Error Estimates for a Semidiscrete Finite Element Method for Nonlinear Parabolic Equations in Nonconvex Polygonal Domains

Tamal Pramanick

Department of Mathematics, Indian Institute of Technology Guwahati, Guwahati (India)

ABSTRACT

We consider the nonlinear parabolic problem with homogeneous Dirichlet boundary conditions in a plane nonconvex polygonal domain. A special feature in a polygonal domain is the presence of singularities in the solutions generated by the corners even if the forcing term is smooth. As a result, the rate of convergence which is optimal order in a convex polygonal domain is reduced for the case of nonconvex polygonal domain. However, it is possible to get a same rate of convergence as for the domain with smooth boundary, by introducing a proper re_nement of the elements around the corners. The re_nement were introduced by Babu_ska [Computing, 6 (1970), pp. 264-273]. We analyze the convergence properties in the L1(L2) norm for the semidiscrete method.

Keywords. Nonlinear parabolic problem, nonconvex polygonal domain, singularity, reen-trant corner, renement, semidiscrete, order of convergence

AMS subject classications. 65M60, 65N15, 65N30

I. INTRODUCTION

In this paper, we focus our attention on nonlinear parabolic initial-boundary value problems in domains with nonsmooth boundaries. Parabolic partial differential equations on nonconvex domains appear in many applications such as heat conduction in chip design, environment modeling, porous media flow and modeling of complex technical engines (cf. [1]).

We consider the nonlinear parabolic problem, for $u = u(x,t)$,

$$\begin{aligned} u_t - \nabla\cdot(a(u)\nabla u) &= f(u) && \text{in } \Omega,\ t\in J,\\ u &= 0 && \text{on } \partial\Omega,\ t\in J,\\ \text{with } u(\cdot,0) &= v && \text{in } \Omega, \end{aligned} \tag{1.1}$$

where $J = (0,T]$, $T > 0$, be a finite interval in time and Ω be a nonconvex polygonal domain in $\mathbb{R}^2$, with boundary $\partial\Omega$. Also, define the smooth functions a and f on $\mathbb{R}$ such that

$$0 < \mu \leq a(u) \leq M, \quad |a'(u)| + |f'(u)| \leq B, \quad \text{for } u \in \mathbb{R}. \tag{1.2}$$

We assume that the above problem admits a unique solution.

For simplicity, we assume that ω is exactly one interior angle which is reentrant such that $\pi < \omega < 2\pi$. We set $\beta = \pi/\omega$, notice that $\frac{1}{2} < \beta < 1$. In particular, for the case of L-shaped domain, $\omega = 3\pi/2$ and $\beta = 2/3$. Assume that O is the associated vertex at the origin and (r, θ) denotes the polar coordinates describing the domain near O, with $0 < \theta < \omega$. The singularity in the solution will arise at the corner O with a leading term near O of the form

$$\kappa(f) r^{\beta} \sin(\beta\theta), \tag{1.3}$$

where, $\kappa(f) \neq 0$ in general, even when f is smooth. Further details on the singular function and the singular solution, we refer to [2, 3]. Note that, elliptic error estimates play a crucial role in the error analysis for the parabolic problems (1.1), and the regularity of the solution of the elliptic problem for the nonsmooth domain Ω can be found in [4, 5]. We show that, the order of convergence in $L^\infty(L^2)$ norm for the semidiscrete method is reduced from $\mathcal{O}(h^2)$ to $\mathcal{O}(h^{2\beta})$, due to the presence of singularity in the solution of (1.1) at the reentrant corner. However, with a proper refinement of mesh near the corners of the domain one may restored the optimal order convergence (cf. [6]). Finite element method (FEM) for nonlinear parabolic problems in nonconvex polygonal domain are introducing for the first time in the literature.

The paper is organized as follows. In the next section we define some notations and preliminaries which will use throughout this paper. The finite element space corresponding to the triangulations of the domain Ω and the existence and uniqueness of the finite element solutions are presented in this section. The elliptic projection which is used in the error estimates is defined here. Section 3 devoted to the a priori error estimates for the spatially semidiscrete scheme. In this section we introduce a systematical refinement near the nonconvex corner in order to improve the order of convergence for the spatially semidiscrete error estimates. Finally, some concluding remarks are presented in Section 4.

