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FORA TYPE EXTENSIONS OF SOME FIXED POINT THEOREMS

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ABSTRACT

Singal and Lal generalized Nadler's result. Lal and Gupta generalized Fora's result by the setting of Rhoades. In this paper we further generalize some results of Fora by larger class of spaces and for a larger class of mappings.

Keywords: Product Space, uniform space, continuous mapping, fixed point property, complete metric space. Locally contraction in the first variable.

I. INTRODUCTION

Let us recall the result of Fora [4]

1.1 Theorem

Let (X, d) be a complete metric space. Let Z be a topological space with the fixed point property (f. p. p) and let f be a continuous function from XxZ into XxZ. If f is locally contraction mapping in the first variable, then f has a fixed point.

In what follows X will denote a complete Hausdorff uniform space, Z a topological space which has the f.p.p. and f is a mapping from XxZ into XxZ. π_1 is the projection of XxZ on X along Z. (m) for $1 \leq m \leq 125$ will denote the condition (m) in Rhoades [7].

Following this we say that f ϵ (m) locally in the first variable means if for any z ϵ Z, there exists an open set V(z) of Z containing z and a real number $\lambda(z) \in [0,1)$ such that

$d(\pi_1 f(x_1, u), \pi_1 f(x_2, u)) \leq \lambda(z)\, d(x_1, x_2)$ for all $x_1, x_2 \in X$ and all $u \in V(z)$, the mapping $f_u: X \rightarrow X$ satisfies (m), where $f_u(x) = \pi_1 f(x, u)$. Here as earlier when f_u satisfies (m), the constants or the functions that appear in condition (m) depend upon z ϵ Z.

II. MAIN RESULTS

2.1 Theorem

Let (X,ɥ) be a complete Hausdorff uniform space, Z any topological space with the f. p. p. and $f: X\times Z \rightarrow X\times Z$ be a continuous mapping when X is assigned the topology generated by any pseudometric ρ which is uniformly continuous with respect to the uniformity ɥ. If f satisfies condition (23) in the first variable locally, then f has a fixed point.

Proof : Let $\{\rho_\alpha : \alpha \in I\}$ be the collection of all uniformly continuous pseudometrics on X.

Let $x_0 \epsilon X$ be fixed and for any $z \epsilon Z$, we construct a sequence $t_n(z) = t_n$ in X as follows:

$t_0 = x_0$, $t_n = \pi_1 f(t_{n-1}, z)$; $n \geq 1$

Step-I: $\{t_n\}$ is a Cauchy sequence in X

since $f \epsilon$ (23) locally in the first variable, i.e, for each $z \epsilon Z$, there exists an open set V(z) of Z containing z and monotonically decreasing functions $\alpha_i:(0,\infty)\to[0,1)$ (depend upon z) satisfying $\sum \alpha_i(t)< 1 (i=1,2,3,4,5)$ such that for each $x, x_* \epsilon X$, $x \neq x_*$ and all $u \epsilon V(z)$,if we put $x = t_{n-1}$, $x_* = t_n$, then we have

$$\rho_\alpha(\pi_1 f(x, u),\pi_1 f(x_*,u)) \leq \alpha_1.\rho_\alpha(x,(\pi_1 f(x, u)) + \alpha_2.\rho_\alpha(x_*,\pi_1 f(x_*,u)) + \alpha_3.\rho_\alpha(x,\pi_1 f(x_*,u))$$
$$+\alpha_4.\rho_\alpha (x_*,(\pi_1 f(x,u)) + \alpha_5.\rho_\alpha(x,x_*) \qquad (1)$$

Where, for brevity, we let $\alpha_i = \alpha_i(\rho_\alpha(x, x_*))$.

Above equation (1) implies (taking $x = t_{n-1}$, $x_* = t_n$)

$$\rho_\alpha(\pi_1 f(t_{n-1},z),\pi_1 f(t_n, z)) \leq \alpha_1.\rho_\alpha(t_{n-1},\pi_1 f(t_{n-1},z)) + \alpha_2.\rho_\alpha(t_n ,\pi_1 f(t_n, z)) + \alpha_3.\rho_\alpha(t_{n-1},\pi_1 f(t_n, z))$$
$$+\alpha_4.\rho_\alpha(t_n,\pi_1 f(t_{n-1},z)) + \alpha_5.\rho_\alpha(t_{n-1},t_n).$$

$$\text{or } \rho_\alpha(t_n,t_{n+1}) \leq \alpha_1.\rho_\alpha(t_{n-1},t_n) +\alpha_2.\rho_\alpha(t_n,t_{n+1})+\alpha_3.\rho_\alpha(t_{n-1},t_{n+1})+\alpha_5.\rho_\alpha(t_{n-1},t_n) \qquad (2)$$

