Coding Theorems on New 'Useful' Fuzzy Information Measure of Order α and Type β

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ABSTRACT

In this paper, we define a new two parametric 'useful' fuzzy average code-word length $L_\alpha^\beta(A;U)$ *of order* α *and type* β *for a fuzzy set 'A'. Also, with the help of Kraft's inequality, the bounds of the proposed* $L_\alpha^\beta(A;U)$ *are found in terms of the fuzzy information measure* $H_\alpha^\beta(A;U)$.

Keywords: Coding Theorem, Code-Word Length, Holder's Inequality, and Kraft's Inequality.

AMS Classification: 94A17, 94A24

I.INTRODUCTION

Fuzzy set theory is highly important in modelling inexact and complex systems and has been used in an effective manner as a decision making system in the area of control [1, 2]. The fuzzy set theory was developed by L. Zadeh [3] and defined the entropy of a fuzzy event as uncertainty associated with it. Fuzziness being a feature of uncertainty, results due to lack of sharp difference of being a member of a particular set or not. A fuzzy subset '*A*' defined on a universe of discourse $X = \{x_1, x_2, ..., x_n\}$ is given as: $A = \{(x, \mu_A(x)) : x \in X,\ \mu_A(x) \in [0,1]\}$.

Here $\mu_A(x)$ is a membership function of '*A*' which describes the degree of belongingness of '*x*' in '*A*'. When $\mu_A(x)$ is valued either '0' or '1', it is the characteristic function of a crisp set or a non-fuzzy set.

We consider a utility distribution as $U = \{(u_1, u_2, ..., u_n); u_i > 0\ \forall i = 1, 2, ..., n\}$ where u_i denotes the utility of x_i. Corresponding to the information source,

$$U = \begin{bmatrix} x_1 & x_2 & \dots & x_n \\ p_1 & p_2 & \dots & p_n \\ u_1 & u_2 & \dots & u_n \end{bmatrix};\ p_i \geq 0,\ \sum_{i=1}^{n} p_i = 1 \tag{1}$$

Belis and Guiasu [4] gave the following measure of information, and called it as 'useful' information

$$H(P;U) = -\sum_{i=1}^{n} u_i p_i \log p_i \qquad (2)$$

If utility is ignored in (2) (i.e., $u_i = 1 \ \forall \ i = 1, 2, ..., n$), then this measure reduces to Shannon's [5] entropy measure. Guiasu and Picard [6] considered the problem of encoding letter output by the above source (1) by means of a single letter prefix, with code-words $c_1, c_2, ..., c_n$ and lengths $l_1, l_2, ..., l_n$ that satisfy the Kraft's [7] inequality i.e., $\sum_{i=1}^{n} D^{-l_i} \leq 1$ ('D' being the size of code alphabet). They gave the following quantity (3) and called it as 'useful' mean length of the code: $L(U) = \dfrac{\sum_{i=1}^{n} u_i p_i l_i}{\sum_{i=1}^{n} u_i p_i}$ (3)

Guiasu and Picard [6] have given the average length for 'useful' codes and for this Longo [8] proved a noiseless coding theorem. Several other authors have given generalized coding theorems for different fuzzy information measures under the condition of uniquely decipherability. A. H. Bhat and M. A. K. Baig [9, 10], A. H. Bhat, M. A. Bhat, M. A. K. Biag and Saima Manzoor [11], D. S. Hooda and U. S. Bhaker [12], Priti Jain and R. K. Tuteja [13] etc are some of them.

