L^p(0,T;\mathbf{B})} := \left(\int_0^T ||v(t)||_{\mathbf{B}}^p\, dt\right)^{1/p},$$

with the standard modification for $p = \infty$. We write $||v||_{L^p(0,T;\mathbf{B})} = ||v||_{L^p(\mathbf{B})}$.

2.1FULLY DISCRETE FINITE ELEMENT SOLUTION

Let $\mathcal{T}_h = \{K\}$ be the family of quasiuniform triangulations of $\Omega$with $max_{K\in\mathcal{T}_h} diam(K) \le h$, in the sense of Ciarlet [11] and Thomée [2]. Let thus$S_h \subset H_0^1(\Omega)$ be the finite dimensional space corresponding to the triangulations$\mathcal{T}_h$ is defined by

$$S_h = \{\, \chi \in C : \chi|_\tau \text{is linear}, \quad \forall \tau \in \mathcal{T}_h \text{and} \;\; \chi|_{\partial\Omega} = 0\,\},$$

where$C = C(\Omega)$ be the space of continuous functions on $\bar{\Omega}$. We study the semidiscrete solution $u_h : \bar{J} \to S_h$ such that

$$(u_{h,t},\chi) + (\nabla u_h, \nabla\chi) = (f(u_h),\chi) \qquad \forall \chi \in S_h, \;\; t \in J, \quad (2.1)$$

$$\text{With} \quad u_h(0) = v_h,$$

where$v_h \in S_h$ is an approximation of v.

We shall now turn to fully discrete schemes for the backward Euler method. Let k be the constant time step, $t_n = nk$, U^n be the approximation of the exact solution $u(t_n)$in S_h. This method is defined by replacing the time derivative in (2.1) by a backward Euler quotient $\bar{\partial} U^n = (U^n - U^{n-1})/k$,

$$(\bar{\partial} U^n, \chi) + (\nabla U^n, \nabla\chi) = (f(U^n),\chi) \qquad \forall \chi \in S_h, \;\; n \ge 1, \qquad (2.2)$$

$$\text{With} \quad U^0 = v_h.$$

We also consider the linearized version of (2.2) with replacing the term U^n by U^{n-1} in the nonlinear term $f(U^n)$:

$$(\bar{\partial} U^n, \chi) + (\nabla U^n, \nabla\chi) = (f(U^{n-1}),\chi) \qquad \forall \chi \in S_h, \;\; n \ge 1, \qquad (2.3)$$

$$\text{With} \quad U^0 = v_h.$$

The main aim of this paper is to prove the following estimate in $L^\infty(L^2)$ norm for the error between the solutions of the fully discrete problem (2.2) or (2.3) and the continuous problem (1.1).

Theorem 2.1. Let U^n and u be the solutions of (2.2) or (2.3), and (1.1), respectively. Assume that (1.2) hold true. Then, under the appropriate regularity assumptions for u, we have

$$||U^n - u(t_n)|| \le C||v_h - v|| + C(u)\left(h^{2\beta} + k\right) \quad for \;\; \beta < s < 1, \; t_n \in \bar{J}. \qquad (2.4)$$

For the purpose of the proof of Theorem 2.1, we introduce the so called *elliptic* or *Ritz projection*$R_h$onto S_h, defined by

$$(\nabla R_h v, \nabla\chi) = (\nabla v, \nabla\chi) \qquad \forall \chi \in S_h, \;\; for \;\; v \in H_0^1(\Omega). \qquad (2.5)$$

Setting $\chi = R_h v$ in (2.5), it follows that the Ritz projection is stable in $H_0^1(\Omega)$, i.e.,

$$||\nabla R_h v|| \le ||\nabla v|| \qquad \forall v \in H_0^1(\Omega).$$

IJARSE
ISSN: 2319-8354

We therefore have the following error estimate in this projection.

