International Journal of Advance Research in Science and Engineering
Volume No.07, Special Issue No.04, April 2018
www.ijarse.com
IJARSE
ISSN: 2319-8354

GENERALIZED FIBONACCI-TYPE POLYNOMIALS

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ABSTRACT

In this study, we present Fibonacci-type polynomials. We have used their Binet's formula to derive the identities. The proofs of the main theorems are based on simple algebra and give several interesting properties involving them.

Mathematics Subject Classification: 11B39, 11B37

Keywords*: *Generalized Fibonacci polynomials, Binet's formula, Generating function.

I. INTRODUCTION

Fibonacci polynomials are a great importance in mathematics. Large classes of polynomials can be defined by Fibonacci-like recurrence relation and yield Fibonacci numbers [15]. Such polynomials, called the Fibonacci polynomials, were studied in 1883 by the Belgian Mathematician Eugene Charles Catalan and the German Mathematician E. Jacobsthal.

The polynomials $f_n(x)$ studied by Catalan are defined by the recurrence relation

$$f_{n+2}(x) = xf_{n+1}(x) + f_n(x) \tag{1.1}$$

where $f_1(x)=1,\ f_2(x)=x$, and $n \geq 3$. Notice that $f_n(1)=F_n$, the *n*th Fibonacci number.

The Fibonacci polynomials studied by Jacobsthal were defined by

$$J_n(x) = J_{n-1}(x) + xJ_{n-2}(x) \tag{1.2}$$

where $J_1(x)=1=J_2(x)$, and $n \geq 3$.

The Pell polynomials $p_n(x)$ are defined by

$$p_n(x) = 2xp_{n-1}(x) + p_{n-2}(x) \tag{1.3}$$

where $p_0(x)=0,\ p_1(x)=1$, and $n \geq 2$.

The Lucas polynomials $l_n(x)$, originally studied in 1970 by Bicknell, are defined by

$$l_n(x) = xl_{n-1}(x) + l_{n-2}(x) \tag{1.4}$$

where $l_0(x)=2,\ l_1(x)=x$, and $n \geq 2$.

It is well known that the Fibonacci polynomials and Lucas polynomials are closely related. Obviously, they have a deep relationship with the famous Fibonacci and Lucas sequences. That is $f_n(1) = F_n$ and $l_n(1) = L_n$, where F_n and L_n are the Fibonacci and Lucas numbers. Swamy [11] defined the Fibonacci Polynomials and obtained some more identities for these polynomials. Hogget and Lind [17] make a similar "symbolic substitution" of certain sequences into the Fibonacci polynomials, they extend these results to the substitution of any recur rent sequence into any sequence of polynomials obeying a recurrence relation with polynomial coefficients. Since then many problems about the polynomials have been proposed in various issue of the Fibonacci Quarterly. Hoggatt, Philips and Leonard [16] have obtained some more identities involving Fibonacci Polynomials and Lucas polynomials. A. Lupas [3] present many interesting properties of Fibonacci and Lucas Polynomials. C. Berg [4] defined Fibonacci numbers and orthogonal polynomials. S. Falcon and A. Plaza [13] defined the k-Fibonacci polynomials are the natural extension of the k-Fibonacci numbers and many of their properties admit a straightforward proof and many relations for the derivatives of Fibonacci polynomials are proven. K. Kaygisiz and A. Sahin [10] present new generalizations of the Lucas numbers by matrix representation, using Generalized Lucas Polynomials. G. Y. Lee and M. Asci [8], consider the Pascal matrix and define a new generalization of Fibonacci polynomials called (p, q)-Fibonacci polynomials. They obtain combinatorial identities and by using Riordan method they get a factorizations of Pascal matrix involving (p, q)-Fibonacci polynomials. Many authors have studied Fibonacci polynomials. Panwar, Singh and Gupta [18] derived many fundamental properties and sums of generalized Fibonacci Polynomials. In this paper, we present Fibonacci-type Polynomials by changing the initial terms and recurrence relation.

