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Strong Coupled Fixed point For (ϕ, ψ)-Contraction Type Coupling in Metric Spaces

Tawseef Rashid [1] and Q. H. Khan [2]

Abstract. *The aim of this research article is to give answer to an open problem presented by Choudhury et al. [4] concerning the investigation of strong coupled fixed point and related properties for couplings satisfying other type of inequalities. In this direction we define (ϕ, ψ)-contraction type coupling and then proved the existence and uniqueness theorem of strong coupled fixed point for (ϕ, ψ)-contraction type coupling. We give examples to illustrate our main result.*

Keywords : Coupled Fixed Point; strong coupled fixed point, altering distance function, (ϕ, ψ)-contraction type coupling.

2010MSC : 47H10, 54H25,

1 Introduction and Mathematical Preliminaries

T. Gnana Bhaskar and V. Lakshmikantham [5] introduced the concept of coupled fixed point of mapping $F : X \times X \to X$. The results on existence of coupled fixed point were studied in many papers [1, 2, 8, 9, 10, 11] . The concept of coupling was introduced by Choudhury et al.[4]. Choudhury et al. [4] proved the existence and uniqueness of strong coupled fixed point for couplings using Kannan type contractions for complete metric spaces. In this paper we prove the existence and uniqueness of strong coupled fixed point for (ϕ, ψ)-contraction type coupling in complete metric spaces which was put as an open problem by Choudhury et al.[4].

Definition 1.1. (Coupled Fixed Point) [5]. An element $(x, y) \in X \times X$, where X is any non-empty set, is called a coupled fixed point of the mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.

Definition 1.2. (Strong Coupled Fixed Point) [4]. An element $(x, y) \in X \times X$, where X is any non-empty set, is called a strong coupled fixed point of the mapping $F : X \times X \to X$ if (x, y) is coupled fixed point and $x = y$; that is if $F(x, x) = x$.

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[1]Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India.
Email : tawseefrashid123@gmail.com
[2]Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India.
Email : qhkhan.ssitm@gmail.com

Definition 1.3. (Cyclic Mapping) [6]. Let A and B be two non-empty subsets of a given set X. Any function $f : X \to X$ is said to be cyclic (with respect to A and B) if

$$f(A) \subset B \quad and \quad f(B) \subset A.$$

Definition 1.4. (Coupling) [4]. Let (X, d) be a metric space and A *and* B be two non-empty subsets of X. Then a function $F : X \times X \to X$ is said to be a coupling with respect to A *and* B if

$$F(x, y) \in B \; and \; F(y, x) \in A$$

whenever $x \in A$ and $y \in B$.

Definition 1.5. [7]. A function $\psi : [0, \infty) \to [0, \infty)$ is called an altering distance function, if the following properties are satisfied:
(i) ψ is monotone increasing and continuous,
(ii) $\psi(t) = 0$ iff $t = 0$.

2 Main Result

Before going to the main theorem, we define (ϕ, ψ)-contraction type coupling in metric spaces.

Definition 2.1. ((ϕ, ψ)-Contraction Type Coupling). Let A and B be two non-empty subsets of a metric space (X, d) and ϕ, ψ are two altering distance functions. Then a coupling $F : X \times X \to X$ is said to be (ϕ, ψ)-contraction type coupling (with respect to A and B) if it satisfies the following inequality :

$$\psi(d(F(x, y), F(u, v))) \leq \psi(max\{d(x, u), d(y, v)\}) - \phi(max\{d(x, u), d(y, v)\}).$$

for any $x, v \in A$ and $y, u \in B$.