II.BOUNDS FOR NEW 'USEFUL' FUZZY INFORMATION MEASURE

In this paper, we consider the 'useful' fuzzy measure given by Saima [14] et. al. in the paper entitled 'A New Generalized 'Useful' Fuzzy Information Measure and its Properties' as:

$$H_\alpha^\beta(A;U) = \frac{\beta}{1-\alpha} \log_D \left[\frac{\sum_{i=1}^{n} u_i \left\{ \mu_A^{\beta(1-\alpha)}(x_i) + \left(1-\mu_A(x_i)\right)^{\beta(1-\alpha)} \right\}}{\sum_{i=1}^{n} u_i} \right];$$

$$0 < \alpha < 1, 0 < \beta \leq 1, \beta > \alpha \,\&\, u_i > 0 \qquad (4)$$

and then obtain the bounds of 'useful' fuzzy average code-word length i.e.,

$$L_\alpha^\beta(A;U) = \frac{\alpha\beta}{1-\alpha} \log_D \left[\frac{\sum_{i=1}^{n} u_i \left\{ \mu_A^{\beta(1-\alpha)}(x_i) + \left(1-\mu_A(x_i)\right)^{\beta(1-\alpha)} \right\} D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}}{\sum_{i=1}^{n} u_i} \right];$$

$$0 < \alpha < 1, 0 < \beta \leq 1, \beta > \alpha \,\&\, u_i > 0 \qquad (5)$$

in terms of (4) under the following condition:

$$\sum_{i=1}^{n} u_i \left\{ \mu_A^{\beta(1-\alpha)}(x_i) + \left(1 - \mu_A(x_i)\right)^{\beta(1-\alpha)} \right\} D^{-l_i} \le 1 \tag{6}$$

which is the generalization of fuzzy Kraft's [7] inequality. Here '*D*' is the size of code alphabet.

Theorem 1: For all integers '*D*' $(D > 1)$, let the code-word length l_i $(i = 1, 2, ..., n)$ satisfies (6), then the average 'useful' fuzzy code-word length satisfies:

$$H_\alpha^\beta(A;U) \le L_\alpha^\beta(A;U);\ 0 < \alpha < 1, 0 < \beta \le 1, \beta > \alpha \,\&\, u_i > 0 \tag{7}$$

Equality holds in (7) iff

$$l_i = -\log_D \left[\frac{\sum_{i=1}^{n} u_i}{\sum_{i=1}^{n} u_i \left\{ \mu_A^{\beta(1-\alpha)}(x_i) + \left(1 - \mu_A(x_i)\right)^{\beta(1-\alpha)} \right\}} \right] \tag{8}$$

Or we can write $l_i = -\log_D \left[\frac{\sum_{i=1}^{n} u_i}{\sum_{i=1}^{n} u_i \left\{ f\left(\mu_A(x_i), \mu_{A'}(x_i)\right) \right\}} \right]$

where $f\left(\mu_A(x_i), \mu_{A'}(x_i)\right) = \mu_A^{\beta(1-\alpha)}(x_i) + \left(1 - \mu_A(x_i)\right)^{\beta(1-\alpha)}$ (9)

Proof: By Holder's inequality, we have

$$\sum_{i=1}^{n} x_i y_i \ge \left(\sum_{i=1}^{n} x_i^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} y_i^q \right)^{\frac{1}{q}} , \tag{10}$$

for all $x_i, y_i > 0; i = 1, 2, ..., n \;\&\; \frac{1}{p} + \frac{1}{q} = 1;\ p < 1(\ne 0), q < 0 \text{ or } q < 1(\ne 0), p < 0$.

The equality will hold if and only if there exists a positive constant '*c*' such that $x_i^p = c y_i^q$.

Substituting $p = \frac{\alpha - 1}{\alpha}$, $q = 1 - \alpha$, $x_i = \left[\frac{u_i f\left(\mu_A(x_i), \mu_{A'}(x_i)\right)}{\sum_{i=1}^{n} u_i} \right]^{\left(\frac{\alpha}{\alpha - 1}\right)} D^{-l_i}$