II. FIBONACCI-TYPE POLYNOMIALS

The Fibonacci-type polynomials defined by

$$y_n(x) = y_{n-1}(x) + 2x\, y_{n-2}(x) \;\; ; \; n \geq 2 \tag{2.1}$$

with $y_0(x) = 2 \;\; and \;\; y_1(x) = 2$

First few polynomials are

$y_2(x) = 2 + 4x$

$y_3(x) = 2 + 8x$

$y_4(x) = 2 + 12x + 8x^2$

$y_5(x) = 2 + 16x + 24x^2$

$y_6(x) = 2 + 20x + 48x^2 + 16x^3$

…

III. PROPERTIES OF FIBONACCI-TYPE POLYNOMIALS

1. First Explicit Formula for Fibonacci-type polynomials

In the 19th century, the French mathematician Binet devised two remarkable analytical formulas for the Fibonacci and Lucas numbers. In our case, Binet's formula allows us to express the Fibonacci-type Polynomials in function of the roots $\Re_1$ & $\Re_2$ of the following characteristic equation, associated to the recurrence relation (2.1):

$$t^2 = t + 2x \tag{3.1}$$

Theorem 1: (Binet's formula). The nth Fibonacci-type Polynomial is given by

$$y_n(x) = 2\frac{\Re_1^{n+1} - \Re_2^{n+1}}{\Re_1 - \Re_2} \tag{3.2}$$

where $\Re_1$ & $\Re_2$ are the roots of the characteristic equation (3.1) , $\Re_1 > \Re_2$ and

$\Re_1 = \dfrac{1+\sqrt{1+8x}}{2}$ and $\Re_2 = \dfrac{1-\sqrt{1+8x}}{2}$.

Proof: we use the Principle of Mathematical Induction (PMI) on n. It is clear the result is true for $n = 0$ *and* $n = 1$ by hypothesis. Assume that it is true for i such that $0 \le i \le r+1$, then

$$y_i(x) = 2\frac{\Re_1^{i+1} - \Re_2^{i+1}}{\Re_1 - \Re_2}$$

It follows from definition of Fibonacci-type Polynomials and from equation (3.2),

$$y_{r+2}(x) = y_{r+1}(x) + 2x\, y_r(x) = 2\frac{\Re_1^{r+3} - \Re_2^{r+3}}{\Re_1 - \Re_2} \tag{3.3}$$

Thus, the formula is true for any positive integer n.

Proposition 2: For any integer $n \ge 1$,

$$\begin{aligned} \Re_1^{n+2} &= \Re_1^{n+1} + 2x\Re_1^n \\ \Re_2^{n+2} &= \Re_2^{n+1} + 2x\Re_2^n \end{aligned} \tag{3.4}$$

Proof: Since $\Re_1$ & $\Re_2$ are the roots of the characteristic equation (3.1), then

$$\Re_1^2 = \Re_1 + 2x$$
$$\Re_2^2 = \Re_2 + 2x$$

now, multiplying both sides of these equations by $\Re_1^n$ & $\Re_2^n$ respectively, we obtain the desired result.

Theorem 3: For any integer $n \geq 1$,

$$y_n(x) + y_{n-2}(x) = y_{n-1}(x) + (2x+1) y_{n-2}(x) \tag{3.5}$$

Proof: By using Eq. (3.2) in the R.H.S. of Eq. (3.5) and taking in to account that $\Re_1 \Re_2 = -2x$, it is obtained

$$\begin{aligned}
(\text{RHS}) &= 2\frac{\Re_1^{n+1} - \Re_2^{n+1}}{\Re_1 - \Re_2} + 2\frac{\Re_1^{n-1} - \Re_2^{n-1}}{\Re_1 - \Re_2} \\
&= \frac{2}{\Re_1 - \Re_2}\left\{\Re_1^n\left(\Re_1 + \frac{1}{\Re_1}\right) - \Re_2^n\left(\Re_2 + \frac{1}{\Re_2}\right)\right\} \\
&= \frac{2}{\Re_1 - \Re_2}\left\{\Re_1^n - \Re_2^n + \left(\Re_1^{n-1} - \Re_2^{n-1}\right)(2x+1)\right\} \\
&= 2\left(\frac{\Re_1^n - \Re_2^n}{\Re_1 - \Re_2}\right) + 2\left(\frac{\Re_1^{n-1} - \Re_2^{n-1}}{\Re_1 - \Re_2}\right)(2x+1) \\
&= y_{n-1}(x) + (2x+1) y_{n-2}(x)
\end{aligned}$$

This completes the proof.

Corollary 3.1: $y_n(x) + y_{n-2}(x) = 2\left[J_n(x) + (2x+1) J_{n-1}(x)\right]$ (3.6)

Theorem 4: For any integer $n \geq 1$,

$$(1+8x)\, y_{-1}^2(x) = \begin{cases} 4\left(\Re_1^n - \Re_2^n\right)^2 & ; \ if\ n\ is\ even \\ 4\left(\Re_1^n + \Re_2^n\right)^2 & ; \ if\ n\ is\ odd \end{cases} \tag{3.7}$$

Proof: From the Binet's formula of Fibonacci-type Polynomials

$$y_{n-1}^2(x) = \frac{4}{\left(\Re_1 - \Re_2\right)^2}\left\{\Re_1^{2n} - 2(\Re_1\Re_2)^n + \Re_2^{2n}\right\}$$

If n is even $(1+8x)\, y_{-1}^2(x) = 4(\Re_1^n - \Re_2^n)^2$

If n is odd $(1+8x)\, y_{-1}^2(x) = 4(\Re_1^n + \Re_2^n)^2$

Let us denote $(\Re_1^n + \Re_2^n)$ by $j_n(x)$.