Theorem 2.2. Let A and B be two non-empty closed subsets of a complete metric space (X,d) and $F : X \times X \to X$ is a (ϕ,ψ)-contraction type coupling (with respect to A and B) i.e. there exists altering distance functions ψ, ϕ s.t.

$$\psi(d(F(x,y),F(u,v))) \leq \psi(max\{d(x,u),d(y,v)\}) - \phi(max\{d(x,u),d(y,v)\}). \qquad (1)$$

for any $x,v \in A$ and $y,u \in B$, Then
(i) $A \cap B \neq \emptyset$,
(ii) F has a unique strong coupled fixed point in $A \cap B$.
Proof : Since A and B are non-empty subsets of X and F is a coupling (w.r.t. A and B), then for $x_0 \in A$ and $y_0 \in B$ we define sequences $\{x_n\}$ and $\{y_n\}$ in A and B resp. such that,

$$x_{n+1} = F(y_n, x_n) \quad and \quad y_{n+1} = F(x_n, y_n). \qquad (2)$$

If for some n, $x_{n+1} = y_n$ and $y_{n+1} = x_n$, then by using (2), we have

$$x_n = y_{n+1} = F(x_n, y_n) \;\; and \;\; y_n = x_{n+1} = F(y_n, x_n).$$

This shows that (x_n, y_n) is a coupled fixed point of F. So, we are done in this case.
Now assume $x_{n+1} \neq y_n$ or $y_{n+1} \neq x_n$, $\forall\, n$.
Let us define a sequence $\{D_n\}$ by

$$D_n = max\{d(x_{n+1}, y_n), d(y_{n+1}, x_n)\}. \qquad (3)$$

Clearly $\{D_n\} \subseteq [0,\infty)$, $\forall\, n$.
Now using (1), (2) and fact that $x_n \in A$ and $y_n \in B$ $\forall\, n$, we have

$$\begin{aligned}\psi(d(x_n, y_{n+1})) &= \psi[d(F(y_{n-1}, x_{n-1}), F(x_n, y_n))] \\ &= \psi[d(F(x_n, y_n), F(y_{n-1}, x_{n-1}))] \\ &\leq \psi[max\{d(x_n, y_{n-1}), d(y_n, x_{n-1})\}] \\ &\quad -\phi[max\{d(x_n, y_{n-1}), d(y_n, x_{n-1})\}]. \qquad (4)\end{aligned}$$

Using properties of ϕ, we have

$$\psi(d(x_n, y_{n+1})) \leq \psi[max\{d(x_n, y_{n-1}), d(y_n, x_{n-1})\}].$$

Again using properties of ψ, we get

$$d(x_n, y_{n+1}) \leq max\{d(x_n, y_{n-1}), d(y_n, x_{n-1})\}. \tag{5}$$

Now again using (1), (2) and fact that $x_n \in A$ and $y_n \in B\ \forall\ n$, we have

$$\begin{aligned} \psi(d(y_n, x_{n+1})) &= \psi[d(F(x_{n-1}, y_{n-1}), F(y_n, x_n)] \\ &\leq \psi[max\{d(x_{n-1}, y_n), d(y_{n-1}, x_n)\}] \\ &\quad -\phi[max\{d(x_{n-1}, y_n), d(y_{n-1}, x_n)\}]. \end{aligned} \tag{6}$$

Now using properties of ϕ and ψ, we get

$$d(y_n, x_{n+1}) \leq max\{d(x_{n-1}, y_n), d(y_{n-1}, x_n)\}. \tag{7}$$

By using (5) and (7), we get

$$max\{d(y_n, x_{n+1}), d(x_n, y_{n+1})\} \leq max\{d(x_n, y_{n-1}), d(y_n, x_{n-1})\}.$$

i.e.

$$max\{d(x_{n+1}, y_n), d(y_{n+1} x_n)\} \leq max\{d(x_n, y_{n-1}), d(y_n, x_{n-1})\}. \tag{8}$$

From (3), we have

$$D_n \leq D_{n-1}\ \ \forall\ \ n \geq 1.$$

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Therefore $\{D_n\}$ is monotonic decreasing sequence of non-negative real numbers.
Thus $\exists\ r \geq 0$, s.t. $\lim\limits_{n\to\infty} D_n = r$,
i.e.

