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Mathematical Model of Pulsatile Flow of Blood through a Circular Tube with Transient Term

Anand Bansal[1], P. R. Sharma[2], P. Chaturani[3]

1Department of Mathematics, Government Engineering College, Ajmer (India)

2Department of Mathematics (Retd.), University of Rajasthan, Jaipur (India)

3Department of Mathematics (Retd.), IIT Powai, Mumbai (India)

ABSTRACT

Aim of this paper is to derive the transient term of velocity profile for pulsatile flow of blood. This term has not been given proper attention in available literature. It is assumed to be negligible compared to other terms. The effects of initial conditions on the analytic solution of the pulsatile flow of blood are investigated. The governing equation of motion is solved using two different initial and boundary conditions given by Lightfoot [1] and Sud and Sekhon [2] The change in velocity expression is observed only in transient term. Comparative values of the transient term for two different initial conditions are given in Table-1. Comparison of the magnitudes of transient, steady state and oscillatory forms in the velocity expression when body force G = 0.0 and 1.0g are made through Tables-2 and 3, respectively. Variation of velocity transient term with viscosity is shown through Fig. 1 and variation of its magnitude with time when body force G = 0.0 and 1.0g are presented through Fig. 2 to 5.

Keywords:Blood, body force, circular tube, pressure gradient, pulsatile, transient term.

I.INTRODUCTION

In the literature of pulsatile flow of blood, two mathematical methods have been used to study the flow characteristics. The first one is method of separation of variables, used by Womersley[3], Lightfoot [1], Verma and Sharma [4], Milnor[5], Sharma, Ariel and Chaturani [6] and Sharma and Mishra [7] etc. In this method, flow velocity was considered of the form:

$$u_z(r,t) = e^{i\omega t} u_z(r). \qquad \ldots(1)$$

The other method is namely integral transform technique, which has been used by Chaturani and Rathod [8] and Sud and Sekhon [2] etc. It is noticed that the solutions obtained by above mentioned methods are exactly same with only one major difference. The second method (integral transform technique) gives transition term, which is many a time quite important. Where as, in the first method, it has been assumed that steady state pulsatile flow is of the form of the total force.

It appears that some researchers (Sud and Sekhon [2], Chaturani and Palanisamy [9 and 10] etc.) had a feeling that the duration of the transient time and the magnitude of the flow variables in transient time is small in comparison to oscillatory period and the magnitude of the amplitude of the steady state oscillatory part of the flow variables, hence the transient quantities have not been computed, in literature.

In this paper, aim is to derive transient term of velocity profile for pulsatile flow of blood through a circular tube. Then the effects of initial conditions on the analytic solution of the pulsatile flow of blood are investigated.

II. GOVERNING EQUATIONS OF MOTION

Assuming blood to be viscous, incompressible and Newtonian fluid, and the tube wall to be rigid and very long compared to its diameter, flow to be symmetric about tube axis, the variation of velocity along the tube length to be small in comparison to the rate of change of velocity with respect to time and radial distance, and the frequency of body acceleration be small so that the wave effect can be neglected. Under these assumptions, the governing equation of motion for the flow of blood in the presence of body force through a circular tube in a cylindrical polar coordinate system is given by:

$$\rho \frac{\partial u_z}{\partial t} = \rho G - \frac{\partial p}{\partial z} + \mu_f \left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r} \frac{\partial u_z}{\partial r} \right), \qquad \ldots(2)$$

where ρ *and* μ_f are the density and viscosity of blood respectively, u_z is the axial component of velocity, z is axial distance and t is time and r is the radial coordinate. G is the body force in axial direction in terms of Fourier series (Kreyszig [11]) given by:

$$\text{G=} a_0 + \sum_{m=1}^{\infty} \left[a_m \cos(m\omega_b t + \phi_m) + b_m \sin(m\omega_b t + \phi_m) \right], \qquad \ldots(3)$$

$\omega_b = 2\pi f_b$, f_b is its frequency with $\partial p / \partial z$, ϕ_m is its phase difference, a_0, a_m and b_m are Fourier coefficients and p is the pressure which is a function of t and z:

$$-\frac{\partial p}{\partial z} = A_0 + \sum_{n=1}^{\infty} \left[A_n \cos(n\omega_p t) + B_n \sin(n\omega_p t) \right], \qquad t \geq o \qquad \ldots(4)$$

$\omega_p = 2\pi f_p$, f_p is heart pulse frequency and A_0, A_n and B_n are Fourier coefficients.

