COMMON FIXED POINT OF WEAKLY AND SEMI COMPATIBLE MAPS IN FUZZY METRIC SPACE

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Abstract. The purpose of this paper is to obtain common fixed point theorem in fuzzy metric space using the concept of semi-compatibility and weak compatibility for six self maps with another functional inequality and only one map is needed to be continuous, which generalizes the result of Singh and Chauhan[13] and Cho[1].

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1. INTRODUCTION.

Zadeh [15] proposed a mathematical way by defining the notion of fuzzy set in 1965.The special feature of fuzzy set is that it assign partial membership for elements in its domain, while in ordinary set theory particular element has either full membership or no membership, intermediate situation is not considered. Fuzzy metric space was defined by several researchers to use this concept in Analysis and Topology. In this paper, we are considering the fuzzy metric space defined by Kramosil and Michalek [9] and modified by George and Veeramani [4].

Jungck[6] proposed the concept of compatibility. The concept of compatibility in fuzzy metric space was proposed by Mishra et al.[11]. Later on, Jungck[7] generalized the concept of compatibility by introducing the concept of weak compatibility. Cho et al.[3] introduced

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the concept of semi-compatible maps in d-topological space. Singh and Jain[14] defined the concept of semi-compatible maps in fuzzy metric space.

Singh and Chauhan [13] and Cho [1] provided fixed point theorems in fuzzy metric space for four self maps using the concept of compatibility where two mappings needed to be continuous. In this paper, we prove a common fixed point theorem for six self maps in fuzzy metric space using the concept of semi-compatibility and weak compatibility with another functional inequality, where only one map is needed to be continuous as a generalization of the result of Singh and Chauhan[13] and Cho[1].

Definition 1.1. [15] Let X be any set. A fuzzy set A in X is a function with domain in X and values in [0, 1].

Definition 1.2. [12] A binary operation $*$: $[0, 1] \times [0, 1] \to [0, 1]$ is called a continuous t-norm if it satisfies the following conditions:

(i) $*$ is associative and commutative,

(ii) $*$ is continuous,

(iii) $a*1 = a$, for all $a \in [0, 1]$,

(iv) $a * b \leq c * d$, whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Examples of t-norms are

$a * b = \min \{a, b\}$ (minimum t-norm),

$a * b = ab$ (product t-norm).

Definition 1.3. [4] The 3-tuple (X, M, $*$) is called a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions :

(FM-1) $M(x, y, t) > 0$,
(FM-2) $M(x, y, t) = 1$ if and only if $x = y$,
(FM-3) $M(x, y, t) = M(y, x, t)$,
(FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(FM-5) $M(x, y, .) : (0, \infty) \to [0, 1]$ is continuous,

for all $x, y, z \in X$ and $t, s > 0$.

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Let (X, d) be a metric space and let a*b = a b or a*b = min{a, b} for all a, b $\in$ [0, 1]. Let $M(x, y, t) = \frac{t}{t+d(x,y)}$ for all x, y $\in$ X and t > 0.Then (X, M, *) is a fuzzy metric space, and this fuzzy metric M induced by d is called the standard fuzzy metric [4].

Definition 1.4. [5] A sequence $\{x_n\}$ in a fuzzy metric space (X, M, *) is said to be convergent to a point x $\in$ X if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all t >0.

Further, the sequence $\{x_n\}$ is said to be Cauchy if $\lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1$, for all t > 0 and p > 0.

The space (X, M, *) is said to be complete if every Cauchy sequence in X is convergent in X.

Lemma 1.5. [5] *Let (X, M, *) be a fuzzy metric space. Then M is non-decreasing for all x, y $\in$ X.*

Lemma 1.6. [10] *Let (X, M, *) be a fuzzy metric space. Then M is a continuous function on $X^2 \times (0, \infty)$.*

Throughout this paper (X, M, *) will denote the fuzzy metric space with the following condition :

(FM-6) $\lim_{n\to\infty} M(x, y, t) = 1$ for all x, y $\in$ X and t > 0.

Lemma 1.7.[11] *If there exists k$\in$ (0,1) such that M(x, y, kt) $\geq$ M(x, y, t) for all x, y $\in$ X and t > 0, then x = y.*

Lemma 1.8.[8] *The only t-norm * satisfying r * r $\geq$ r for all r $\in$ [0,1] is the minimum t-norm, that is a*b = min{a, b} for all a, b $\in$ (0, 1).*

Lemma 1.9.[2] *Let $\{y_n\}$ be a sequence in a fuzzy metric space (X, M, *) with condition (FM-6). If there exists a number k $\in$ (0, 1), such that*

$$M(y_{n+2}, y_{n+1}, kt) \geq M(y_{n+1}, y_n, t) \text{ for all } t > 0$$

Then $\{y_n\}$ is a Cauchy sequence in X.