$$\lim_{n\to\infty} max\{d(x_{n+1}, y_n), d(y_{n+1}, x_n)\} = r. \tag{9}$$

Since $\psi : [0,\infty) \to [0,\infty)$ is non-decreasing, then $\forall\ a, b \in [0,\infty)$, we have

$$max\{\psi(a), \psi(b)\} = \psi(max\{a, b\}). \tag{10}$$

On using (4), (6) and (10), we get

$$\begin{aligned}\psi[max\{d(x_n, y_{n+1}), d(y_n, x_{n+1})\}] &= max\{\psi(d(x_n, y_{n+1}), \psi(d(y_n, x_{n+1}))\}\\ &\leq \psi[max\{d(x_n, y_{n-1}), d(y_n, x_{n-1})\}]\\ &\quad -\phi[max\{d(x_n, y_{n-1}), d(y_n, x_{n-1})\}].\end{aligned}$$

On using (4), (6) and (10), we get

$$\begin{aligned}\psi[max\{d(x_n, y_{n+1}), d(y_n, x_{n+1})\}] &= max\{\psi(d(x_n, y_{n+1}), \psi(d(y_n, x_{n+1}))\}\\ &\leq \psi[max\{d(x_n, y_{n-1}), d(y_n, x_{n-1})\}]\\ &\quad -\phi[max\{d(x_n, y_{n-1}), d(y_n, x_{n-1})\}].\end{aligned}$$

Letting $n \to \infty$ in above inequality, using(9) and continuities of ϕ and ψ, we have

$$\psi(r) \leq \psi(r) - \phi(r) \leq \psi(r)$$

$\Rightarrow \phi(r) = 0$, since ϕ is altering distance function, so $r = 0$.
Hence $\lim\limits_{n\to\infty} D_n = 0$, i.e.

$$\lim_{n\to\infty} max\{d(x_n, y_{n+1}), d(y_n, x_{n+1})\} = 0.$$

Now we define a sequence $\{R_n\}$ by $R_n = d(x_n, y_n)$,
we show that $R_n \to 0 \ \ as \ \ n \to \infty$.
By using (1) and (2), we get

$$\begin{aligned} \psi(R_n) &= \psi(d(x_n, y_n)) \\ &= \psi(d(F(y_{n-1}, x_{n-1}), F(x_{n-1}, y_{n-1})) \\ &\leq \psi[max\{d(x_{n-1}, y_{n-1}), d(x_{n-1}, y_{n-1})\}] \\ &\quad -\phi[max\{d(x_{n-1}, y_{n-1}), d(x_{n-1}, y_{n-1})\}] \\ &= \psi(d(x_{n-1}, y_{n-1})) - \phi((d(x_{n-1}, y_{n-1}))). \end{aligned} \tag{12}$$

By properties of ϕ and ψ, we have

$$R_n \leq d(x_{n-1}, y_{n-1}) = R_{n-1},$$

i.e.

$$R_n \leq R_{n-1} \ \ \forall \ \ n \geq 1.$$

Thus $\{R_n\}$ is monotonic decreasing sequence of non-negative real numbers.
Therefore $\exists \ s \geq 0$, s.t.

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} d(x_n, y_n) = s \tag{13}$$

take $n \to \infty$ in (12), using (13) and continuities of ϕ and ψ, we have

$$\psi(s) \leq \psi(s) - \phi(s) \leq \psi(s)$$

$\Rightarrow \phi(s) = 0$, but since ϕ is altering distance function, so we have $s = 0$.
i.e.

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} d(x_n, y_n) = 0. \tag{14}$$

Now using triangular inequality, (11) and (14), we have

$$\lim_{n\to\infty} d(x_n, x_{n+1}) \leq \lim_{n\to\infty} [d(x_n, y_n) + d(y_n, x_{n+1})] = 0 \tag{15}$$

and

$$\lim_{n\to\infty} d(y_n, y_{n+1}) \leq \lim_{n\to\infty} [d(y_n, x_n) + d(x_n, y_{n+1})] = 0. \tag{16}$$