II. NOTATIONS AND PRELIMINARIES

In this paper, we will use some standard notation. We denote the standard Lebesgue spaces by $L^p(\Omega)$, $1 \leq p \leq \infty$, with the norm $\|\cdot\|_{L^p(\Omega)}$. In particular, for $p = 2$, $L^2(\Omega)$ is a Hilbert space with the norm $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$ induced by the inner product $\langle u, v\rangle = \int_\Omega u(x)v(x)dx$. For an integer $m > 0$ and $1 \leq p < \infty$, $W^{m,p}(\Omega)$ denotes the standard Sobolev space of real valued functions with their weak derivatives of order up to m in the Lebesgue space $L^p(\Omega)$ (cf. [7]). The space $W^{m,p}(\Omega)$ is equipped with the norm $\|\cdot\|_{W^{m,p}(\Omega)}$. For $p = 2$, we denote the Hilbert

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space $W^{m,2}(\Omega)$ by $H^m(\Omega)$ with the norm $\|\cdot\|_{H^m(\Omega)}$. For an integer $m \geq 0$, set $s = m + \sigma$, $0 < \sigma < 1$, and then $H^s = H^s(\Omega)$ denote the sobolev spaces of fractional order with the norm defined by

$$\|u\|_{H^s} = \left(\|u\|^2_{H^m} + \sum_{|\alpha|=m} \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{2+2\sigma}} dxdy \right)^{1/2}.$$

For a given Banach space $\mathbf{B}$ and for $1 \leq p < +\infty$, we define

$$L^p(0,T;\mathbf{B}) = \left\{ v : [0,T] \to \mathbf{B} \;\middle|\; v(t) \in \mathbf{B} \text{ for almost all } t \in [0,T] \text{ and } \int_0^T \|v(t)\|^p_{\mathbf{B}} dt < \infty \right\}$$

equipped with the norm

$$\|v\|_{L^p(0,T;\mathbf{B})} := \left(\int_0^T \|v(t)\|^p_{\mathbf{B}} \right)^{1/p},$$

with the standard modification for $p = \infty$. We write $\|v\|_{L^p(0,T;\mathbf{B})} = \|v\|_{L^p(\mathbf{B})}$.

2.1 Finite element solution

In order to introduce the finite element space, let $\mathcal{T}_h = \{\tau\}$ be the family of quasiuniform triangulation of Ω with $\max_{\tau \in \mathcal{T}_h} \text{diam}(\tau) \leq h$ (see, e.g. [8, 9]). The triangulations are quasiuniform in the sense that there is some constant $c > 0$ such that $\min_{\tau \in \mathcal{T}_h} \text{diam}(\tau) \geq ch$. Let S_h be the finite dimensional space corresponding to the triangulations $\mathcal{T}_h$ is defined by

$$S_h = \{\chi \in \mathcal{C} \;:\; \chi|_\tau \text{ is linear}, \;\forall \tau \in \mathcal{T}_h \;\text{ and }\; \chi|_{\partial\Omega} = 0\},$$

where $\mathcal{C} = \mathcal{C}(\Omega)$ be the space of continuous functions on $\bar{\Omega}$. Then the approximation with the finite elements leads to the semidiscrete problem to find $u_h : \bar{J} \to S_h$ such that

$$\begin{aligned} (u_{h,t}, \chi) + (a(u_h)\nabla u_h, \nabla\chi) = (f(u_h), \chi) \quad \forall \chi \in S_h,\; t \in J, \\ \text{with} \quad u_h(0) = v_h, \end{aligned} \tag{2.1}$$

where v_h is an approximation of v in S_h. Let $\{\phi_j\}_{j=1}^{N_h}$ be the standard nodal basis functions for S_h. Then writing the solution as $u_h(x,t) = \sum_{j=1}^{N_h} \alpha_j(t)\phi_j(x)$, we have from (2.1),

$$\begin{aligned} \sum_{j=1}^{N_h} \alpha_j'(t)(\phi_j, \phi_k) + \sum_{j=1}^{N_h} \alpha_j(t) \Big(a(\sum_{l=1}^{N_h} \alpha_l(t)\phi_l)\nabla\phi_j, \nabla\phi_k \Big) = \Big(f(\sum_{l=1}^{N_h} \alpha_l(t)\phi_l), \phi_k \Big), \\ \text{with} \quad \alpha_j(0) = \gamma_j, \quad \text{for } j,k = 1,2,...,N_h, \end{aligned} \tag{2.2}$$