The governing equation of motion is solved by using two different initial and boundary conditions as mentioned below

(i) Initial and boundary conditions considered by Lightfoot [1] are:

When t = o; $u_z(r,0) = u_0$ (const.),

When t > 0; r = 0: $u_z(0,t)$ is finite, r = R: $u_z(R,t)$ = 0. …(5)

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(ii) Initial and boundary conditions considered by Sud and Sekhon [2]) are:

When t = 0; $u_z(r,0) = \dfrac{(R^2 - r^2)\left(A_0 + \sum_{n=1}^{\infty} A_n\right)}{4\mu_f}$,

When t > 0; r = 0: $u_z(0,t)$ is finite, r = R: $u_z(R,t)$ = 0. …(6)

III.METHOD OF SOLUTION

Applying Laplace transformation (Sneddon [12] and Kreyszig [11]), the solution of the equation (2) is obtained of the form given by:

$$u(r,t) = (\rho a_0 + A_0)\frac{(R^2 - r^2)}{4\mu_f} +$$

$$\sum_{m=1}^{\infty}\left\{\frac{a_m \sin(m\omega_b t + \phi_m) - b_m \cos(m\omega_b t + \phi_m)}{m\omega_b} - \frac{(a_m - ib_m)}{2im\omega_b}\frac{J_0\left(r\sqrt{\frac{-i\rho m\omega_b}{\mu_f}}\right)}{J_0\left(R\sqrt{\frac{-i\rho m\omega_b}{\mu_f}}\right)}e^{i(m\omega_b t + \phi_m)}\right.$$

$$\left. + \frac{(a_m + ib_m)}{2im\omega_b}\frac{J_0\left(r\sqrt{\frac{i\rho m\omega_b}{\mu_f}}\right)}{J_0\left(R\sqrt{\frac{i\rho m\omega_b}{\mu_f}}\right)}e^{-i(m\omega_b t + \phi_m)}\right\} + \sum_{n=1}^{\infty}\left\{\frac{A_n \sin(n\omega_p t) - B_n (n\omega_p t)}{\rho n\omega_p}\right.$$

$$- \frac{(A_n - iB_n)}{2in\rho\omega_b}\frac{J_0\left(r\sqrt{\frac{-i\rho n\omega_p}{\mu_f}}\right)}{J_0\left(R\sqrt{\frac{-i\rho n\omega_p}{\mu_f}}\right)}e^{in\omega_p t} + \frac{(A_n + iB_n)}{2in\rho\omega_b}\frac{J_0\left(r\sqrt{\frac{i\rho n\omega_p}{\mu_f}}\right)}{J_0\left(R\sqrt{\frac{i\rho n\omega_p}{\mu_f}}\right)}e^{-in\omega_p t}$$

+ Transient term (u_t). …(7)

where i = $\sqrt{-1}$, J_0 is the Bessel function of zeroth order with complex argument.

For first initial and boundary conditions (5), the transient term u_{t1} is obtained and given by:

$$u_{t1} = 2\sum_{k=1}^{\infty}\frac{e^{-\eta\lambda_k^2 t}}{\rho\eta\lambda_k^3 J_1(\lambda_k)}J_0\left(\frac{r}{R}\lambda_k\right)\left[\left(-\rho a_0 - A_0 + \rho u_0\eta\lambda_k^2\right) - \right.$$

$$\rho\sum_{m=1}^{\infty}\frac{\left[a_m\left(\eta^2\lambda_k^4\cos\phi_m + m\omega_b\eta\lambda_k^2\sin\phi_m\right) + b_m\left(-m\omega_b\eta\lambda_k^2\cos\phi_m + \eta^2\lambda_k^4\sin\phi_m\right)\right.}{\left(\eta^2\lambda_k^4 + m^2\omega_b^2\right)} -$$

$$\sum_{n=1}^{\infty}\left[\frac{\eta^2\lambda_k^4 A_n - n\omega_p\eta\lambda_k^2 B_n}{\left(\eta^2\lambda_k^4 + n^2\omega_b^2\right)}\right]\Bigg\}, \qquad \ldots(8)$$

where $\eta = \mu_f / \rho R^2$ and $\lambda_1,\ \lambda_2, \ldots\ldots\ \lambda_k$ are the zeros of the Bessel function J_0 given by:

$$J_0\left(R\sqrt{\frac{i^2 s\rho}{\mu_f}}\right)=0.$$

When the equation (2) is solved under initial and boundary conditions (6), then steady pulsatile part of the solution remains same but the expression for transient term changes and the changed form of transient term u_{t2} is given by:

$$u_{t2} = 2\sum_{k=1}^{\infty}\frac{e^{-\eta\lambda_k^2 t}}{\rho\eta\lambda_k^3 J_1(\lambda_k)}J_0\left(\frac{r}{R}\lambda_k\right)\Bigg\{\left(-\rho a_0 + \sum_{n=1}^{\infty}A_n\right) -$$

$$\rho\sum_{m=1}^{\infty}\frac{\left[a_m\left(\eta^2\lambda_k^4\cos\phi_m + m\omega_b\eta\lambda_k^2\sin\phi_m\right) + b_m\left(-m\omega_b\eta\lambda_k^2\cos\phi_m + \eta^2\lambda_k^4\sin\phi_m\right)\right]}{\left(\eta^2\lambda_k^4 + m^2\omega_b^2\right)} - .$$

$$\sum_{n=1}^{\infty}\left[\frac{\eta^2\lambda_k^4 A_n - n\omega_p\eta\lambda_k^2 B_n}{\left(\eta^2\lambda_k^4 + n^2\omega_b^2\right)}\right]\Bigg\}. \qquad \ldots(9)$$

Now both transient terms u_{t1} and u_{t2} can be written as given below:

$$u_{t1} = 2\sum_{k=1}^{\infty}\left[D_k\left(-\rho a_0 - A_0 + \rho u_0\eta\lambda_k^2\right) + C_k\right], \qquad \ldots(10)$$

$$u_{t2} = 2\sum_{k=1}^{\infty}\left[D_k\left(-\rho a_0 + \sum_{n=1}^{\infty}A_n\right) + C_k\right], \qquad \ldots(11)$$

where

$$D_k = \frac{e^{-\eta\lambda_k^2 t}}{\rho\eta\lambda_k^3 J_1(\lambda_k)}J_0\left(\frac{r}{R}\lambda_k\right), \qquad \ldots(12)$$

and

$$C_k = 2\sum_{k=1}^{\infty}\frac{e^{-\eta\lambda_k^2 t}}{\rho\eta\lambda_k^3 J_1(\lambda_k)}J_0\left(\frac{r}{R}\lambda_k\right)\Bigg[-\rho\sum_{m=1}^{\infty}\Big[a_m\left(\eta^2\lambda_k^4\cos\phi_m + m\omega_b\eta\lambda_k^2\sin\phi_m\right) +$$

$$b_m\left(-m\omega_b\eta\lambda_k^2\cos\phi_m + \eta^2\lambda_k^4\sin\phi_m\right)\Big]\Big/\left(\eta^2\lambda_k^4 + m^2\omega_b^2\right) -$$

$$\sum_{n=1}^{\infty}\frac{\left(\eta^2\lambda_k^4 A_n - n\omega_p\eta\lambda_k^2 B_n\right)}{\eta^2\lambda_k^4 + n^2\omega_p^2}. \qquad \ldots(13)$$

It is observed that both the transient terms u_{t1} and u_{t2} are different to each other. For the first initial condition $u_z(r,0) = u_0$, the difference of term in u_{t1} is $\left(-A_0 + \rho u_0 \eta \lambda_k^2\right)$. While for the second initial condition $u_z(r,0) = \dfrac{\left(R^2 - r^2\right)\left(A_0 + \sum_{n=1}^{\infty} A_n\right)}{4\mu_f}$, the difference of term in u_{t2} is $\left(\sum_{n=1}^{\infty} A_n\right)$.

IV. RESULTS AND DISCUSSION

It is very interesting to note that the steady state and oscillatory part of the solution are unaffected by the initial conditions, only transient term is influenced by initial conditions. It is further noticed that the analysis (variable separable) used by Womersley [3] gives only steady state oscillatory part of the flow; it does not give transient part of the solution. The transient part of the flow can be obtained by transform techniques (Sud and Sekhon [2])

A comparison of two transient terms [Equations (10) and (11)] for the two different initial conditions is given in Table -1. The only change is in the transient term and the steady pulsatile part of the solution remains unaffected. This appears reasonable from physical viewpoint, since the driving force will force the flow to be steady pulsatile of a form similar to driving force in due course, irrespective of the initial conditions.