Now we will prove that sequences $\{x_n\}$ and $\{y_n\}$ are cauchy sequences in A and B resp. If possible, let $\{x_n\}$ or $\{y_n\}$ is not a Cauchy sequence. Then there exists an $\varepsilon > 0$, and sequence of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that $\forall$ positive integer k, with $n(k) > m(k) > k$, we have

$$T_k = max\{d(x_{m(k)}, x_{n(k)}), d(y_{m(k)}, y_{n(k)})\} \geq \varepsilon. \tag{17}$$

and

$$max\{d(x_{m(k)}, x_{n(k)-1}), d(y_{m(k)}, y_{n(k)-1})\} < \varepsilon. \tag{18}$$

Now we show that

$$d(y_{n(k)}, x_{m(k)+1}) \leq max\{d(x_{m(k)}, y_{n(k)-1}), d(y_{m(k)}, x_{n(k)-1})\}$$

By using (1) and (2), we get

$$\begin{aligned} \psi[d(y_{n(k)}, x_{m(k)+1})] &= \psi[d(F(x_{n(k)-1}, y_{n(k)-1}), F(y_{m(k)}, x_{m(k)}))] \\ &\leq \psi[max\{d(x_{n(k)-1}, y_{m(k)}), d(y_{n(k)-1}, x_{m(k)})\}] \\ &\quad -\phi[max\{d(x_{n(k)-1}, y_{m(k)}), d(y_{n(k)-1}, x_{m(k)})\}]. \end{aligned}$$

Using properties of ϕ and ψ, we have

$$d(y_{n(k)}, x_{m(k)+1}) \leq max\{d(x_{n(k)-1}, y_{m(k)}), d(y_{n(k)-1}, x_{m(k)})\}. \tag{19}$$

Similarly we can show by same pattern that,

$$d(x_{n(k)}, y_{m(k)+1}) \leq max\{d(y_{n(k)-1}, x_{m(k)}), d(x_{n(k)-1}, y_{m(k)})\}. \tag{20}$$

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From (19) and (20), we have

$$\begin{aligned} max\{d(y_{n(k)}, x_{m(k)+1}), d(x_{n(k)}, y_{m(k)+1})\} &\leq max\{d(x_{m(k)}, y_{n(k)-1}), d(y_{m(k)}, x_{n(k)-1})\} \\ &= \lambda. \end{aligned} \tag{21}$$

Where $\lambda = max\{d(x_{m(k)}, y_{n(k)-1}), d(y_{m(k)}, x_{n(k)-1})\}$.

It is fact that for $a, b, c \in R^+$, $max\{a + c, b + c\} = c + max\{a, b\}$.
Therefore by triangular inequality, (18) and the above fact, we have

$$\begin{aligned} \lambda &= max\{d(x_{m(k)}, y_{n(k)-1}), d(y_{m(k)}, x_{n(k)-1})\} \\ &\leq max\{d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, y_{n(k)-1}), d(y_{m(k)}, y_{n(k)-1}) + d(y_{n(k)-1}, x_{n(k)-1})\} \\ &= d(x_{n(k)-1}, y_{n(k)-1}) + max\{d(x_{m(k)}, x_{n(k)-1}), d(y_{m(k)}, y_{n(k)-1})\} \\ &< d(x_{n(k)-1}, y_{n(k)-1}) + \varepsilon. \end{aligned} \tag{22}$$

Thus from (21) and (22), we get

$$max\{d(y_{n(k)}, x_{m(k)+1}), d(x_{n(k)}, y_{m(k)+1})\} < d(x_{n(k)-1}, y_{n(k)-1}) + \varepsilon. \tag{23}$$

Now again by traingular inequality, we have

$$d(x_{n(k)}, x_{m(k)}) \leq d(x_{n(k)}, y_{n(k)}) + d(y_{n(k)}, x_{m(k)+1}) + d(x_{m(k)+1}, x_{m(k)}). \tag{24}$$

and

$$d(y_{n(k)}, y_{m(k)}) \leq d(y_{n(k)}, x_{n(k)}) + d(x_{n(k)}, y_{m(k)+1}) + d(y_{m(k)+1}, y_{m(k)}). \tag{25}$$