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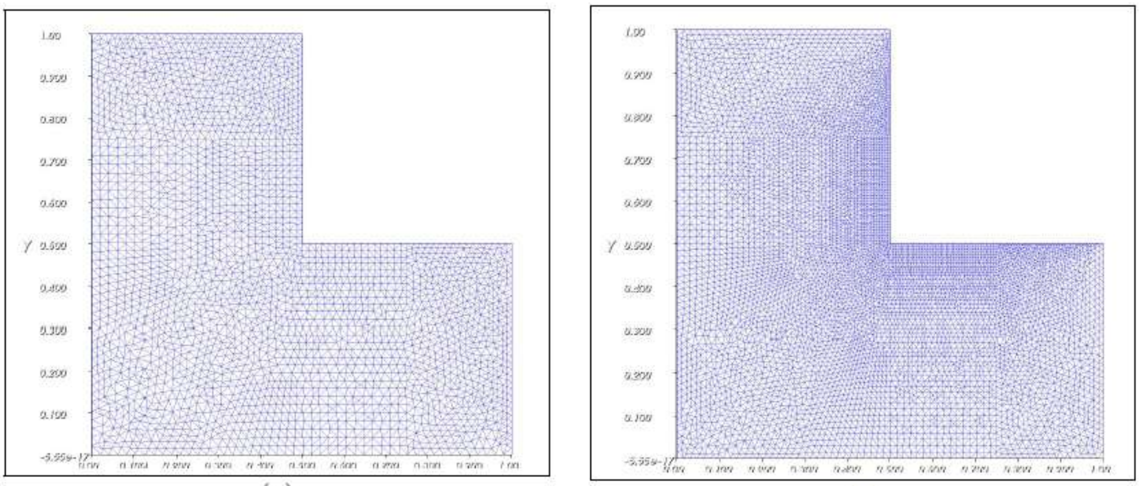

(a) (b)

Figure 1: (a) Finite element discretizations for the L-shaped domain, # triangles= 5178 and # dof= 2702. (b) Further refinement made towards the nonconvex corner for the L-shaped domain, # triangles= 8646 and # dof= 4468.

where γ_j the components of the given initial approximation of v_h and $N_h = \dim(S_h)$. Now setting $\alpha = \alpha(t) = (\alpha_1(t), \alpha_2(t), ..., \alpha_{N_h}(t))^T$, (2.2) may be written in the matrix form

$$A\alpha' + B(\alpha)\alpha = \tilde{f}(\alpha) \quad \text{for} \quad t \in J, \quad \text{with} \quad \alpha(0) = \gamma, \tag{2.3}$$

where $A = (a_{jk})$ and $B(\alpha) = (b_{jk}(\alpha))$ with elements

$$a_{jk} = (\phi_j, \phi_k) \quad \text{and} \quad b_{jk}(\alpha) = \left(a(\sum_{l=1}^{N_h} \alpha_l \phi_l) \nabla \phi_j, \nabla \phi_k \right),$$

respectively, $\tilde{f}(\alpha) = (f_k(\alpha))$ be the vector with entries $f_k(\alpha) = \left(f(\sum_{l=1}^{N_h} \alpha_l \phi_l), \phi_j \right)$ and $\gamma = (\gamma_k)$. Using our assumption (1.2), it can be easily derive that the matrices A and $B(\alpha)$ are positive definite and also $B(\alpha)$ and $\tilde{f}(\alpha)$ are globally Lipschitz continuous on $\mathbb{R}^{N_h}$. Therefore the system has a unique solution for $t \in J$.

Before we start the semidiscrete error analysis for the semidiscrete problem (2.1), introduce the elliptic projection $\tilde{u}_h = \tilde{u}_h(t)$ in S_h of the exact solution u is defined by

$$(a(u(t))\nabla(\tilde{u}_h(t) - u(t)), \nabla \chi) = 0, \quad \forall \chi \in S_h, \; t \geq 0. \tag{2.4}$$

In order to have some estimates for the error in this projection, we first derive the following auxiliary result.