Using symmetry in $f \epsilon$ (23) locally in the first variable we have

$$\rho_\alpha(t_{n+1},t_n) \leq \alpha_1.\rho_\alpha(t_n,t_{n+1})+ \alpha_2.\rho_\alpha(t_{n-1},t_n)+ \alpha_4.\rho_\alpha(t_{n-1},t_{n+1})+ \alpha_5.\rho_\alpha(t_n,t_{n-1}) \qquad (3)$$

Adding equation (2) and (3) we get

$$2 \rho_\alpha (t_n,t_{n+1}) \leq (\alpha_1+\alpha_2+2\alpha_5).\rho_\alpha(t_{n-1},t_n) + (\alpha_1+\alpha_2).\rho_\alpha(t_n,t_{n+1}) + (\alpha_3+\alpha_4).\rho_\alpha(t_{n-1},t_{n+1})$$
$$\leq(\alpha_1+\alpha_2+2\alpha_5).\rho_\alpha(t_{n-1},t_n)+(\alpha_1+\alpha_2).\rho_\alpha(t_n,t_{n+1})+(\alpha_3+\alpha_4).[\rho_\alpha(t_{n-1},t_n) +\rho_\alpha(t_n,t_{n+1})]$$
$$= (\alpha_1+ \alpha_2 + \alpha_3 + \alpha_4 + 2\alpha_5).\rho_\alpha(t_{n-1}, t_n) + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4).\rho_\alpha(t_n,t_{n+1})$$

$$\text{or } \rho_\alpha(t_n, t_{n+1}) \leq \left(\frac{\alpha_1+\alpha_2+\alpha_3+\alpha_4+2\alpha_5}{2-\alpha_1-\alpha_2-\alpha_3-\alpha_4}\right)\rho_\alpha(t_{n-1},t_n)$$

$$\leq \lambda_\alpha. \rho_\alpha(t_{n-1}, t_n) \qquad (4)$$

$$\text{where } \lambda_\alpha = \left(\frac{\alpha_1+\alpha_2+\alpha_3+\alpha_4+2\alpha_5}{2-\alpha_1-\alpha_2-\alpha_3-\alpha_4}\right) < 1.$$

By induction we obtain

$$\rho_\alpha (t_n,t_{n+1}) \leq \lambda_\alpha^{\,n}. \rho_\alpha (t_0 ,t_1)$$

Using the triangle inequality, we find for m>n,

$$\rho_\alpha (t_n, t_m) \leq \rho_\alpha (t_n, t_{n+1}) + \rho_\alpha (t_{n+1}, t_{n+2}) +\ldots.+ \rho_\alpha (t_{m-1}, t_m)$$
$$\leq (\lambda_\alpha^{\,n} + \lambda_\alpha^{\,n+1} +\ldots\ldots..+ \lambda_\alpha^{\,m-1}).\rho_\alpha(t_0, t_1)$$
$$= \frac{\lambda_\alpha^{\,n}(1-\lambda_\alpha^{\,m-n})}{(1-\lambda_\alpha)}\rho_\alpha(t_0,t_1)$$
$$< \frac{\lambda_\alpha^{\,n}}{(1-\lambda_\alpha)}\rho_\alpha(t_0,t_1)$$

Since $\lambda_\alpha^{\,n} \to 0$ as $n\to\infty$, then above inequality shows that $\{t_n\}$ is a ρ_α-Cauchy Sequence in X (i.e a Cauchy sequence in ρ_α– topology).

Let B ={S_p: p ϵ N} where S_p={t_n: n ≥ p} be a Cauchy filter base in (X, ῃ). To see this we first note that the family{H(α ,∈):α ϵ I,∈>0}is a base for ῃ as A^* (ῃ)={ρ_α:α ϵ I} an augmented associated family for ῃ on X. Now let H ϵ ῃ be an arbitrary entourage. Then there exist a υ ϵ I and ∈ > 0 such that H(υ , ∈) ⊂ H. Now since {t_n} is a ρ_α - Cauchy sequence in X, there exists a positive integer p such that ρ_υ (t_n, t_m) < ∈ for all m ≥ p, n ≥ p .This implies that S_p x S_p ⊂ H(υ , ∈).Thus given any H ϵ ῃ we can find a S_p ϵ B such that S_p x S_p ⊂ H. Hence B is a Cauchy filter in (X, ῃ). Since (X, ῃ) is complete and Hausdorff, the Cauchy filter B = {S_p} converges to a point say t_z in X.