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Definition 1.10. [11] Let f and g be self mappings on a fuzzy metric space (X, M, *). The pair (f, g) is said to compatible if $\lim_{n\to\infty}(fgx_n, gfx_n, t) = 1$ for all t > 0, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$, for some $z \in X$.

Definition 1.11. [7] Let f and g be self mappings on a fuzzy metric space (X, M, *). Then the mappings are said to be weakly compatible if they commute at their coincidence point, that is,
fx = gx implies fgx = gfx.

It is known that a pair (f, g) of compatible maps is weakly compatible but converse is not true in general.

Definition 1.12. [14] A pair (A, B) of self maps of a fuzzy metric space (X, M, *) is said to be semi-compatible if $\lim_{n\to\infty} ABx_n = Bx$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x$.

It follows that if (A, B) is semi-compatible and Ax = Bx then ABx = BAx that means every semi-compatible pair of self maps is weak compatible but the converse is not true in general.

Cho[1] generalized the result of Singh and Chauhan[13] as follows:

Theorem 1.13. [1] *Let (X, M, *) be a complete fuzzy metric space and let A, B, S and T be mappings from X into itself such that the following conditions are satisfied :*

(i) $AX \subset TX$, $BX \subset SX$,

(ii) S and T are continuous,

(iii) the pairs [A,S] and [B,T] are compatible,

(iv) there exists $q \in (0, 1)$ *such that for every* $x, y \in X$ *and* $t > 0$,

$$M(Ax, By, qt) \geq M(Sx, Ty, t) * M(Ax, Sx, t) * M(By, Ty, t) * M(Ax, Ty, t)$$

Then A, B, S and T have a unique common fixed point in X.

II MAIN RESULT

Our result generalizes the results of Singh and Chauhan [13] and Cho[1] as we are using the concept of semi-compatibility and weak compatibility which are lighter conditions than that of compatibility, also only one map is needed to be continuous. We are proving the result for six self maps using another inequality.

Theorem 2.1. *Let (X, M, *) be a complete fuzzy metric space and let A, B, S, T, P and Q be mappings from X into itself such that the following conditions are satisfied :*

(2.1.1) $A(X) \subset ST(X)$, $B(X) \subset PQ(X)$

(2.1.2) either A or PQ is continuous;

(2.1.3) (A, PQ) is semi-compatible and (B, ST) is weakly compatible;

(2.1.4) $PQ = QP$, $ST = TS$, $AQ = QA$ *and* $BT = TB$;

(2.1.5) there exists $q \in (0, 1)$ *such that for every x, y in X and* $t > 0$,

$$M(Ax, By, qt) \geq M(Ax, STy, t) * M(Ax, PQx, t) * M(By, STy, t) * M(PQx, STy, t) * M(PQx, By, 2t).$$

Then A, B, S, T, P and Q have a unique common fixed point in X.

Proof. Let x_0 be an arbitary point in X. As $A(X) \subset ST(X)$ and $B(X) \subset PQ(X)$, then there exists $x_1, x_2 \in X$ such that $Ax_0 = STx_1 = y_0$ and $Bx_1 = PQx_2 = y_1$.

We can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_{2n} = STx_{2n+1} = Ax_{2n} \quad \text{and} \quad y_{2n+1} = Bx_{2n+1} = PQx_{2n+2} \text{ for } n = 0, 1, 2, \ldots$$

Now, we first show that $\{y_n\}$ is a Cauchy sequence in X.

From (2.1.5), we have

$$\begin{aligned} M(y_{2n}, y_{2n+1}, qt) &= M(Ax_{2n}, Bx_{2n+1}, qt) \\ &\geq M(Ax_{2n}, STx_{2n+1}, t) * M(Ax_{2n}, PQx_{2n}, t) * M(Bx_{2n+1}, STx_{2n+1}, t) \\ &\quad * M(PQx_{2n}, STx_{2n+1}, t) * M(PQx_{2n}, Bx_{2n+1}, 2t). \\ &= M(y_{2n}, y_{2n}, t) * M(y_{2n}, y_{2n-1}, t) * M(y_{2n+1}, y_{2n}, t) * M(y_{2n-1}, y_{2n}, t) \\ &\quad * M(y_{2n-1}, y_{2n+1}, 2t) \end{aligned}$$