From (17) (23), (24) and (25), we get

$$\begin{aligned} T_k &= max\{d(x_{n(k)}, x_{m(k)}), d(y_{n(k)}, y_{m(k)})\} \\ &\leq d(x_{n(k)}, y_{n(k)}) + max\{d(x_{m(k)}, x_{m(k)+1}), d((y_{m(k)}, y_{m(k)+1})\} \\ &\quad + max\{d(y_{n(k)}, x_{m(k)+1}), d(x_{n(k)}, y_{m(k)+1})\} \\ &< d(x_{n(k)}, y_{n(k)}) + max\{d(x_{m(k)}, x_{m(k)+1}), d((y_{m(k)}, y_{m(k)+1})\} \\ &\quad + d(x_{n(k)-1}, y_{n(k)-1}) + \varepsilon. \end{aligned} \tag{26}$$

Take $k \to \infty$ in (26) and using (14), (15), (16) and (17), we have

$$\varepsilon \leq T_k < \varepsilon.$$

Which is a contradiction, Hence $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in A and B respectively. Since A and B are closed subsets of complete metric space X, therefore $\{x_n\}$ and $\{y_n\}$ are convergent in A and B respec. Thus $\exists\ x \in A$ and $y \in B$, s.t.

$$x_n \to x \quad and \quad y_n \to y. \tag{27}$$

Also by using (14) in (27), we get

$$x = y. \tag{28}$$

Thus $x = y \in A \cap B$, which shows that $A \cap B \neq \emptyset$.
Using (1), (2) and fact that $x_n, x \in A$ and $y_n, y \in B\ \forall\ n$, we have

$$\begin{aligned}
\psi(d(x_{n+1}, F(x,y))) &= \psi(d(F(y_n, x_n), F(x,y)) \\
&= \psi(d(F(x,y), F(y_n, x_n)) \\
&\leq \psi[max\{d(x,y_n), d(y,x_n)\}] - \phi[max\{d(x,y_n), d(y,x_n)\}].
\end{aligned}$$

Using properties of ψ and ϕ, we get

$$d(x_{n+1}, F(x,y)) \leq max\{d(x,y_n), d(y,x_n)\}. \tag{29}$$

Now using triangular inequality, (27), (28) and (29), we get

$$\begin{aligned}
d(x, F(x,x)) &= d(x, F(x,y)) \\
&\leq d(x, x_{n+1}) + d(x_{n+1}, F(x,y)) \\
&\leq d(x, x_{n+1}) + max\{d(x,y_n), d(y,x_n)\} \\
&\to 0 \quad as \quad n \to \infty.
\end{aligned}$$

$\Rightarrow F(x,x) = x$, hence F has a strong coupled fixed point in $A \cap B$.

Uniqueness: Let if possible F has two strong fixed points l, m in $A \cap B$, then

$$F(l,l) = l \quad and \quad F(m,m) = m. \tag{30}$$

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Now using (1), (30), properties of ϕ and ψ and $l, m \in A \cap B$, we have

$$\begin{aligned} \psi(d(l,m)) &= \psi(d(F(l,l), F(m,m)) \\ &\leq \psi[max\{d(l,m), d(l,m)\}] - \phi[max\{d(l,m), d(l,m)\}] \\ &\leq \psi(d(l,m)) - \phi(d(l,m)). \end{aligned}$$

Which gives $\phi(d(l,m)) = 0$, as ϕ is altering distance function so $d(l,m) = 0$.
Hence $l = m$, which proves the uniqueness.

Note 2.3. If A and B are two non-empty subsets of a metric space (X, d) and $F : X \times X \to X$ is a coupling with respect to A and B. Then by definition of coupling for $a \in A$ and $b \in B$, we have $F(a, b) \in B$ and $F(b, a) \in A$.

Now let (a, b) be the coupled fixed point of F, then $F(a, b) = a$ and $F(b, a) = b$. But in general this is absurd because $F(a, b) \in B$ *and* $a \in A$. Similarly $F(b, a) \in A$ *and* $b \in B$. This is only possible for $a, b \in A \cap B$.

