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Number of Zeros of a Polynomial in a Disk

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ABSTRACT

In this paper we discuss the number of zeros of a complex algebraic polynomial of degree n with restricted coefficients in a disk centered at origin.

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I. INTRODUCTION AND STATEMENT OF RESULTS

One of the basic theorems of mathematics is the Fundamental Theorem of Algebra, according to which, *"every polynomial of degree $n \geq 1$ has exactly n zeros in the complex plane"*. This theorem does not however say anything regarding the location of zeros of a polynomial. The problem of locating some or all the zeros of a given polynomial as a function of its coefficients is of long standing interest in mathematics. This fact can be deduced by glancing at the references in the comprehensive books of Marden [8] and Milovanovic, Mitrinovic and Rassias [9] and by noting the abundance of recent publications on the subject.

Historically speaking, the subject dates from about the time when the geometric representation of the complex numbers was introduced into mathematics, and the first contributors to the subject were Gauss and Cauchy. Cauchy [3] improved the result of Gauss and proved:

Theorem A: *If $p(z) = \sum_{j=0}^{n} a_j z^j, a_n \neq 0$, is a polynomial of degree n with complex coefficients, then all the zeros of $p(z)$ lie in the circle,*

$$|z| \leq 1 + \max_{0 \leq j \leq n-1} \left|\frac{a_j}{a_n}\right|.$$

Other results of similar type were obtained among others by Aziz [1], Q.G. Mohammad [10] etc. Now we mention the following elegant result which is commonly known as Eneström-Kakeya Theorem in the theory of distribution of zeros of polynomials.

Theorem B: *If* $p(z) = \sum_{j=0}^{n} a_j z^j, a_n \neq 0$, *is a polynomial of degree* n *with real coefficients satisfying* $a_n \geq a_{n-1} \geq a_{n-2} \geq \cdots \geq a_0 > 0$, *then all the zeros of* $p(z)$ *lie in the circle* $|z| \leq 1$.

Theorem B was proved by Eneström [4] and independently by Kakeya [7].

By using Schwartz lemma, Aziz and Mohammad [2] generalized Eneström-Kakeya theorem and proved the following:

Theorem C: *Let* $f(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *with positive and real coefficients. If* $t_1 > t_2 \geq 0$ *can be found such that*

$$a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2} \geq 0, r = 1,2, \ldots, n+1, a_{-1} = a_{n+1} = 0$$

Then all the zeros of $p(z)$ *lie in* $|z| \leq t_1$.

Regarding the number of zeros in $|z| \leq \frac{1}{2}$ of the polynomial $p(z) = \sum_{j=0}^{n} a_j z^j$, Mohammed [10] proved the following:

Theorem D: *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *such that*

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0,$$

then the number of zeros of $p(z)$ in $|z| \leq {}^{1}/_{2}$ does not exceed

$$1 + \frac{1}{log2} log \frac{a_n}{a_0}.$$

In this paper, we relax the restriction on the coefficients of polynomial and prove the more general result from which the other results follows by fairly uniform procedure.

As a generalization of Theorem D, we prove the following result.

Theorem 1: *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n with real coefficients . If for some* $t > 0$,

$$t^n a_n \geq t^{n-1} a_{n-1} \geq \cdots \geq t \ \ a_1 \geq a_0 > 0,$$

then the number of zeros of $p(z)$ in $|z| < t/2$ does not exceed

$$1+\frac{1}{log2}log\frac{a_n t^n}{a_0}.$$

Remark 1. For $t=1$, Theorem 1 reduces to Theorem D.

Theorem 2: *Let $p(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n with real coefficients. If $a_j>0, j=0,1,2,\dots,n$ and for some $t_1>t_2\geq 0$*

$$a_jt_1t_2+a_{j-1}(t_1-t_2)-a_{j-2}\geq 0, j=1,2,\dots,n$$

with $a_{-1}=0$, then the number of zeros of $p(z)$ in $|z|<t_1/2$ does not exceed

$$\frac{1}{log2}log\frac{2a_nt_1^{n+2}}{a_0t_1t_2}.$$

Remark 2: For $t_1=1$ and $t_2=0$, Theorem 2 reduces to Theorem D.

Finally, we prove the following:

Theorem 3: *Let $p(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n, If $a_j>0,\ j=0,1,2,\dots,n$ and for some $t_1>t_2\geq 0$*

$$a_jt_1t_2+a_{j-1}(t_1-t_2)-a_{j-2}\geq 0, \quad j=1,2,\dots,k$$

and

$$a_jt_1t_2+a_{j-1}(t_1-t_2)-a_{j-2}\leq 0, \quad j=k+1,K+2,\dots,n$$

With $a_{-1}=0$, then the number of zeros of $p(z)$ in $|z|<t_1/2$ does not exceed

$$1+\frac{1}{\log 2}log\frac{M_1}{a_0t_2},$$

Where

$$M_1=a_kt_1^kt_2+a_{k-1}t_1^k+a_nt_1^{n+1}-a_nt_1^nt_2-a_{n-1}t_1^n.$$

Lemma

For the proof of some of these results we need the following lemma (see page 171 of second edition) [11].