Lemma 2.1. *Assume $b = b(x)$ be a smooth function in Ω with $0 < \mu \leq b(x) \leq M$ for $x \in \Omega$.*

Consider $\xi \in H^{1+s}(\Omega) \cap H_0^1(\Omega)$, *and let* ξ_h *be defined by*

$$(b\nabla(\xi_h - \xi), \nabla\chi) = 0 \quad \forall\chi \in S_h. \tag{2.5}$$

Then

$$\|\nabla(\xi_h - \xi)\| \leq C_1 h^\beta \|\Delta\xi\|_{H^{-1+s}} \quad \textit{for } \beta < s \leq 1, \tag{2.6}$$

and

$$\|\xi_h - \xi\| \leq C_2 h^{2\beta} \|\Delta\xi\|_{H^{-1+s}} \quad \textit{for } \beta < s \leq 1. \tag{2.7}$$

The constants C_1 *and* C_2 *depends on* μ *and* M *and on the family of triangulations* $\mathcal{T}_h$. *Also* C_2 *depends on an upper bound for* ∇b.

Proof. Consider any $\chi \in S_h$, we have

$$\begin{aligned} \mu \|\nabla(\xi_h - \xi)\|^2 &\leq (b\nabla(\xi_h - \xi), \nabla(\xi_h - \xi)) \\ &= (b\nabla(\xi_h - \xi), \nabla(\chi - \xi)) \\ &\leq M \|\nabla(\xi_h - \xi)\| \|\nabla(\chi - \xi)\|, \end{aligned}$$

which implies

$$\|\nabla(\xi_h - \xi)\| \leq (M/\mu) \|\nabla(\chi - \xi)\|. \tag{2.8}$$

Now, define the *Ritz projection* $R_h : H_0^1(\Omega) \to S_h$ as the orthogonal projection (cf. [10]),

$$(\nabla R_h v, \nabla\chi) = (\nabla v, \nabla\chi) \quad \forall\chi \in S_h, \ \text{ for } v \in H_0^1(\Omega).$$

Following [3, Lemma 2.5] and choosing $\chi = R_h\xi$ in (2.8), we obtain

$$\|\nabla(\xi_h - \xi)\| \leq C_1 h^\beta \|\Delta\xi\|_{H^{-1+s}} \quad \text{for } \beta < s < 1,$$

which proofs (2.6). In order to show (2.7) we use the duality argument. For this purpose, we consider the problem

$$-\nabla \cdot (b\nabla\psi) \equiv -b\Delta\psi - \nabla b \cdot \nabla\psi = \varphi \quad \text{in } \Omega, \quad \psi = 0 \quad \text{on } \partial\Omega, \tag{2.9}$$

and since, $\|\psi\| \leq C \|\nabla\psi\|$ for $\psi \in H_0^1$, we have

$$\mu \|\nabla\psi\|^2 \leq (b\nabla\psi, \nabla\psi) = (\varphi, \psi) \leq \|\varphi\| \|\psi\| \leq C \|\varphi\| \|\nabla\psi\|,$$

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which gives $\|\nabla\psi\| \leq C\|\varphi\|$. Therefore, using the elliptic regularity estimate (see, e.g. [3, Lemma 2.5], or Bacuta, Brahmble and Xu [4]) and for boundedness of ∇b together with equation (2.9),

$$\|\Delta\psi\|_{H^{-1+s}} \leq C\|\varphi + \nabla b \cdot \nabla\psi\| \leq C\|\varphi\|. \tag{2.10}$$

Hence, with $\chi = R_h\psi$,

$$\begin{aligned}
(\xi_h - \xi, \varphi) &= (b\nabla(\xi_h - \xi), \nabla\psi) \\
&= (b\nabla(\xi_h - \xi), \nabla(\psi - \chi)) \\
&\leq M\|\nabla(\xi_h - \xi)\|\,\|\nabla(\psi - \chi)\| \\
&\leq (Ch^\beta\|\Delta\xi\|_{H^{-1+s}})(Ch^\beta\|\Delta\psi\|_{H^{-1+s}}) \\
&\leq C_2h^{2\beta}\|\Delta\xi\|_{H^{-1+s}}\|\varphi\|,
\end{aligned}$$

and this completes the proof of the lemma. □

III. SPATIALLY SEMIDISCRETE ERROR ANALYSIS

In this section we have concerned on some error estimates for the spatially semidiscrete finite element approximation (2.1) of the parabolic problem (1.1). For this purpose, we split the error term using the so called elliptic projection $\tilde{u}_h$ defined in (2.4) as a sum of two terms,

$$u_h - u = (u_h - \tilde{u}_h) + (\tilde{u}_h - u) = \theta + \rho. \tag{3.1}$$

Hence, for estimates the error we need to first estimate the term ρ and ρ_t, which is given in the following lemma.