The most important fact to be noted is that for any coupling $F : X \times X \to X$ (with respect to A and B), where A and B be any two non-empty subsets of metric space (X, d), if we investigate for coupled fixed point (x, y) in product space $A \times B$, then we should directly investigate in product subspace $(A \cap B) \times (A \cap B)$. Similarly for strong coupled fixed point, we should investigate it in $A \cap B$.

Example 2.4. : Let $X = [0, 3]$ be the complete metric space with respect to usual metric 'd' on X i.e. $d(x, y) =| x - y |$. Let $A = \{1\}$ and $B = \{1, 2\}$ be the closed subsets of X. We define $F : X \times X \to X$ by

$$F(x, y) = min\{x, y\}, \forall x, y \in X. \tag{31}$$

Also we define $\phi, \psi : [0, \infty) \to [0, \infty)$ by

$$\phi(t) = t^2 \quad and \quad \psi(t) = t^3 \quad \forall\, t. \tag{32}$$

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Then clearly ϕ *and* ψ are altering distance functions and A and B are closed subsets of a complete metric space $[0,3]$.
First we show that F is a coupling (w.r.t. A and B).
Let $x \in A$, $y \in B$, i.e. $x = 1$ *and* $y = 1, 2$, we have by (72)

$$F(x,y) = 1 \in B \quad and \quad F(y,x) = 1 \in A.$$

This shows that F is a coupling (w.r.t. A and B).
Now we show that F is (ϕ, ψ)-contraction type coupling.
Let $x, v \in A$ and $y, u \in B$, then four cases arise
case(i) $x = v = 1$ and $y = u = 1$,
case(ii) $x = v = 1$ and $y = 1, u = 2$,
case(iii) $x = v = 1$ and $y = 2, u = 1$,
case(iv) $x = v = 1$ and $y = 2, u = 2$.
For case(i) when $x = v = 1$ and $y = u = 1$, we have from (72)
$F(x,y) = F(1,1) = 1$ and $F(u,v) = F(1,1) = 1$.
using above and (32), we get

$$\psi(d(F(x,y), F(u,v))) = \psi(0) = 0. \tag{33}$$

Also $d(x,u) = 0$ and $d(y,v) = 0$, so $max\{d(x,u), d(y,v)\} = 0$.
Then from (32), we get

$$\psi(max\{d(x,u), d(y,v)\}) - \phi(max\{d(x,u), d(y,v)\}) = \psi(0) - \phi(0) = 0. \tag{34}$$

Thus (33) and (34) gives

$$\psi(d(F(x,y), F(u,v))) \leq \psi(max\{d(x,u), d(y,v)\}) - \phi(max\{d(x,u), d(y,v)\}).$$

We are done in case(i).
For case(ii) when $x = v = 1$ and $y = 1, u = 2$, we have from (72)
$F(x, y) = F(1, 1) = 1$ and $F(u, v) = F(2, 1) = 1$.
Using above and (32), we get

$$\psi(d(F(x, y), F(u, v))) = \psi(0) = 0. \tag{35}$$

Also $d(x, u) = 1$ and $d(y, v) = 0$, so $max\{d(x, u), d(y, v)\} = 1$.
Then from (32), we get

$$\psi(max\{d(x, u), d(y, v)\}) - \phi(max\{d(x, u), d(y, v)\}) = \psi(1) - \phi(1) = 0. \tag{36}$$

Thus (35) and (36) gives

$$\psi(d(F(x, y), F(u, v))) \leq \psi(max\{d(x, u), d(y, v)\}) - \phi(max\{d(x, u), d(y, v)\}).$$

Thus (35) and (36) gives

$$\psi(d(F(x, y), F(u, v))) \leq \psi(max\{d(x, u), d(y, v)\}) - \phi(max\{d(x, u), d(y, v)\}).$$