Lemma 2.2. Let R_h be defined by (2.5). Then for $v \in H^{1+s}(\Omega) \cap H_0^1(\Omega)$, $\beta < s \leq 1$with $C = C_s$, we have

$$\|R_h v - v\| + h^\beta \|\nabla(R_h v - v)\| \leq Ch^{2\beta} \|\Delta v\|_{H^{-1+s}}.$$

*Proof.*The proof is easily follows from [1, Lemma 2.5].

3. PROOFOF THEOREM 2.1

We first decompose the error in a standard way as

$$U^n - u(t_n) = (U^n - R_h u(t_n)) + (R_h u(t_n) - u(t_n)) = \theta^n + \rho^n, \tag{3.1}$$

whereR_h is defined by (2.5). In view of Lemma 2.2, the bound for $\rho^n = \rho(t_n)$ is given by

$$\|\rho^n\| \leq Ch^{2\beta} \|\Delta u(t_n)\|_{H^{-1+s}} = C(u)h^{2\beta}, \tag{3.2}$$

hence it only remains to estimate θ^n. From (2.2) we have using (2.5), $\chi \in S_h$.

$$(\bar{\partial}\theta^n, \chi) + (\nabla\theta^n, \nabla\chi) = (\bar{\partial}U^n, \chi) + (\nabla U^n, \nabla\chi) - (\bar{\partial}R_h u^n, \chi) - (\nabla R_h u^n, \nabla\chi)$$

$$= (f(U^n), \chi) - (u_t^n, \chi) - (\bar{\partial}R_h u^n - u_t^n, \chi) - (\nabla R_h u^n, \nabla\chi)$$

$$= (f(U^n), \chi) - (u_t^n, \chi) - (\bar{\partial}\rho^n, \chi) - (\bar{\partial}u^n - u_t^n, \chi) - (\nabla u^n, \nabla\chi)$$

$$= (f(U^n) - f(u^n), \chi) - (\bar{\partial}\rho^n, \chi) - (\bar{\partial}u^n - u_t^n, \chi),$$

or,

$$(\bar{\partial}\theta^n, \chi) + (\nabla\theta^n, \nabla\chi) = (f(U^n) - f(u^n), \chi) - (\bar{\partial}\rho^n, \chi) - (\bar{\partial}u^n - u_t^n, \chi). \tag{3.3}$$

Therefore, choosing $\chi = \theta^n$ and using (1.2) and (3.1), we obtain

$$\frac{1}{2}\bar{\partial}\|\theta^n\|^2 + \|\nabla\theta^n\|^2 \leq C\left(\|U^n - u^n\| + \|\bar{\partial}\rho^n\| + \|\bar{\partial}u^n - u_t^n\|\right)\|\theta^n\|$$

$$\leq C(\|\theta^n\|^2 + \|\rho^n\|^2 + \|\bar{\partial}\rho^n\|^2 + \|\bar{\partial}u^n - u_t^n\|^2)$$

$$= C\left(\|\theta^n\|^2 + R_n\right),$$

where$R_n = \|\rho^n\|^2 + \|\bar{\partial}\rho^n\|^2 + \|\bar{\partial}u^n - u_t^n\|^2$. This yields

$$(1 - Ck)\|\theta^n\|^2 \leq \|\theta^{n-1}\|^2 + CkR_n,$$

which gives, for small k,

$$\|\theta^n\|^2 \leq (1 + Ck)\|\theta^{n-1}\|^2 + CkR_n,$$

by repeated application we have

$$\|\theta^n\|^2 \leq (1 + Ck)^n\|\theta^0\|^2 + Ck\sum_{j=1}^{n}(1 + Ck)^{n-j}R_j$$

$$\leq C\|\theta^0\|^2 + Ck\sum_{j=1}^{n} R_j \qquad for \quad t_n \in J. \tag{3.4}$$