Then previous formula become: $(1+8x)\, y_{-1}^2(x) = 4 j_n^2(x)$ (3.8)

2. Catalan's Identity

Catalan's identity for Fibonacci numbers was found in 1879 by Eugene Charles Catalan a Belgian mathematician who worked for the Belgian Academy of Science in the field of number theory.

Theorem 5: (Catalan's identity) $y_{n-1}^2(x) - y_{n+r-1}(x)y_{n-r-1}(x) = (-2x)^{n-r}\, y_{r-1}^2(x)$ (3.9)

Proof: $y_{n-1}^2(x) - y_{n+r-1}(x)y_{n-r-1}(x) = 4\left(\frac{\Re_1^n - \Re_2^n}{\Re_1 - \Re_2}\right)^2 - 4\left(\frac{\Re_1^{n+r} - \Re_2^{n+r}}{\Re_1 - \Re_2}\right)\left(\frac{\Re_1^{n-r} - \Re_2^{n-r}}{\Re_1 - \Re_2}\right)$

$$= \frac{4}{(\Re_1 - \Re_2)^2}\left(\frac{\Re_1^{2r} + \Re_2^{2r}}{(\Re_1\Re_2)^r} - 2\right)(\Re_1\Re_2)^n$$

$$= (-2x)^{n-r}\left(2\frac{\Re_1^r - \Re_2^r}{\Re_1 - \Re_2}\right)^2$$

$$y_{n-1}^2(x) - y_{n+r-1}(x)y_{n-r-1}(x) = (-2x)^{n-r}\, y_{r-1}^2(x)$$

This completes the proof.

Corollary 5.1: $y_{n-1}^2(x) - y_{n+r-1}(x)y_{n-r-1}(x) = (-2x)^{n-r}\, 4J_r^2(x)$ (3.10)

3. Cassini's Identity

This is one of the oldest identities involving the Fibonacci numbers. It was discovered in 1680 by Jean-Dominique Cassini a French astronomer.

Theorem 6: (*Cassini's identity or Simpson's identity*)

$$y_{n-1}^2(x) - y_n(x)y_{n-2}(x) = (-2x)^{n-1}4 \qquad (3.11)$$

Proof: Taking $r = 1$ in Catalan's identity (3.10) the proof is completed.

4. Limit of the quotient of two consecutive terms

A useful property in these polynomials is that the limit of the quotient of two consecutive terms is equal to the positive root of the corresponding characteristic equation.

Theorem 7: $\lim_{n\to\infty}\left(\frac{y_{n-1}(x)}{y_{n-2}(x)}\right) = \Re_1$ (3.12)

Proof: By Binet's formula (3.2), we have

$$\lim_{n\to\infty}\left(\frac{y_{n-1}(x)}{y_{n-2}(x)}\right)=\lim_{n\to\infty}\frac{\Re_1^n-\Re_2^n}{\Re_1^{n-1}-\Re_2^{n-1}}=\lim_{n\to\infty}\frac{1-\left(\frac{\Re_2}{\Re_1}\right)^n}{\frac{1}{\Re_1}-\left(\frac{\Re_2}{\Re_1}\right)^n\frac{1}{\Re_2}}$$

and taking into account that $\lim_{n\to\infty}\left(\frac{\Re_2}{\Re_1}\right)^n=0$, since $|\Re_2|<\Re_1$, Eq. (3.12) is obtained.

Theorem 8: If $S_n=\sum_{i=0}^{n} y_{i-1}(x)$, then $S_n=y_{n-1}(x)+\left\{\frac{2-y_n(x)}{(-2x)}\right\}$ (3.13)

Proof: By Binet's formula (3.2), we have

$$S_n=2\sum_{i=0}^{n}\frac{\Re_1^i-\Re_2^i}{\Re_1-\Re_2}$$

$$=\frac{2}{(\Re_1-\Re_2)}\left(\frac{\Re_1^{n+1}-1}{\Re_1-1}-\frac{\Re_2^{n+1}-1}{\Re_2-1}\right)$$

$$=\frac{2}{(\Re_1-1)(\Re_2-1)}\left\{\Re_1\Re_2\left(\frac{\Re_1^n-\Re_2^n}{\Re_1-\Re_2}\right)+\frac{\Re_1-\Re_2}{\Re_1-\Re_2}-\frac{\Re_1^{n+1}-\Re_2^{n+1}}{\Re_1-\Re_2}\right\}\backslash$$

$$S_n=y_{n-1}(x)+\left\{\frac{2-y_n(x)}{(-2x)}\right\}$$

This completes the proof.