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$$\& \ \ y_i = \left[\frac{u_i f\left(\mu_A(x_i), \mu_{A'}(x_i)\right)}{\sum_{i=1}^{n} u_i} \right]^{\left(\frac{1}{1-\alpha}\right)} \qquad \text{in (10), we get}$$

$$\left[\frac{\sum_{i=1}^{n} u_i \{ f\left(\mu_A(x_i), \mu_{A'}(x_i)\right)\} D^{-l_i}}{\sum_{i=1}^{n} u_i} \right] \geq$$

$$\left[\frac{\sum_{i=1}^{n} u_i \{ f\left(\mu_A(x_i), \mu_{A'}(x_i)\right)\} D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}}{\sum_{i=1}^{n} u_i} \right]^{\left(\frac{\alpha}{\alpha-1}\right)} \left[\frac{\sum_{i=1}^{n} u_i \{ f\left(\mu_A(x_i), \mu_{A'}(x_i)\right)\}}{\sum_{i=1}^{n} u_i} \right]^{\left(\frac{1}{1-\alpha}\right)} \tag{11}$$

Using condition (6), we get

$$\left[\frac{\sum_{i=1}^{n} u_i \{ f\left(\mu_A(x_i), \mu_{A'}(x_i)\right)\} D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}}{\sum_{i=1}^{n} u_i} \right]^{\left(\frac{\alpha}{\alpha-1}\right)} \left[\frac{\sum_{i=1}^{n} u_i \{ f\left(\mu_A(x_i), \mu_{A'}(x_i)\right)\}}{\sum_{i=1}^{n} u_i} \right]^{\left(\frac{1}{1-\alpha}\right)} \leq 1$$

$$\Rightarrow \left[\frac{\sum_{i=1}^{n} u_i \{ f\left(\mu_A(x_i), \mu_{A'}(x_i)\right)\}}{\sum_{i=1}^{n} u_i} \right]^{\left(\frac{1}{1-\alpha}\right)} \leq \left[\frac{\sum_{i=1}^{n} u_i \{ f\left(\mu_A(x_i), \mu_{A'}(x_i)\right)\} D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}}{\sum_{i=1}^{n} u_i} \right]^{\left(\frac{\alpha}{1-\alpha}\right)} \tag{12}$$

Applying logarithms with base '*D*' to both sides of (12) and then multiplying both sides by $\beta > 0 \ (0 < \beta \leq 1)$, we get

$$\frac{\beta}{1-\alpha} \log_D \left[\frac{\sum_{i=1}^{n} u_i \{ f\left(\mu_A(x_i), \mu_{A'}(x_i)\right)\}}{\sum_{i=1}^{n} u_i} \right] \leq \frac{\alpha\beta}{1-\alpha} \log_D \left[\frac{\sum_{i=1}^{n} u_i \{ f\left(\mu_A(x_i), \mu_{A'}(x_i)\right)\} D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}}{\sum_{i=1}^{n} u_i} \right] \tag{13}$$

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Using (9) in (13), we get

$$\frac{\beta}{1-\alpha}\log_D\left[\frac{\sum_{i=1}^{n}u_i\left\{\mu_A^{\beta(1-\alpha)}(x_i)+\left(1-\mu_A(x_i)\right)^{\beta(1-\alpha)}\right\}}{\sum_{i=1}^{n}u_i}\right]\leq$$

$$\frac{\alpha\beta}{1-\alpha}\log_D\left[\frac{\sum_{i=1}^{n}u_i\left\{\mu_A^{\beta(1-\alpha)}(x_i)+\left(1-\mu_A(x_i)\right)^{\beta(1-\alpha)}\right\}D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}}{\sum_{i=1}^{n}u_i}\right]$$

Thus, the above can be written as:

$$H_\alpha^\beta(A;U)\leq L_\alpha^\beta(A;U)$$

Hence, the result is established.

Theorem 2: For every code with lengths $l_1,l_2,...,l_n$ satisfying the condition (7), $L_\alpha^\beta(A;U)$ can be made to satisfy the inequality:

$$L_\alpha^\beta(A;U)<H_\alpha^\beta(A;U)+\beta;\ 0<\alpha<1,0<\beta\leq1,\beta>\alpha\,\&u_i>0 \tag{14}$$