Using Lemma 2.2, we have

$$\|\rho^j\| \leq C(u)h^{2\beta},$$

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ISSN: 2319-8354

$$||\bar{\partial}\rho^j|| = ||k^{-1}(\rho^j - \rho^{j-1})|| = ||k^{-1}\int_{t_{j-1}}^{t_j} \rho_t ds|| \leq C(u)h^{2\beta},$$

and let $w^j = \bar{\partial}u^j - u_t^j$. Then

$$kw^j = u(t_j) - u(t_{j-1}) - ku_t(t_j) = -\int_{t_{j-1}}^{t_j}(s - t_{j-1})u_{tt}(s)ds,$$

or,

$$||\bar{\partial}u^j - u_t^j|| = ||k^{-1}\int_{t_{j-1}}^{t_j}(s - t_{j-1})u_{tt}(s)ds|| \leq C(u)k.$$

Altogether these estimates, we obtain $R_j \leq C(u)\left(h^{2\beta} + \mathrm{k}\right)^2, \beta < s < 1$. Again

$$||\theta^0|| = ||v_h - R_h v|| \leq ||v_h - v|| + ||R_h v - v|| \leq ||v_h - v|| + Ch^{2\beta}||\Delta v||_{H^{-1+s}}.$$

Therefore, equation (3.4) yields

$$||\theta^n|| \leq C||v_h - v|| + C(u)(h^{2\beta} + \mathrm{k}).$$

Altogether (3.1), (3.2) and (3.5), shows the required estimate of (2.4). In the same argument this estimates can be easily obtained for linearized modification form (2.3)(see e.g., [2, Theorem 13.3]). Hence this completes the proof.

Remark 3.1. Note that, $\mathcal{O}(h^{2\beta})$is the best possible convergence we obtain away from the nonconvex corner as the singularity at the reentrant corner pollutes the finite element solution everywhere in Ω for the case of globally quasiuniform mesh. However, with a systematical refinement of triangulations towards the nonconvex corner we obtain an optimal order convergence $\mathcal{O}(h^2)$in $L^\infty(L^2)$ norm. The refinement were introduced by Babuška[12].

Further refinement towards the nonconvex corner. In order to introduce the refinement of triangulations systematically (cf. [1]), let $d(x)$ be the distance to the nonconvex corner and $d_j = 2^{-j}$, for $j = 0,1,\dots,\hat{J}$. Assume that, for $j = 0,1,\dots,\hat{J}, \Omega_j = \{x \in \Omega : d_j/2 \leq d(x) \leq d_j\}, \Omega_j' = \Omega_{j-1} \cup \Omega_j \cup \Omega_{j+1}$, and $\Omega_J = \{x \in \Omega : d(x) \leq d_J/2\}$.Choose $\hat{J}$ such that $d_J \approx h^{1/\beta}$, where h be the meshsize in the interior of the domain. Furthermore, choose $\nu \geq 1/\beta$ such that

$$h_j \leq Chd_j^{1-\beta+\epsilon} \text{and} Ch^\nu \leq h_J \leq Ch^{1/\beta}, \quad \text{with } c > 0, \tag{3.6}$$

whereϵ be any small positive number, and h_j denotes the maximal meshsize onΩ_j. Also let the mesh is locally quasiuniform on each Ω_j' so that $h_{min} \geq h^\nu$and $\dim(S_h) \leq Ch^{-2}$. The finite element triangulations for an L-shaped domain is depicted in Figure 1a.

We now have the following auxiliary result.

Lemma 3.2. Let R_h be defined by (2.5). Then with the triangulations above, satisfying (3.6), we have

$$||R_h v - v|| + h||\nabla(R_h v - v)|| \leq Ch^2||\Delta v||.$$

*Proof.*Following Chatzipantelidiset. al. [1, Lemma 2.9]with $s = 1$, the proof is easily follows.

We finally show that the optimal order error bounds for Theorem 2.1 is obtained by refinement towards the nonconvex corner.

Theorem 3.3. Let U^n and u be the solutions of (2.2) or (2.3), and (1.1), respectively. Assume that the triangulations underlying the S_h are refined as in Lemma 3.2, and (1.2) hold true. Then, under the appropriate regularity assumptions for u, we have

$$\|U^n - u(t_n)\| \le C\|v_h - v\| + C(u)(h^2 + k).$$

*Proof.*In view of Lemma 3.2 and following the similar argument as in the proof of Theorem 2.1, the rest of the proof is standard.