Using definition 1.2 and definition 1.3, we get

$\geq M(y_{2n-1}, y_{2n}, t) * M(y_{2n+1}, y_{2n}, t).$

Using lemma 1.5 and lemma 1.8, we get

$$M(y_{2n}, y_{2n+1}, qt) \geq M(y_{2n-1}, y_{2n}, t) \;\; \forall\, t > 0.$$

In general

$$M(y_n, y_{n+1}, qt) \geq M(y_{n-1}, y_n, t) \;\forall\, t > 0.$$

Therefore

$$M(y_n, y_{n+1}, t) \geq M(y_{n-1}, y_n, t/q) \geq M(y_{n-2}, y_{n-1}, t/q^2) \geq \ldots \geq M(y_0, y_1, t/q^n)$$

Using (FM-6), we get

$$\lim_{n\to\infty} M(y_n, y_{n+1}, t) = 1 \;\forall\, t > 0.$$

Now for any positive integer p,

$$M(y_n, y_{n+p}, t) \geq M(y_n, y_{n+1}, t/p) * M(y_{n+1}, y_{n+2}, t/p) * \ldots\ldots * M(y_{n+p-1}, y_{n+p}, t/p).$$

Therefore

$$\lim_{n\to\infty} M(y_n, y_{n+p}, t) = 1 * 1 * 1 * \ldots * 1 = 1.$$

Thus, $\{y_n\}$ is a Cauchy sequence in X. By completeness of (X, M, *), $\{y_n\}$ converges to some point z in X.

Consequently, the subsequences $\{Ax_{2n}\}$, $\{Bx_{2n+1}\}$, $\{STx_{2n+1}\}$ and $\{PQx_{2n+2}\}$ of sequence $\{y_n\}$ also converges to z in X.

Case I. Suppose A is continuous, we have

$$APQx_{2n} \to Az$$

The semi-compatibility of the pair (A, PQ) gives that

$$A(PQ)x_{2n} \to PQz.$$

We know that the limit in a fuzzy metric space is unique, we get

$$Az = PQz$$

Step 1. Putting x = z and y = x_{2n+1} in (2.1.5), we have

$M(Az, Bx_{2n+1}, qt) \geq M(Az, STx_{2n+1}, t) * M(Az, PQz, t) * M(Bx_{2n+1}, STx_{2n+1}, t)$

$* M(PQz, STx_{2n+1}, t) * M(PQz, Bx_{2n+1}, 2t).$

Letting $n\to\infty$ and using above results, we get

$M(Az, z, qt) \geq M(Az, z, t) * M(Az, Az, t) * M(z, z, t) * M(Az, z, t) * M(Az, z, 2t)$

$M(Az, z, qt) \geq M(Az, z, t).$

Now by Lemma 1.7, we get

$Az = z$

Hence $Az = z = PQz$.

Step 2. Putting $x = Qz$ and $y = x_{2n+1}$ in (2.1.5), we have

$M(AQz, Bx_{2n+1}, qt) \geq M(AQz, STx_{2n+1}, t) * M(AQz, PQQz, t)$

$* M(Bx_{2n+1}, STx_{2n+1}, t) * M(PQQz, STx_{2n+1}, t) * M(PQQz, Bx_{2n+1}, 2t).$

As $AQ = QA$ and $PQ = QP$, We have

$A(Qz) = Q(Az) = Qz$ and $PQ(Qz) = QP(Qz) = Q(PQz) = Qz$

Letting $n\to\infty$ and using above results, we get

$M(Qz, z, qt)(PQ)Ax_{2n} = \lim_{n\to\infty} A(PQ)x_{2n} = PQz.$

Step 6. Putting $x = PQx_{2n}$ and $y = x \lim_{n\to\infty} \lim_{n\to\infty} {}_{2n+1}$ in (2.1.5), we have

$M(APQx_{2n}, Bx_{2n+1}, qt) \geq M(APQx_{2n}, STx_{2n+1}, t) * M(APQx_{2n}, PQPQx_{2n}, t)$

$* M(Bx_{2n+1}, STx_{2n+1}, t) * M(PQPQx_{2n}, STx_{2n+1}, t)$

$* M(PQPQx_{2n}, Bx_{2n+1}, 2t).$

Letting $n \to \infty$ and using above results, we get

$M(PQz, z, qt) \geq M(PQz, z, t) * M(PQz, PQz, t) * M(z, z, t) * M(PQz, z, t) * M(PQz, z, 2t).$

$M(PQz, z, qt) \geq M(PQz, z, t).$

Now by Lemma 1.7, we get

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$PQz = z$.