It is observed that the magnitude and duration of transition in the two cases are quite different. Because with the change of state (sitting to walking or during play, the fast changes instate can occur), this term could be important while determining performance or work done on arterial walls. The oscillation in wall shear at higher wall shear could be more injurious to the wall muscle. This term has not been given the desire attention in literature.

It is also observed from Fig. 1 that the transient time for velocity decreases as the blood viscosity increases. A comparison of magnitudes of transient, steady state and oscillatory terms in the velocity expression is shown in Table- 2 and 3. It is observed that magnitude of the transient term is comparable with magnitude of the steady state term. Hence it is not insignificant to be left out.

Fig. 2 to 5 show variation of magnitude of transient term (velocity) with time for the flow with and without body force. It can be seen that the transient time is more than the time period of the flow. Again, it is noticed that transient time and the magnitude of flow variables in transient time are significant for flow with and without body force.

Table-1. Comparison of the transient term for two different initial conditions

k	Transient term u_{t1}	Transient term u_{t2}
1	5416.6 $D_1 + C_1$	11972.32 $D_1 + C_1$
2	18218.99 $D_2 + C_2$	11972.32 $D_2 + C_2$
3	32330.31 $D_3 + C_3$	11972.32 $D_3 + C_3$
4	43992.68 $D_4 + C_4$	11972.32 $D_4 + C_4$

Table-2. Comparison of the magnitudes of transient, steady state and oscillatory terms in the velocity expression when G = 0.0g.

t (sec.)	Magnitude of the transient term (m/sec.)	Magnitude of the steady state term (average) (m/sec.)	Maximum amplitude of the oscillatory term over steady state term (m/sec.)
0	3.61	0.46	0.7
0.89	3.05	0.46	0.7
45.0	1.98×10^{-4}	0.46	0.7

Table-3. Comparison of the magnitudes of transient, steady state and oscillatory terms in the velocity expression when G = 1.0g.

t (sec.)	Magnitude of the transient term (m/sec.)	Magnitude of the steady state term (average) (m/sec.)	Maximum amplitude of the oscillatory term over steady state term (m/sec.)
0	60.43	56.36	2.0
0.89	52.14	56.36	2.0
45.0	3.52×10^{3}	56.36	2.0

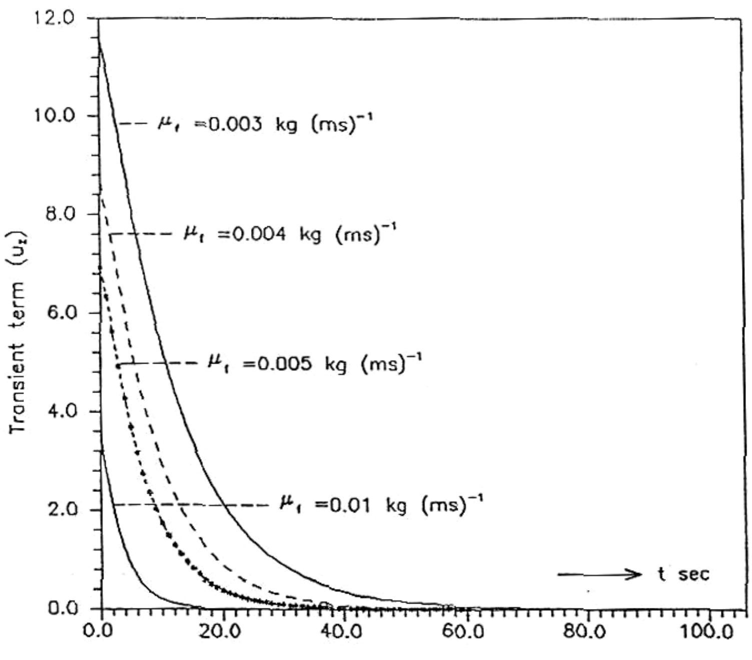

Fig. 1. Variation of velocity transient term with viscosity μ_f when f_p =1.2 Hz.