Lemma 3.1. *Let ρ is defined by (3.1) and $C(u)$ independent of $t \in J$. Then under consideration the appropriate regularity assumptions on u, we have for $t \in J$, $\beta < s < 1$,*

$$\|\rho(t)\| + h^\beta\|\nabla\rho(t)\| \leq C(u)h^{2\beta},$$

and

$$\|\rho_t(t)\| + h^\beta\|\nabla\rho_t(t)\| \leq C(u)h^{2\beta}.$$

Proof. Note that $\nabla a(u) = a'(u)\nabla u$, the first proof easily follows from Lemma 2.1 with $b(x) = a(u(x,t))$.

For the second estimate, differentiating (2.4) we have

$$(a(u)\nabla\rho_t, \nabla\chi) + (a(u)_t\nabla\rho, \nabla\chi) = 0 \quad \forall\chi \in S_h.$$

So, for uniformly boundedness of $a(u)$ and $a(u)_t$,

$$\begin{aligned}\mu\,\|\nabla\rho_t\|^2 &\le (a(u)\nabla\rho_t, \nabla\rho_t)\\ &= (a(u)\nabla\rho_t, \nabla(\chi - u_t)) + (a(u)\nabla\rho_t, \nabla(\tilde{u}_{h,t} - \chi))\\ &= (a(u)\nabla\rho_t, \nabla(\chi - u_t)) + (a(u)_t\nabla\rho, \nabla(\chi - \tilde{u}_{h,t}))\\ &\le C(\|\nabla\rho_t\|\,\|\nabla(\chi - u_t)\| + \|\nabla\rho\|\,\|\nabla(\chi - \tilde{u}_{h,t})\|)\\ &\le C\big(\,\|\nabla\rho_t\|\,\|\nabla(\chi - u_t)\| + \|\nabla\rho\|\,(\|\nabla(\chi - u_t)\| + \|\nabla\rho_t\|)\big),\end{aligned}$$

following [10], with $\chi = R_h u_t$ and using Lemma 2.1 this yields

$$\mu\,\|\nabla\rho_t\|^2 \le \frac{\mu}{2}\,\|\nabla\rho_t\|^2 + C\,\|\nabla\rho\|^2 + C(u)h^{2\beta},$$

together with the previous estimate of $\nabla\rho$ already shown and letting $\beta < s < 1$, we have $\|\nabla\rho_t\| \le C(u)h^{\beta}$.

Now for the estimate of ρ_t, we use the duality argument as in the proof of Lemma 2.1. With $b = a(u)$ and ψ is defined as in (2.9),

$$(\rho_t, \varphi) = (a(u)\nabla\rho_t, \nabla\psi) = (a(u)\nabla\rho_t, \nabla(\psi - \chi)) - (a(u)_t\nabla\rho, \nabla\chi). \tag{3.2}$$

Since $a(u)$ is bounded in view of (1.2), hence using (2.4) the second term of the right hand side of (3.2) gives

$$(a(u)_t\nabla\rho, \nabla\chi) = \frac{a(u)_t}{a(u)}(a(u)\nabla\rho, \nabla\chi) = 0,$$

and therefore, we have

$$(\rho_t, \varphi) = (a(u)\nabla\rho_t, \nabla(\psi - \chi)),$$

choosing $\chi = R_h\psi$, together with (2.10) and with the estimates for $\nabla\rho_t$, we obtain

$$|(\rho_t, \varphi)| \le C\,\|\nabla\rho_t\|\,h^{\beta}\|\Delta\psi\|_{H^{-1+s}} \le C(u)h^{2\beta}\|\varphi\|,$$

which gives, $\|\rho_t\| \le C(u)h^{2\beta}$. This completes the proof. □

The main result for the error estimate between the solution of semidiscrete problem (2.1) and the continuous problem (1.1) given in the following theorem.