We are done in case(ii).
For case(iii) when $x = v = 1$ and $y = 2, u = 1$, we have from (72)
$F(x, y) = F(1, 2) = 1$ and $F(u, v) = F(1, 1) = 1$.
Using above and (32), we get

$$\psi(d(F(x, y), F(u, v))) = \psi(0) = 0. \tag{37}$$

Also $d(x, u) = 0$ and $d(y, v) = 1$, so $max\{d(x, u), d(y, v)\} = 1$.
Then from (32), we get

Thus (37) and (38) gives

$$\psi(d(F(x,y),F(u,v))) \leq \psi(max\{d(x,u),d(y,v)\}) - \phi(max\{d(x,u),d(y,v)\}).$$

case(iii) is proved.
For case(iv) when $x = v = 1$ and $y = 2, u = 2$, we have from (72)
$F(x,y) = F(1,2) = 1$ and $F(u,v) = F(2,1) = 1$.
Using above and (32), we get

$$\psi(d(F(x,y),F(u,v))) = \psi(0) = 0. \tag{39}$$

Also $d(x,u) = 1$ and $d(y,v) = 1$, so $max\{d(x,u),d(y,v)\} = 1$.
Then from (32), we get

$$\psi(max\{d(x,u),d(y,v)\}) - \phi(max\{d(x,u),d(y,v)\}) = \psi(1) - \phi(1) = 0. \tag{40}$$

Thus (39) and (40) gives

$$\psi(d(F(x,y),F(u,v))) \leq \psi(max\{d(x,u),d(y,v)\}) - \phi(max\{d(x,u),d(y,v)\}).$$

Hence F is (ϕ,ψ)-contraction type coupling (with respect to A and B). Thus all the conditions of Theorem 2.2. are satisfied:
then F has a strong coupled fixed point in $A \cap B$.
Clearly $A \cap B = \{1\} \neq \emptyset$ and 1 is the unique strong coupled fixed point of F in $A \cap B$ as $F(1,1) = min\{1,1\} = 1$.

Example 2.5. : Let $X = \mathbb{R}$ and d is a usual metric on X, *i.e* $d(x,y) =| x - y |$. Let $A = \{\frac{\pi}{2}\}$ and $B = \{\frac{\pi}{2}, \pi\}$ be two closed subsets of complete metric space $\mathbb{R}$. Let us define $F : X \times X \to X$ by

$$F(x,y) = \frac{x+y}{2} + \frac{\pi}{4} sin(x+y). \tag{41}$$

Also we define $\phi, \psi : [0,\infty) \to [0,\infty)$ by

$$\phi(t) = 2t \quad and \quad \psi(t) = t^4 \quad \forall\, t \tag{42}$$

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Clearly ϕ and ψ are altering distance functions.
We now show that all the assumptions of Theorem 2.2 are satisfied.
First we show that F is a coupling (with respect to A and B).
Let $x \in A$ and $y \in B$, then $x = \frac{\pi}{2}$ and $y = \frac{\pi}{2}, \pi$.
Case(i) when $x = \frac{\pi}{2}$ and $y = \frac{\pi}{2}$, then from (41), we get

$$F(x,y) = F(\tfrac{\pi}{2}, \tfrac{\pi}{2}) = \tfrac{\pi}{2} \in B \quad \text{and} \quad F(y,x) = F(\tfrac{\pi}{2}, \tfrac{\pi}{2}) = \tfrac{\pi}{2} \in A.$$

Case(ii) when $x = \frac{\pi}{2}$ and $y = \pi$, then from (41), we have

$$F(x,y) = F(\tfrac{\pi}{2}, \pi) = \tfrac{\pi}{2} \in B \quad \text{and} \quad F(y,x) = F(\pi, \tfrac{\pi}{2}) = \tfrac{\pi}{2} \in A.$$

This shows that F is a coupling (with respect to A and B).
Now we show that F is a (ϕ, ψ)-contraction type coupling.
For any $x, v \in A$ and $y, u \in B$, we have four cases