Proof: We have from theorem 1, the equality holds in (7) if and only if

$$D^{-l_i}=\left[\frac{\sum_{i=1}^{n}u_i}{\sum_{i=1}^{n}u_i\left\{f\left(\mu_A(x_i),\mu_{A'}(x_i)\right)\right\}}\right];0<\alpha<1,0<\beta\leq1,\beta>\alpha\,\&u_i>0$$

$$\Rightarrow l_i=\log_D\left[\frac{\sum_{i=1}^{n}u_i\left\{f\left(\mu_A(x_i),\mu_{A'}(x_i)\right)\right\}}{\sum_{i=1}^{n}u_i}\right]$$

We consider a positive integer l_i $(i=1,2,...,n)$ in such a way that it satisfies the inequality:

$$\log_D\left[\frac{\sum_{i=1}^{n}u_i\left\{f\left(\mu_A(x_i),\mu_{A'}(x_i)\right)\right\}}{\sum_{i=1}^{n}u_i}\right]\leq l_i<\log_D\left[\frac{\sum_{i=1}^{n}u_i\left\{f\left(\mu_A(x_i),\mu_{A'}(x_i)\right)\right\}}{\sum_{i=1}^{n}u_i}\right]+1 \tag{15}$$

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Now, considering the interval of length unity:

$$\delta_i = \left[\log_D \left[\frac{\sum_{i=1}^{n} u_i \left\{ f\left(\mu_A(x_i), \mu_{A'}(x_i)\right)\right\}}{\sum_{i=1}^{n} u_i} \right], \log_D \left[\frac{\sum_{i=1}^{n} u_i \left\{ f\left(\mu_A(x_i), \mu_{A'}(x_i)\right)\right\}}{\sum_{i=1}^{n} u_i} \right] + 1 \right]$$

There exists exactly one positive integer l_i in every δ_i such that

$$0 < \log_D \left[\frac{\sum_{i=1}^{n} u_i \left\{ f\left(\mu_A(x_i), \mu_{A'}(x_i)\right)\right\}}{\sum_{i=1}^{n} u_i} \right] \le l_i < \log_D \left[\frac{\sum_{i=1}^{n} u_i \left\{ f\left(\mu_A(x_i), \mu_{A'}(x_i)\right)\right\}}{\sum_{i=1}^{n} u_i} \right] + 1 \tag{16}$$

We will first show that the defined sequence $l_1, l_2, ..., l_n$ satisfies the generalized fuzzy Kraft [7] inequality (6).

Now, from (16), we have

$$\log_D \left[\frac{\sum_{i=1}^{n} u_i \left\{ f\left(\mu_A(x_i), \mu_{A'}(x_i)\right)\right\}}{\sum_{i=1}^{n} u_i} \right] \le l_i$$

$$\Rightarrow D^{-l_i} \le \left[\frac{\sum_{i=1}^{n} u_i}{\sum_{i=1}^{n} u_i \left\{ f\left(\mu_A(x_i), \mu_{A'}(x_i)\right)\right\}} \right] \tag{17}$$

Multiplying both sides of (17) by $\left[\frac{u_i \left\{ f\left(\mu_A(x_i), \mu_{A'}(x_i)\right)\right\}}{\sum_{i=1}^{n} u_i} \right]$ and then summing over $i = 1, 2, ..., n$, we get

$$\frac{\sum_{i=1}^{n} u_i \left\{ f\left(\mu_A(x_i), \mu_{A'}(x_i)\right)\right\} D^{-l_i}}{\sum_{i=1}^{n} u_i} \le 1$$

which is the generalized fuzzy Kraft [7] inequality. The last inequality in (16) gives:

$$l_i < \log_D \left[\frac{\sum_{i=1}^{n} u_i \{ f(\mu_A(x_i), \mu_{A'}(x_i)) \}}{\sum_{i=1}^{n} u_i} \right] + 1$$

$$\Rightarrow D^{l_i} < \left[\frac{\sum_{i=1}^{n} u_i \{ f(\mu_A(x_i), \mu_{A'}(x_i)) \}}{\sum_{i=1}^{n} u_i} \right] D \tag{18}$$