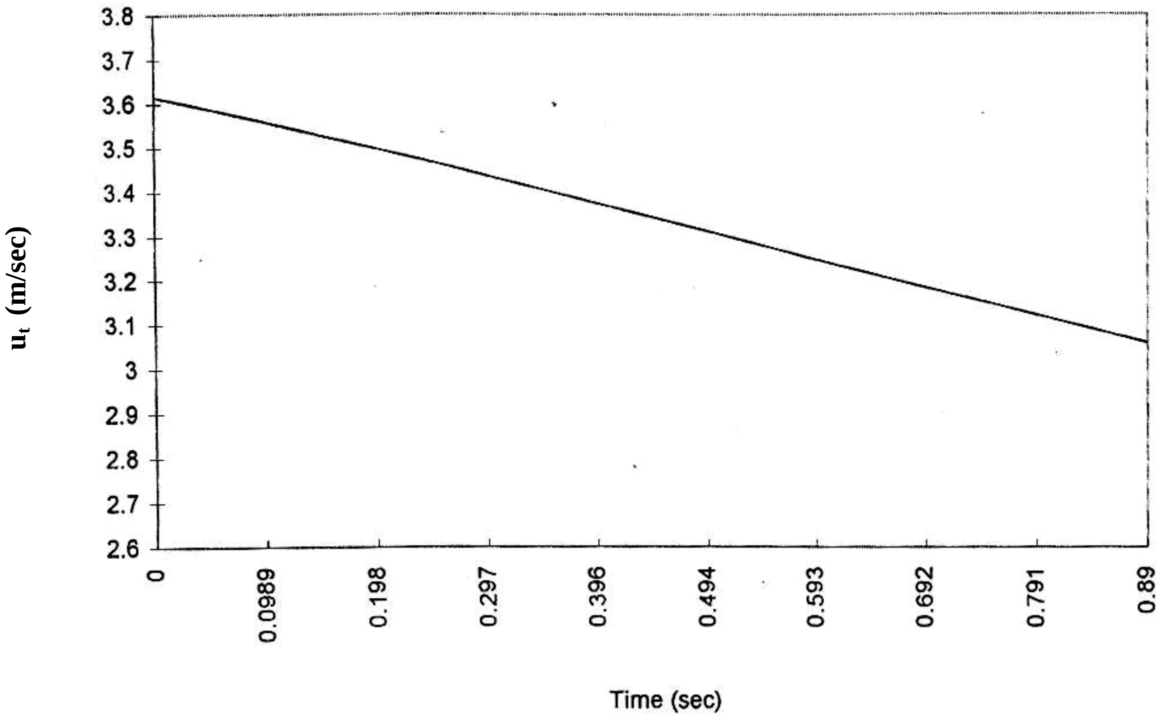

Fig. 2. Variation of magnitude of transient term (velocity) with time when G = 0.0g.