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Theorem 3.2. *Let u_h and u be the solution of (2.1) and (1.1), respectively. Then with $C = C_T$, we have*

$$\|u_h(t) - u(t)\| \leq C\,\|v_h - v\| + C(u)h^{2\beta} \quad \textit{for}\ \beta < s < 1,\ t \in \bar{J}. \tag{3.3}$$

Proof. We first write the error term as in (3.1), and since ρ is bounded in view of Lemma 3.1, so it remains to estimate θ. For $\chi \in S_h$ and using (2.4) yields

$$\begin{aligned}
&(\theta_t, \chi) + (a(u_h)\nabla\theta, \nabla\chi)\\
&= (u_{h,t}, \chi) + (a(u_h)\nabla u_h, \nabla\chi) - (\tilde{u}_{h,t}, \chi) - (a(u_h)\nabla\tilde{u}_h, \nabla\chi)\\
&= (f(u_h), \chi) - (\tilde{u}_{h,t} - u_t, \chi) - (u_t, \chi) - (a(u)\nabla\tilde{u}_h, \nabla\chi) + ((a(u) - a(u_h))\nabla\tilde{u}_h, \nabla\chi)\\
&= (f(u_h), \chi) - (\rho_t, \chi) - (u_t, \chi) - (a(u)\nabla u, \nabla\chi) + ((a(u) - a(u_h))\nabla\tilde{u}_h, \nabla\chi),
\end{aligned}$$

and thus

$$(\theta_t, \chi) + (a(u_h)\nabla\theta, \nabla\chi) = (f(u_h) - f(u), \chi) + ((a(u) - a(u_h))\nabla\tilde{u}_h, \nabla\chi) - (\rho_t, \chi). \tag{3.4}$$

Now, using (1.2) and (2.4),

$$\mu(\nabla\tilde{u}_h, \nabla\chi) \leq (a(u)\nabla\tilde{u}_h, \nabla\chi) = (a(u)\nabla u, \nabla\chi) \leq M(\nabla u, \nabla\chi),$$

which leads to

$$(\nabla\tilde{u}_h, \nabla\chi) \leq (M/\mu)(\nabla u, \nabla\chi),$$

by putting $\chi = \tilde{u}_h$ both side

$$\|\nabla\tilde{u}_h\| \leq C\,\|\nabla u\|\,,$$

and this yields

$$\|\nabla\tilde{u}_h(t)\| \leq C(u). \tag{3.5}$$

Therefore with $\chi = \theta$ in (3.4) and using (1.2), (3.5), we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\theta\|^2 + \mu\,\|\nabla\theta\|^2 &\leq C\,\|u_h - u\|\,(\|\theta\| + \|\nabla\theta\|) + \|\rho_t\|\,\|\theta\|\\
&\leq \mu\,\|\nabla\theta\|^2 + C(\|\theta\|^2 + \|\rho\|^2 + \|\rho_t\|^2),
\end{aligned}$$

after integration this leads to

$$\|\theta(t)\|^2 \leq \|\theta(0)\|^2 + C\int_0^t (\|\theta\|^2 + \|\rho\|^2 + \|\rho_t\|^2)ds.$$

Now, using Gronwall's lemma we obtain

$$\|\theta(t)\|^2 \leq C\,\|\theta(0)\|^2 + C\int_0^t (\|\rho\|^2 + \|\rho_t\|^2)ds, \tag{3.6}$$

where C now depends on T. We have

$$\|\theta(0)\| \leq \|v_h - v\| + \|\tilde{u}_h(0) - v\| \leq \|v_h - v\| + Ch^{2\beta}, \tag{3.7}$$

where, $C = C(v)$. With this and together with Lemma 3.1 we obtain from (3.6)

$$\|\theta(t)\| \leq C\,\|v_h - v\| + C(u)h^{2\beta},$$

and this completes the proof of the theorem. □

Remark 3.3. *Note that, the singularity occur in the finite element solution due to the re-entrant corner in Ω for the case of globally quasiuniform mesh. Hence, $\mathcal{O}(h^{2\beta})$ is the best possible convergence we obtain away from the nonconvex corner. However, to obtain an optimal order convergence $\mathcal{O}(h^2)$, we refine the triangulations systematically towards the nonconvex corner. The refinement was first introduced by Babuška [6].*

In order to introduce the refinement of triangulations systemically (cf. [3]), let $d(x)$ be the distance to the nonconvex corner and $d_j = 2^{-j}$, for $j = 0, 1, ..., \hat{J}$. Assume that, for $j = 0, 1, ..., \hat{J}$,

$$\begin{aligned} \Omega_j &= \{x \in \Omega : \; d_j/2 \leq d(x) \leq d_j\}, \\ \Omega_j' &= \Omega_{j-1} \cup \Omega_j \cup \Omega_{j+1}, \quad \text{and} \\ \Omega_I &= \{x \in \Omega : \; d(x) \leq d_{\hat{j}}/2\}. \end{aligned}$$