Let mapping g: Z→Z defined as g(z) = π_2f(t_z , z) where π_2 is the projection of X x Z on Z along X.

Step II : g : Z→Z is continuous.

Let z ϵ Z and U be an open set containing g(z). Then f(t_z, z) ∈ X x U. Since f is continuous at (t_z , z)when X is assigned the topology τ(ρ) in which ρ ϵ A*(□) implies ρ = ρ_α for some α ϵ I , there exists an open set G ⊂ Z and a real number ∈ > 0 such that

$$(t_z , z) \in S(t_z ,\in, \rho) \text{ x } G \text{ and } f\{S(t_z ,\in, \rho)\}\text{x } G \subset X \text{ x } U$$

Since f ϵ (23) locally contraction in the first variable, therefore there exists an open set W in Z containing z and λϵ [0,1) such that

$$\rho(\pi_1 f(x , v) , \pi_1 f(x_*,v)) \le \lambda.\rho(x , x_*)$$

for all x, x_* ϵ X and all v ϵ W.

Since λ^m→0 as m→∞, we all choose n ≥ 1 such that

$$\lambda^n < \frac{\in}{8}\left(\frac{1-\lambda}{\rho(t_0,t_1)+(\in/8)}\right) \text{ and } \rho(t_z , t_m) < \frac{\in}{8} \text{ for all } m \ge n$$

Since f(t_n, z) ϵ X x U and f is continuous at (t_n , z) , there exists a basic open set U_n x V_n in X x Z such that

$$(t_n , z) \in U_n \text{ x } V_n \ , U_n \subset S(\frac{\in}{8}, t_z ,\rho) , V_n \subset G \cap W \text{ and } f(U_n \text{ x } V_n) \subset X \text{ x } U.$$

Since f is continuous at (t_{n-1}, z) and f(t_{n-1} , z) ϵ U_n x Z , there exists a basic open set U_{n-1} x V_{n-1} in X x Z such that

$$(t_{n-1}, z) \in U_{n-1} \text{ x}V_{n-1}, U_{n-1} \subset S(\frac{\in}{8}, t_{n-1} , \rho) , V_{n-1} \subset V_n \text{ and } f(U_{n-1} \text{ x}V_{n-1}) \subset U_n \text{ x } Z.$$

Continuing this way we construct sets U_n , U_{n-1} ,..., U_0 , V_n ,V_{n-1},----,V_0 such that, for 0≤ i ≤ (n-1)

$$(t_i , z) \in U_i \text{ x } V_i \ , U_i \subset S(\frac{\in}{8}, t_i , \rho) , V_i \subset V_{i+1} \text{ and } f (U_i \text{ x}V_i) \subset U_{i+1} \text{ x } Z.$$

It remains to show that g(V_0) ⊂ U :

Let y ϵ V_0. Then from the above mention properties we have (t'_0, y) ϵ U_0 x V_0,

Where t'_0 = xo. Thus f(t'_0, y) ϵ U_1 x Z ie. , t'_1 = π_1f(t'_0, y) ϵ U_1, consequently $\rho(t'_1, t_1) < \frac{\in}{8}$.

Using the triangular inequality we have

$\rho(t'_0, t'_1) = \rho(t_0, t_1') \le \rho(t_0, t_1) + \rho(t_1, t'_1) < \rho(t_0, t_1) + \frac{\in}{8}$.

Since $f(U_1 x V_1) \subset U_2 x Z$ and $(t'_1, y) \in U_1 x V_1$ therefore $f(t'_1, y) \in U_2 x Z$

ie $t'_2 = \pi_1 f(t'_1, y) \in U_2$.

In this way we find the sequence $t'_n(y) = t'_n$, for which $t'_i = \pi_1 f(t'_{i-1}, y) \in Ui$; i =1,2,..., n.

Moreover, $t'_n \in U_n$ and $U_n \subset S(\frac{\in}{8}, t_z, \rho)$, therefore $\rho(t'_n, t_z) < \frac{\in}{8}$.

Using the triangle inequality we find, for $m \ge n$.