For $0 < \alpha < 1$, raising power $\left(\frac{1-\alpha}{\alpha} \right)$ on both sides of (18), we get

$$D^{l_i \left(\frac{1-\alpha}{\alpha} \right)} < \left[\frac{\sum_{i=1}^{n} u_i \{ f(\mu_A(x_i), \mu_{A'}(x_i)) \}}{\sum_{i=1}^{n} u_i} \right]^{\left(\frac{1-\alpha}{\alpha} \right)} D^{\left(\frac{1-\alpha}{\alpha} \right)} \tag{19}$$

Multiplying both sides of (19) by $\frac{u_i f \{\mu_A(x_i), \mu_{A'}(x_i)\}}{\sum_{i=1}^{n} u_i}$ and summing over $i = 1, 2, ..., n$, we get:

$$\left[\frac{\sum_{i=1}^{n} u_i \{ f(\mu_A(x_i), \mu_{A'}(x_i)) \} D^{l_i \left(\frac{1-\alpha}{\alpha} \right)}}{\sum_{i=1}^{n} u_i} \right] < \left[\frac{\sum_{i=1}^{n} u_i \{ f(\mu_A(x_i), \mu_{A'}(x_i)) \}}{\sum_{i=1}^{n} u_i} \right]^{\left(\frac{1}{\alpha} \right)} D^{\left(\frac{1-\alpha}{\alpha} \right)} \tag{20}$$

Applying logarithms to the base '*D*' on both sides of (20), we get:

$$\log_D \left[\frac{\sum_{i=1}^{n} u_i \{ f(\mu_A(x_i), \mu_{A'}(x_i)) \} D^{l_i \left(\frac{1-\alpha}{\alpha} \right)}}{\sum_{i=1}^{n} u_i} \right] < \frac{1}{\alpha} \log_D \left[\frac{\sum_{i=1}^{n} u_i \{ f(\mu_A(x_i), \mu_{A'}(x_i)) \}}{\sum_{i=1}^{n} u_i} \right] + \frac{1-\alpha}{\alpha}$$

.

Since, $0 < \alpha < 1$ & $0 < \beta \leq 1$, after suitable operations, we get:

$$\frac{\alpha\beta}{1-\alpha}\log_D\left[\frac{\sum_{i=1}^{n}u_i\{f(\mu_A(x_i),\mu_{A'}(x_i))\}D^{l_i\left(\frac{1-\alpha}{\alpha}\right)}}{\sum_{i=1}^{n}u_i}\right]<\frac{\beta}{1-\alpha}\log_D\left[\frac{\sum_{i=1}^{n}u_i\{f(\mu_A(x_i),\mu_{A'}(x_i))\}}{\sum_{i=1}^{n}u_i}\right]+\beta\,\mathrm{Th}$$

e above can be written as

$$\frac{\alpha\beta}{1-\alpha}\log_D\left[\frac{\sum_{i=1}^{n}u_i\left\{\mu_A^{\beta(1-\alpha)}(x_i)+(1-\mu_A(x_i))^{\beta(1-\alpha)}\right\}D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}}{\sum_{i=1}^{n}u_i}\right]<$$

$$\frac{\beta}{1-\alpha}\log_D\left[\frac{\sum_{i=1}^{n}u_i\left\{\mu_A^{\beta(1-\alpha)}(x_i)+(1-\mu_A(x_i))^{\beta(1-\alpha)}\right\}}{\sum_{i=1}^{n}u_i}\right]+\beta$$

Thus, we can write $L_\alpha^\beta(A)<H_\alpha^\beta(A)+\beta;\ 0<\alpha<1, 0<\beta\le 1, \beta>\alpha\ \&u_i>0$

III. CONCLUSION

In this paper, we define a new generalized 'useful' fuzzy mean code word length i.e., $L_\alpha^\beta(A;U)$ corresponding to $H_\alpha^\beta(A;U)$ and also show that

$$H_\alpha^\beta(A;U)\le L_\alpha^\beta(A;U)<H_\alpha^\beta(A;U)+\beta;\quad 0<\alpha<1, 0<\beta\le 1, \beta>\alpha\ \&u_i>0.$$

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