Lemma 1: Let $f(z)$ be regular and $|f(z)| \le M$, in the circle $|z| \le R$ and suppose that $f(0) \neq 0$, then the number of zeros of $f(z)$ in the circle $|z| \le \frac{1}{2}R$ does not exceed

$$\frac{1}{\log 2}\log\frac{M}{|f(0)|}.$$

Proof of the Theorems:

Proof of Theorem 1. Consider the polynomial

$$F(z) = (t-z)p(z)$$

$$= (t-z)(a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} + a_n z^n)$$

$$= -a_n z^{n+1} + ta_0 + \varphi(z)$$

(3.1)

where

$$\varphi(z) = \sum_{j=1}^{n}(ta_j - a_{j-1})z^j.$$

Therefore for $|z| < t$, we have

$$|\varphi(z)| \le \sum_{j=1}^{n}|ta_j - a_{j-1}|t^i$$

$$= |ta_1 - a_0|t + |ta_2 - a_1|t^2 + \cdots + |ta_n - a_{n-1}|t^n$$

$$= t^{n+1}a_n - ta_0$$

Since $\varphi(0) = 0$, therefore by Schwartz Lemma,

$|\varphi(z)| \le (t^n a_n - a_0)|z|$ for $|z| < t$.

Now from (3.1), we have for $|z| < t$,

$$|F(z)| \le |a_n|t^{n+1} + t|a_0| + |\varphi(z)|$$

$$\le a_n t^{n+1} + ta_0 + t^{n+1}a_n - ta_0$$

$$= 2a_n t^{n+1}$$

Since $|F(z)| \leq 2a_n t^{n+1}$ for $|z| \leq t$ and $F(0) = a_0 t \neq 0$, therefore by Lemma 1 it follows that the number of zeros of $F(z)$ (hence of p) in $|z| \leq t/2$ does not exceed

$$\frac{1}{\log 2} \log \frac{2a_n t^{n+1}}{a_0 t}$$

Proof of Theorem 2: Consider the polynomial

$$F(z) = (t_2 + z)(t_1 - z)p(z)$$

$$= \{t_1 t_2 + (t_1 - t_2)z - z^2\}(a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n)$$

$$= -a_n z^{n+2} + \{a_n(t_1 - t_2) - a_{n-1}\}z^{n+1} + a_0 t_1 t_2 + \varphi(z) \quad \ldots\ldots$$

(3.2)

where

$$\varphi(z) = \sum_{j=1}^{n} (t_1 t_2 a_j + (t_1 - t_2)a_{j-1} - a_{j-2})z^j.$$

Since $a_j t_1 t_2 + a_{j-1}(t_1 - t_2) - a_{j-2} \geq 0, j = 1,2, \ldots, n$, therefore for $|z| < t_1$, we have

$$|\varphi(z)| \leq \sum_{j=1}^{n} |t_2 a_j + (t_1 - t_2)a_{j-1} - a_{j-2}| t_1^j.$$

$$= |a_1 t_1 t_2 + (t_1 - t_2)a_0| t_1 + |a_2 t_1 t_2 + (t_1 - t_2)a_1 - a_0| t_1^2 + \cdots$$

$$+ |a_n t_1 t_2 + (t_1 - t_2)a_{n-1} - a_{n-2}| t_1^n$$

$$= (a_1 t_1 t_2 + (t_1 - t_2)a_0)\, t_1 + (a_2 t_1 t_2 + (t_1 - t_2)a_1 - a_0)\, t_1^2 + \cdots$$

$$+ (a_n t_1 t_2 + (t_1 - t_2)a_{n-1} - a_{n-2})\, t_1^n$$

$$= t_1^{n+1}(a_n t_2 + a_{n-1}) - a_0 t_1 t_2.$$

Since $\varphi(0) = 0$, therefore by Schwartz Lemma

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$|\varphi(z)| \le (t_1^n a_n t_2 + t_1^n a_{n-1} - a_0 t_2)|z| \quad for \quad |z| < t_1,$