Case(i) when $x = v = \frac{\pi}{2}$ and $y = u = \frac{\pi}{2}$, then

$$d(x,u) = d(y,v) = 0 \quad \Rightarrow max\{d(x,u), d(y,v)\} = 0 \tag{43}$$

Also from (41)

$$F(x,y) = F(\frac{\pi}{2}, \frac{\pi}{2}) = \frac{\pi}{2} \ \ and \ \ F(u,v) = F(\frac{\pi}{2}, \frac{\pi}{2}) = \frac{\pi}{2}.$$

Therefore

$$d(F(x,y), F(u,v)) = d(\frac{\pi}{2}, \frac{\pi}{2}) = 0 \tag{44}$$

Thus from (42), (43) and (44), we have

$$\psi(d(F(x,y), F(u,v))) = \psi(max\{d(x,u), d(y,v)\}) - \phi(max\{d(x,u), d(y,v)\}). \tag{45}$$

Thus we are done in case(i).

Case(ii) when $x = v = \frac{\pi}{2}$ and $y = u = \pi$, then

$$d(x,u) = d(y,v) = \frac{\pi}{2} \quad \Rightarrow max\{d(x,u), d(y,v)\} = \frac{\pi}{2}. \tag{46}$$

Also using (41), we have

$$F(x,y) = F(\frac{\pi}{2}, \pi) = \frac{\pi}{2} \quad and \quad F(u,v) = F(\pi, \frac{\pi}{2}) = \frac{\pi}{2}.$$

Therefore

$$d(F(x,y), F(u,v)) = d(\frac{\pi}{2}, \frac{\pi}{2}) = 0. \tag{47}$$

From (42), (46) and (47), we get

$$\begin{aligned} \psi(max\{d(x,u), d(y,v)\}) - \phi(max\{d(x,u), d(y,v)\}) &= \psi(\frac{\pi}{2}) - \phi(\frac{\pi}{2}) \\ &= (\frac{\pi}{2})^4 - \pi \\ &> 0 \\ &= \psi(d(F(x,y), F(u,v))). \end{aligned}$$

or

$$\psi(d(F(x,y), F(u,v))) < \psi(max\{d(x,u), d(y,v)\}) - \phi(max\{d(x,u), d(y,v)\}).$$

Which proved case(ii).

Case(iii) when $x = v = \frac{\pi}{2}$ and $y = \pi, u = \frac{\pi}{2}$ then

$$d(x,u) = 0 \quad and \quad d(y,v) = \frac{\pi}{2} \quad \Rightarrow max\{d(x,u), d(y,v)\} = \frac{\pi}{2}. \tag{48}$$

Also using (41), we have

$$F(x,y) = F(\frac{\pi}{2}, \pi) = \frac{\pi}{2} \quad and \quad F(u,v) = F(\frac{\pi}{2}, \frac{\pi}{2}) = \frac{\pi}{2}.$$

Therefore

$$d(F(x,y), F(u,v)) = d(\frac{\pi}{2}, \frac{\pi}{2}) = 0. \tag{49}$$

From (42), (48) and (49), we get

$$\begin{aligned} \psi(max\{d(x,u), d(y,v)\}) - \phi(max\{d(x,u), d(y,v)\}) &= \psi(\frac{\pi}{2}) - \phi(\frac{\pi}{2}) \\ &= (\frac{\pi}{2})^4 - \pi \\ &> 0 \\ &= \psi(d(F(x,y), F(u,v))). \end{aligned}$$

or

$$\psi(d(F(x,y), F(u,v))) < \psi(max\{d(x,u), d(y,v)\}) - \phi(max\{d(x,u), d(y,v)\}).$$

Which proved case(iii).
Similarly we can prove case(iv) when when $x = v = \frac{\pi}{2}$ and $y = \frac{\pi}{2}, u = \pi$.
Hence all the assumptions of Theorem 2.2 are satisfied, thus F has a unique strong coupled fixed point in $A \cap B$. From above we can easily see that $A \cap B = \{\frac{\pi}{2}\} \neq \emptyset$ and $F(\frac{\pi}{2}, \frac{\pi}{2}) = \frac{\pi}{2}$, thus $\frac{\pi}{2}$ is the unique strong coupled fixed point of F in $A \cap B$.

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