$\rho(t'_n, t_z) \le \rho(t_z, t'_n) + \rho(t'_n, t'_{n+1}) + \ldots\ldots\ldots + \rho(t'_{m-1}, t'_m)$

$$< \frac{\in}{8} + \lambda^n \cdot \rho(t'_0, t'_1) + \text{----} + \lambda^{m-1} \rho(t'_0, t'_1)$$

$$= \frac{\lambda^n}{1-\lambda} \cdot \rho(t'_0, t'_1) + \frac{\in}{8}$$

$$< \left(\frac{\lambda^n}{1-\lambda}\right).[\rho(t'_0, t'_1) + \frac{\in}{8}] + \frac{\in}{8}$$

$$< \frac{\in}{8} + \frac{\in}{8} = \frac{\in}{4}$$

If $t_y = \lim t'_n$, then the above inequality shows that $\rho(t_y, t_z) \le \in/4$. Therefore $(t_y, y) \in S(\in, t_z, \rho)$ and consequently $f(t_y, y) \in X x U$, ie. , $g(y) = \pi_1 f(t_y, y) \in U$. Therefore, our claim is proved and hence g is continuous.

Step –III : $\pi_1 f(t_z, z) = t_z$

If possible , let $u = \pi_1 f(t_z, z) \ne t_z$. Since the uniform space X is Hausdorff, there exists a pseudometric ρ on X such that $\rho(u, t_z) = \in > 0$.

Since f is continuous on X x Z and X is assigned the topology $\tau(\rho)$, we have open sets U and V such that

$(t_z, z) \in U x V$, $U \subset S(\frac{\in}{4}, t_z, \rho)$ and $f(U x V) \subset S(\frac{\in}{4}, u, \rho) x Z$.

Since $\lim t_n = t_z$, there is a natural number $k \ge 1$ such that $t_n \in U$ for all $n \ge k$. Therefore

$f(t_k, z) \in S(\frac{\in}{4}, u, \rho) x Z$, ie $t_{k+1} = \pi_1 f(t_k, z) \in S(\frac{\in}{4}, u, \rho)$.

Also $t_{k+1} \in U \subset S(\frac{\in}{4}, t_z, \rho)$.This contradicts the fact that $\rho(t_z, u) = \in$. Therefore our assumption is false and consequently we have the required conclusion.

Now as in step II of the theorem 2.1, g: Z→Z is continuous mapping. Since *Z* has the fixed point property, therefore there exist $z_0 \in Z$ such that $g(z_0) = z_0$. As in step-III above we have $\pi_1 f(t_{z0}, z_0) = t_{z0}$. But $z_0 = g(zo) = \pi_2 f(t_{z0}, z_0)$.

Hence $f(t_{z0}, z_0) = (t_{z0}, z_0)$ ie, (t_{z0}, z_0) is a fixed point of f. This completes the proof.

Now we formulate following results of [9]:

2.2 Theorem

Let (X, ɥ)) be a complete Hausdorff uniform space, Z any topological space with the f.p.p. and $f : X\times Z \to X\times Z$ be a continuous mapping when X is assigned the topology generated by any pseudometric ρ which is uniformly continuous with respect to the uniformity ɥ. If f satisfies condition (22) locally in the first variable and for some $x_0 \in X$, for each $z \in Z$ the sequence $t_n(z) = t_n = \pi_1 f(t_{n-1}, z)$; $(n \geq 1)$, has a cluster point, then f has a fixed point.

2.3 Theorem

Let (X, ɥ) be a complete Hausdorff uniform space, Z any topological space with the f.p.p. and $f : X\times Z \to X\times Z$ be a continuous mapping when X is assigned the topology generated by pseudometric ρ which is uniformly continuous with respect to the uniformity ɥ. If f satisfies condition (24) locally in the first variable, then f has a fixed point.

Proofs are by the help of theorem 2.1 and [9].

III. CONCLUSION

(i) Rhoades [7] proved that conditions (1), (2), (4), (7), (8), (11), (15) and (18) are all stronger than (23), therefore the above theorem 2.1 has eight corollaries corresponding to these eight conditions

(ii) In view of Theorem 1, of Rhoades [7], the Theorem 2.2 above has as many as eleven corollaries corresponding to conditions (1), (4), (5), (6), (7), (9), (11), (18), (19), (20), and (21). The corollaries corresponding to conditions (1), (4), (7), (11), and (18) are also corollaries to theorem 2.1, which are mentioned above, therefore remaining corollaries are corresponding to (5), (6), (9), (19), (20), and (21).

(iii) In view of Theorem 1, of Rhoades [7], our theorem 2.3 above has twelve corollaries corresponding to conditions (1), (4), (5), (7), (9), (11), (12), (14), (16) (18), (19), (20), and (21). Out of these only three are new corollaries corresponding to conditions (12), (14), and (16).

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