Corollary 8.1: $S_n=2\left[J_n(x)+\left\{\frac{1-J_{n+1}(x)}{(-2x)}\right\}\right]$ (3.14)

IV. SUMS OF FIBONACCI-TYPE POLYNOMIALS

In this section, we study the sums of Fibonacci-type Polynomials. This enables us to give in a straightforward way several formulas for the sums of such Polynomials.

Theorem 9: For fixed integers $p,\ q\ with\ 0\le q\le p-1$, the following equality holds

$$y_{p(n+2)+q-1}(x)=j_p(x)y_{p(n+1)+q-1}(x)-(-2x)^p\,y_{pn+q-1}(x) \quad (4.1)$$

Proof: From the Binet's formula of Fibonacci-type and jacobsthal-Lucas

Polynomials, $j_p(x)y_{p(n+1)+q-1}(x) = 2\left(\Re_1^n + \Re_2^n\right)\left(\dfrac{\Re_1^{p(n+1)+q} - \Re_2^{p(n+1)+q}}{\Re_1 - \Re_2}\right)$

$$= \frac{2}{\Re_1 - \Re_2}\left[\left\{\Re_1^{p(n+2)+q} - \Re_2^{p(n+2)+q}\right\} + (-\Re_1\Re_2)^p\left(\Re_1^{pn+q} - \Re_2^{pn+q}\right)\right]$$

$$= y_{p(n+2)+q-1}(x) + (-2x)^p y_{pn+q-1}(x)$$

then, the equality becomes,

$$y_{p(n+2)+q-1}(x) = j_p(x)y_{p(n+1)+q-1}(x) - (-2x)^p y_{pn+q-1}(x)$$

This completes the proof.

Theorem 10: For fixed integers $p,\ q\ with\ 0 \le q \le p-1$, the following equality holds

$$\sum_{i=0}^{n} y_{pi+q-1}(x) = \frac{y_{q-1}(x) - y_{p(n+1)+q-1}(x) + (-2x)^p\left\{y_{pn+q-1}(x) - y_{q-p-1}(x)\right\}}{1 - j_p(x) + (-2x)^p} \tag{4.2}$$

Proof: Applying Binet's formula of generalized Fibonacci Polynomials,

$$\sum_{i=0}^{n} y_{pi+q-1}(x) = 2\sum_{i=0}^{n}\frac{\Re_1^{pi+q} - \Re_2^{pi+q}}{\Re_1 - \Re_2}$$

$$= \frac{2}{\Re_1 - \Re_2}\left[\sum_{i=0}^{n}\Re_1^{pi+q} - \sum_{i=0}^{n}\Re_2^{pi+q}\right]$$

$$= \frac{2}{\Re_1 - \Re_2}\left[\frac{\Re_1^{pn+q+p} - \Re_1^q}{\Re_1^p - 1} - \frac{\Re_2^{pn+q+p} - \Re_2^q}{\Re_2^p - 1}\right]$$

$$= \frac{1}{(-2x)^p - j_p(x) + 1}\left[(-2x)^p\left\{y_{pn+q-1}(x) - y_{q-p-1}(x)\right\} - y_{p(n+1)+q-1}(x) + y_{q-1}(x)\right]$$

$$= \frac{y_{q-1}(x) - y_{p(n+1)+q-1}(x) + (-2x)^p\left\{y_{pn+q-1}(x) - y_{q-p-1}(x)\right\}}{1 - j_p(x) + (-2x)^p}$$

This completes the proof.

Corollary 10.1: $\displaystyle\sum_{i=0}^{n} y_{pi+q-1}(x) = \frac{\left[J_q(x) - J_{p(n+1)+q}(x) + (-2x)^p\left\{J_{pn+q}(x) - J_{q-p}(x)\right\}\right]}{1 - j_p(x) + (-2x)^p}$ (4.3)

V. CONCLUSION

We have derived many fundamental properties in this paper. We describe sums of generalized Fibonacci-type Polynomials. This enables us to give in a straightforward way several formulas for the sums of such Polynomials. These identities can be used to develop new identities of polynomials.

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