With h be the meshsize in the interior of the domain, choose $\hat{J}$ such that $d_{\hat{J}} \approx h^{1/\beta}$, and $\nu \geq 1/\beta$ such that

$$h_j \leq Chd_j^{1-\beta+\epsilon} \quad \text{and} \quad ch^{\nu} \leq h_I \leq Ch^{1/\beta},$$

with $c > 0$, ϵ be a small positive number, and h_j denotes the maximal meshsize on Ω_j. Also let the mesh is locally quasiuniform on each Ω_j' so that $h_{\min} \geq h^{\nu}$ and $\dim(S_h) \leq Ch^{-2}$. The finite element triangulations for an L-shaped domain are shown in Figure 1(a). Also, further refinement on the triangulations are made towards the nonconvex corner to improve the order of convergence, and this is shown in Figure 1(b). With the above refinements we therefore have the following auxiliary result.

Lemma 3.4. *Let ρ is defined by (3.1) and $C(u)$ independent of $t \in J$. Then with the triangu-*

lations above, we have

$$\|\rho(t)\| + h\,\|\nabla\rho(t)\| \leq C(u)h^2 \quad \textit{for } t \in J,$$
$$\|\rho_t(t)\| + h\,\|\nabla\rho_t(t)\| \leq C(u)h^2 \quad \textit{for } t \in J.$$

Proof. Following Chatzipantelidis et al. [3, Lemma 2.9] with $s = 1$, and using the similar arguments as in the proof of Lemma 3.1, the proof is easily follows. □

Next, we shall show that Theorem 3.2 now gives the optimal order convergence by a proper refinements towards the nonconvex corner introduced above.

Theorem 3.5. *Let u_h and u be the solution of (2.1) and (1.1), respectively. Assume that the triangulations underlying the S_h are refined as in Lemma 3.4. Then with $C = C_T$, we have*

$$\|u_h(t) - u(t)\| \leq C\,\|v_h - v\| + C(u)h^2 \quad \textit{for } t \in \bar{J}. \tag{3.8}$$

Proof. Following the similar argument as in the proof of Theorem 3.2 and in view of Lemma 3.4, the rest of the proof is standard. □

IV. CONCLUSIONS

In this paper, we have presented the finite element method for nonlinear parabolic problems in nonconvex polygonal domains. A priori error bounds in the $L^\infty(L^2)$ norm are derived for the spatially semidiscrete method. The derivation gives the convergence rate $\mathcal{O}(h^{2\beta})$. The reduction of the convergence rate from optimal order to $\mathcal{O}(h^{2\beta})$ caused by the presence of the singularity in the solution due to the reentrant corner in the domain. However, with a proper mesh refinement near the corners the order of convergence is improved to the optimal order.

REFERENCES

[1] T. Gao, W. H. Zhang, J. H. Zhu, Y. J. Xu, and D. H. Bassir, *Topology optimization of heat conduction problem involving design-dependent heat load effect*, Finite Elem. Anal. Des., 44(14), 2008, 805–813.

[2] P. Grisvard, *Singularities in boundary value problems*, Springer-Verlag, 1992.

[3] P. Chatzipantelidis, R. D. Lazarov, V. Thomée, and L. B. Wahlbin, *Parabolic finite element equations in nonconvex polygonal domains*, BIT, 46(1), 2006, 113–143.

[4] C. Bacuta, J. Bramble, and J. Xu, *Regularity estimates for elliptic boundary value problems in Besov spaces*, Math. Comp., 72(244), 2003, 1577–1595.

[5] P. Grisvard, *Elliptic problems in nonsmooth domains*, Class. Appl. Math., vol. 69, SIAM, 1985.

[6] I. Babuška, *Finite element method for domains with corners*, Computing, 6(3-4), 1970, 264–273.

[7] R. A. Adams, and J. J. Fournier, *Sobolev spaces*, vol. 140, Academic press, 2003.

[8] S. C. Brenner, and R. Scott, *The mathematical theory of finite element methods*, Texts Appl. Math., vol. 15, Springer, New York, 2008.

[9] P. G. Ciarlet, *The finite element method for elliptic problems*, Class. Appl. Math., vol. 40, SIAM, 2002.

[10] V. Thomée, *Galerkin finite element methods for parabolic problems*, vol. 1054, Springer, Berlin, 2006.