4. NUMERICAL EXPERIMENTS

In this section we perform numerical experiments of test problems to validate the theoretical rates of convergence (ROC).

Example 4.1. Let us consider the problem on the L-shaped domain,$\Omega = (0,1)^2 \times [0.6,1)^2$:

$u_t - \Delta u = u - u^3 \text{ in } \Omega \times (0,0.1]$,

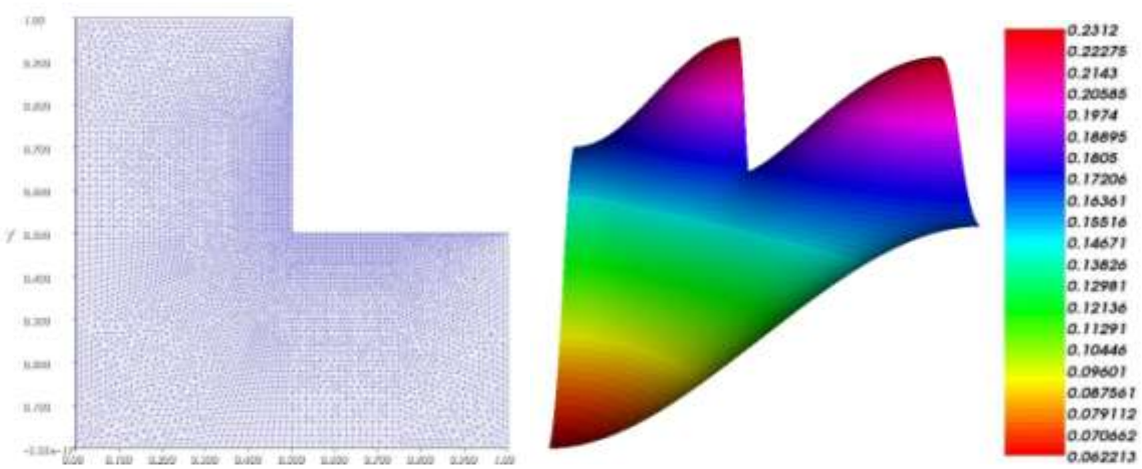
$u = 0 \quad \text{on } \partial\Omega \times (0,0.1]$,

with $u(x,y,0) = xy \quad \text{in } \Omega$,

We consider the backward Euler method with $\mathbb{P}_1$ finite elements and choose the initial mesh size $h = 0.152$ and $k = 0.02$. The ROC is given in Table 1, which shows that our numerical results gives an optimal order convergence which coincides with the theoretical rates of convergence. The finite element solution is depicted in Figure 1b.

Table 1: ROC in $L^\infty(L^2)$ norm for Example 4.1

h	#dof	u_h	$\|\|u_h - u\|\|_{L^\infty(L^2)}$	ROC
0.152	129	0.230134	3.670431e-01	----
0.076	461	0.231145	9.600612e-02	1.935
0.038	1762	0.231187	2.425745e-02	1.984
0.019	6860	0.231201	5.464576e-03	2.151



(a) (b)

Figure 1: (a) Finite element discretizations for the L-shaped domain, further refinement made near the nonconvex corner. (b) Finite element solution for the L-shaped domain for $h = 0.019$, $k = 0.02$, maximum value of solution= 0.2312.

5. CONCLUSIONS

We have presented an approach for the solution of Semilinear parabolic equations in nonconvex polygonal domains. A priori error estimates in $L^\infty(L^2)$norms for the fully discrete case are discussed and analyzed. Starting from a convergence rate $\mathcal{O}(h^{2\beta} + k)$in $L^\infty(L^2)$norm for the nonconvex polygon, we have obtained an optimal order convergence $\mathcal{O}(h^2 + k)$with a proper mesh refinement near the re-entrant corners of the domain.

ACKNOWLEDGEMENTS

The author wish to thank Professor Rajen Kumar Sinha, Indian Institute of Technology Guwahati, for discussing some of the results presented in this paper and inspiring for this research work.

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