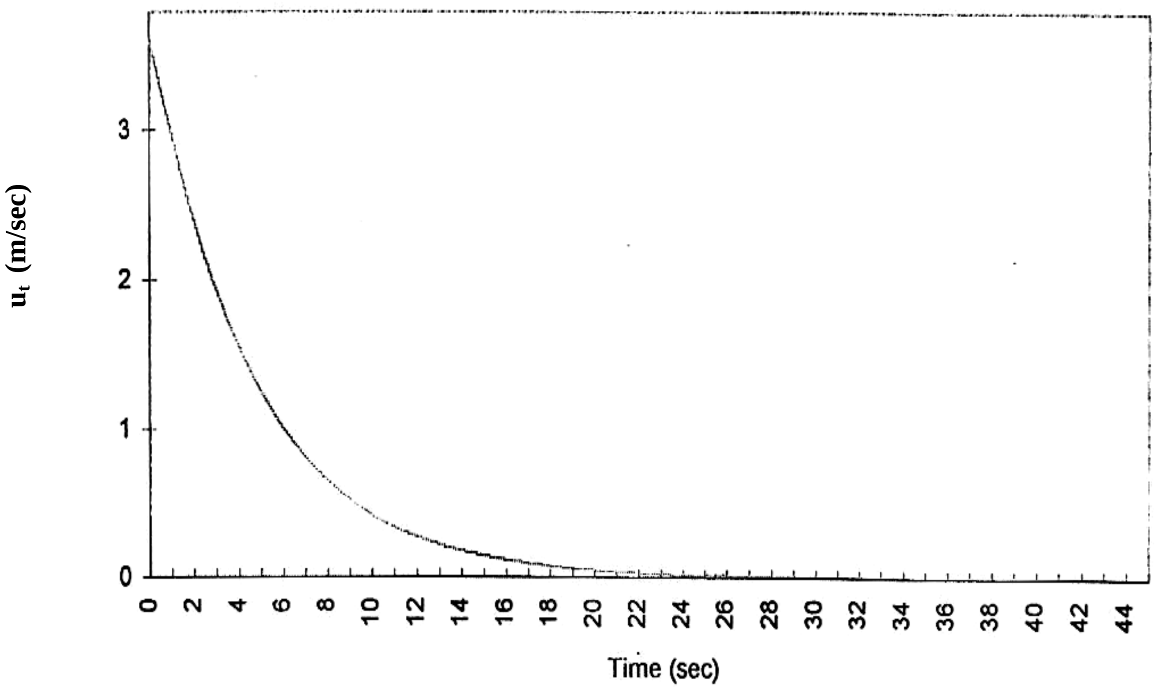

Fig. 3. Variation of magnitude of transient term (velocity) with time when G = 0.0g.

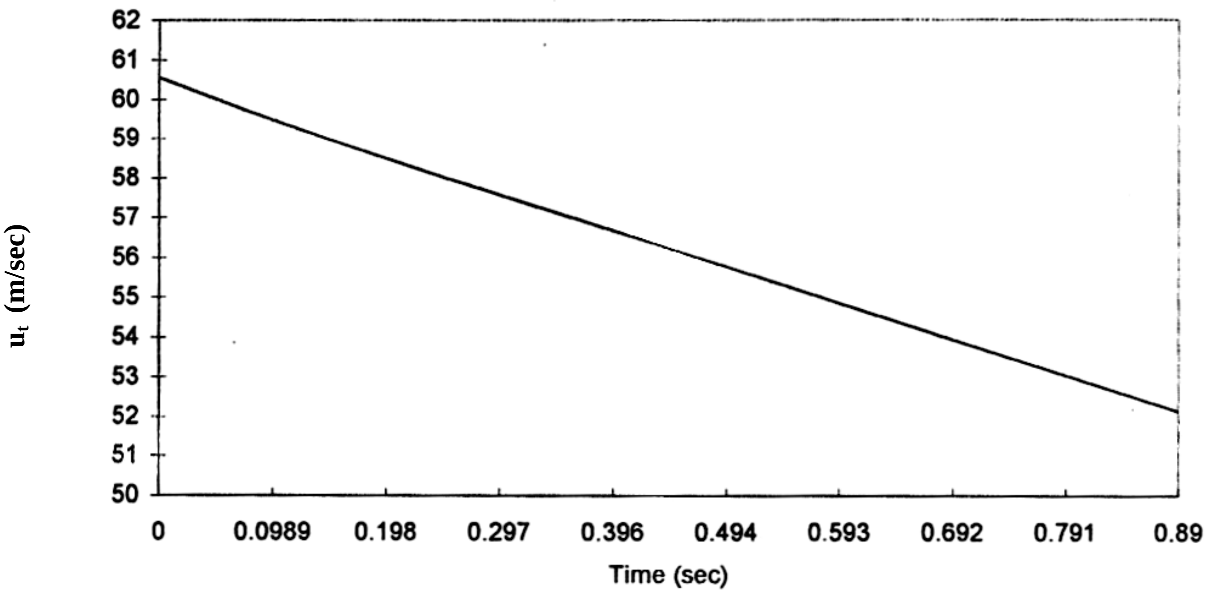

Fig. 4. Variation of magnitude of transient term (velocity) with time when G =1.0g.

V.CONCLUSION

It is observed that the flow expression consists of two parts, the basic steady pulsatile part (steady part and oscillatory part) and the transient part. In this paper, mathematical model with transient term is discussed for pulsatile flow of blood through a circular tube. By applying two different initial conditions (Lightfoot [1] and Sud and Sekhon [2]), it is shown that a change in the initial condition leads to a change only in the transient term, the basic steady pulsatile term remains unaffected. It is observed that the effects of transient term are significant, in comparison to steady pulsatile term, before the transient time approximately 60 sec.

Many researchers [Sud and Sekhon [2] etc.] argued that the transient time is very small in comparison to the period of the system. Other researchers (Chaturani and Palanisamy [9 and 10]) found the transient time is quite large for pulsatile flow with and without body forces. Further, it is observed that the magnitude of the velocity in transient time is not small compared to amplitude of the oscillatory part (Table- 2 and 3) in some situations.

It is observed from Fig. 1 that the effects of the blood viscosity on the transition time of all the flow variables are quite significant and the blood viscosity appears to be an important factor in this study. The quantitative details of transient flow are provided in this paper. Hence, the inputs for the mathematical models are improved and efforts are made to make the models more realistic.

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