Step 7. Putting $x = z$ and $y = x_{2n+1}$ in (2.1.5), we get

$$M(Az, Bx_{2n+1}, qt) \geq M(Az, STx_{2n+1}, t) * M(Az, PQz, t) * M(Bx_{2n+1}, STx_{2n+1}, t) * M(PQz, STx_{2n+1}, t) * M(PQz, Bx_{2n+1}, 2t).$$

Letting $n \to \infty$ and using above results, we get

$$M(Az, z, qt) \geq M(Az, z, t) * M(Az, z, t) * M(z, z, t) * M(z, z, t) * M(z, z, 2t)$$

$$M(Az, z, qt) \geq M(Az, z, t).$$

By Lemma 1.7, we get

$$Az = z$$

Using step 2, we get $Qz = z$.

Now, $PQz = z$ implies $Pz = z$.

Therefore $Az = Qz = Pz = z$.

Now, applying steps 3, 4 and 5, we get

$$Bz = Sz = Tz = z$$

Hence, $Az = Bz = Sz = Tz = Pz = Qz = z$

Thus z is a common fixed point of A, B, S, T, P and Q.

Uniqueness.

Let v be another common fixed point of A, B, S, T, P and Q, then

$$v = Av = Bv = Sv = Tv = Pv = Qv.$$

Putting $x = z$ and $y = w$ in (2.1.5), we get

$$M(Az, Bv, qt) \geq M(Az, STv, t) * M(Az, PQz, t) * M(Bv, STv, t) * M(PQz, STv, t) * M(PQz, Bv, 2t)$$

$$M(z, v, qt) \geq M(z, v, t) * M(z, z, t) * M(v, v, t) * M(z, v, t) * M(z, v, 2t)$$

$$M(z, v, qt) \geq M(z, v, t).$$

Now by Lemma 1.7, we get

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z = v

Therefore, z is unique common fixed point of A, B, S, T, P and Q.

Remark 2.2. If we take Q = T = I in theorem 2.1 then the condition (2.1.4) is satisfied trivially and we get the following result.

Corollary 2.3. *Let (X, M, *) be a complete fuzzy metric space and let A, B, S and P be mappings from X into itself such that the following conditions are satisfied :*

(2.1.6) $A(X) \subset S(X)$, $B(X) \subset P(X)$

(2.1.7) either A or P is continuous;

(2.1.8) (A, P) is semi-compatible and (B, S) is weakly compatible;

(2.1.9) there exists $q \in (0, 1)$ *such that for every x, y in X and* $t > 0$,

$$M(Ax, By, qt) \geq M(Ax, Sy, t) * M(Ax, Px, t) * M(By, Sy, t) * M(Px, Sy, t) * M(Px, By, 2t).$$

Then A, B, S and P have a unique common fixed point in X.

Remark 2.4. If we take a * b = min {a, b} where a, b $\in [0, 1]$, then in view of remark 2.2, corollary 2.3 is a generalization of the result of Singh and Chauhan [13], as only one mapping of the first pair in (2.1.8) is needed to be continuous, also first pair of self maps is taken semi-compatible and second pair of self maps is weakly compatible in (2.1.8) which are lighter conditions than that of compatibility.

Remark 2.5. In view of remark 2.2, corollary 2.3 is also a generalization of the result of Cho [1] in the sense of another functional inequality (2.1.9), semi-compatibility for first pair and weak compatibility for second pair and continuity for only one mapping in the first pair of (2.1.8).

Corollary 2.6. *Let (X, M, *) be a complete fuzzy metric space and let A, B, S, T, P and Q be mappings from X into itself satisfying the conditions (2.1.1), (2.1.2), (2.1.4), (2.1.5) and*

The pair (A, PQ) is semi-compatible and (B, ST) is semi- compatible.

Then A, B, S, T, P and Q have a unique common fixed point in X.

Corollary 2.7. *Let (X, M, *) be a complete fuzzy metric space and let A, B, S, T, P and Q be mappings from X into itself satisfying the conditions (2.1.1), (2.1.2), (2.1.4), (2.1.5) and*

The pair (A,PQ) is compatible and (B,ST) is weak compatible

Then A, B, S, T, P and Q have a unique common fixed point in X.

Corollary 2.8. *Let (X, M, *) be a complete fuzzy metric space and let A, B, S, T, P and Q be mappings from X into itself satisfying the conditions (2.1.1), (2.1.2), (2.1.4), (2.1.5) and*

The pair (A, PQ) is compatible and (B, ST) is compatible.

Then A, B, S, T, P and Q have a unique common fixed point in X.

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