Now from (3.2), we have for $|z| < t_1$,

$$|F(z)| \le |a_n z^{n+2}| + |\{a_n(t_1 - t_2) - a_{n-1}\}z^{n+1}| + |a_0 t_1 t_2| + |\varphi(z)|$$

$$= a_n t_1^{n+2} + \{a_n(t_1 - t_2) - a_{n-1}\}t_1^{n+1} + a_0 t_1 t_2 + t_1^{n+1}(a_n t_2 + a_{n-1}) - a_0 t_1 t_2$$

$$= 2a_n t_1^{n+2}.$$

Since $|F(z)| \le 2a_n t_1^{n+2}$ for $|z| < t_1$ and $F(0) = a_0 t_1 t_2$, therefore by Lemma 1 it follows that the number of zeros of $F(z)$ (hence of p) in $|z| \le \frac{t_1}{2}$ does not exceed

$$\frac{1}{\log 2} log \frac{2a_n t_1^{n+2}}{a_0 t_1 t_2}.$$

Which proves Theorem 2.

Proof of Theorem 3: Consider the polynomial

$$F(z) = (t_2 + z)(t_1 - z)p(z)$$

$$= \{t_1 t_2 + (t_1 - t_2)z - z^2\}\{a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n\}$$

$$= -a_n z^{n+2} + \{a_n(t_1 - t_2) - a_{n-1}\}z^{n+1} + a_0 t_1 t_2 + \varphi(z) \qquad \ldots. (3.3)$$

where

$$\varphi(z) = \sum_{j=1}^{n} (t_1 t_2 a_j + (t_1 - t_2)a_{j-1} - a_{j-2})z^j.$$

Since $a_j t_1 t_2 + a_{j-1}(t_1 - t_2) - a_{j-2} \ge 0, j = 1,2,\ldots,k,$ and $a_j t_1 t_2 + a_{j-1}(t_1 - t_2) - a_{j-2} \le 0$ for $j = k+1, k+2, \ldots, n,$ therefore for $|z| < t_1$, we have

$$|\varphi(z)| \leq \sum_{j=1}^{k} |t_1t_2a_j + (t_1 - t_2)a_{j-1} - a_{j-2}|t_1^j + \sum_{j=k+1}^{n} |t_1t_2a_j + (t_1 - t_2)a_{j-1} - a_{j-2}|t_1^j.$$

$$= |a_1t_1t_2 + (t_1 - t_2)a_0|t_1 + |a_2t_1t_2 + (t_1 - t_2)a_1 - a_0|t_1^2 + \cdots$$

$$+|a_kt_1t_2 + (t_1 - t_2)a_{k-1} - a_{k-2}|t_1^k + |a_{k+1}t_1t_2 + (t_1 - t_2)a_k - a_{k-1}|t_1^{k+1} + \cdots$$

$$+|a_nt_1t_2 + (t_1 - t_2)a_{n-1} - a_{n-2}|t_1^n$$

$$=(a_1t_1t_2 + (t_1 - t_2)a_0)\ t_1 + (a_2t_1t_2 + (t_1 - t_2)a_1 - a_0)\ t_1^2 + \cdots$$

$$+(a_kt_1t_2 + (t_1 - t_2)a_{k-1} - a_{k-2})\ t_1^k - (a_{k+1}t_1t_2 + (t_1 - t_2)a_k - a_{k-1})t_1^{k+1} -$$

$$\ldots - (a_nt_1t_2 + (t_1 - t_2)a_{n-1} - a_{n-2})\ t_1^n$$

$$= -a_0t_1t_2 + 2a_kt_1^{k+1}t_2 + 2a_{k-1}t_1^{k+1} - a_nt_1^{n+1}t_2 - a_{n-1}t_1^{n+1}.$$

Since $\varphi(0) = 0$, therefore by Schwartz Lemma

$$|\ \varphi(z)| \leq \left|-a_0t_2 + 2a_kt_1^kt_2 + 2a_{k-1}t_1^k - a_nt_1^nt_2 - a_{n-1}t_1^n\right||z| \ \ for \ \ |z| < t_1.$$

Hence from (3.3), we have for $|z| < t_1$,

$$|F(z)| \leq |a_n|t_1^{n+2} + |a_n(t_1 - t_2) - a_{n-1}|t_1^{n+1} + |a_0t_1t_2| + |\varphi(z)|$$

$$= 2a_kt_1^{k+1}t_2 + 2a_{k-1}t_1^{k+1} + 2a_nt_1^{n+2} - 2a_nt_1^{n+1}t_2 - 2a_{n-1}t_1^{n+1} = M(say).$$

Since $|F(z)| \leq M$ for $|z| < t_1$ and $F(0) = a_0t_1t_2 \neq 0$, therefore by Lemma 1, it follows that the number of zeros of $F(z)$(hence of p) in $|z| \leq \frac{t_1}{2}$ does not exceed

$$\frac{1}{\log 2} log \frac{M}{a_0t_1t_2},$$

This completes proof of Theorem 3.

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