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# A study of almost tangential intersection curves of four parametric hypersurfaces in $\mathbb{R}^5$

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**Abstract**

In this paper, we present the algorithms for calculating the differential geometric properties of the non-transversal intersection of almost tangential type of four parametric hypersurfaces in $\mathbb{R}^5$. In non-transversal intersection the normals of the surfaces at the intersection point are linearly dependent, while as in nontransversal intersection the normals of the surfaces at the intersection point are linearly independent.

## 1. Introduction

The surface-surface intersection problem is a fundamental process needed in modeling shapes in CAD/CAM system. It is useful in designing of complex objects and animations. The two types of surfaces mostly used in geometric designing are parametric and implicit surfaces. For that reason, different methods have been given for either parametric-parametric or implicit-implicit surface intersection curves in $\mathbb{R}^3$. Differential geometry of a parametric curve in $\mathbb{R}^3$ can be found in textbooks such as Struik [19], Willmore [20], whereas differential geometry of parametric curves in $\mathbb{R}^n$ can be found in the textbook such as in klingenberg [22] and in the contemporary literature on Geometric Modeling [2]. On the other hand, for the differential geometry of intersection curves, there exists a little literature. Willmore [20] and Aléssio [13] presented algorithms to obtain the unit tangent, unit principal normal, unit binormal, curvature and torsion of the transversal intersection curve of two implicit surfaces. Ye and Maekawa [24] presented algorithms for computing the differential geometric properties of both transversal and tangential intersection curves of two surfaces. Aléssio [11] formulated the algorithms for obtaining the geometric properties of intersection curves of three implicit hypersurfaces in $\mathbb{R}^4$. Based on the work of Aléssio [11], Mustufa Düldül [7] worked with three parametric hypersurfaces in $\mathbb{R}^4$ to derive the algorithms for differential geometric properties of transversal intersection. To obtain the first geodesic curvature ($\kappa_{1g}^{S_i}$) and the first geodesic torsion ($\tau_{1g}^{S_i}$) for the transversal intersection curve of 4 parametric hypersurfaces in $\mathbb{R}^5$, we need to derive the Darboux frame $\left\{U_1^{M_i},\cdots,U_5^{M_i}\right\}$. The Darboux frame is obtained by using the Gram-Schmidt orthogonalization process.

S. Yilmaz and M. Turgut [17] obtained the Frenet apparatus of a curve in Euclidean curve in Euclidean space $\mathbb{R}^5$. M. Turgut, et al.[8] derived the Frenet apparatus of non-null curves in Lorentzian space $\mathbb{L}^5$ and in [18] S. Yilmaz and M. Turgut derived the same for Partially Null and Pseudo Null Curves in Minkowski Space-time. Looking at their importance in physics, curves in Euclidean space in $\mathbb{R}^5$ and $\mathbb{L}^5$ are greater importance [1, 14, 23]. In this connection we have not seen any paper on discussion of intersection curves of parametric hypersurfaces in Euclidean space $\mathbb{R}^5$. Thus in this paper, we extend these to obtain the Frenet frame $\{t,n,b_1,b_2,b_3\}$ and curvatures $\{\kappa_1,\kappa_2,\kappa_3,\kappa_4\}$ of almost tangential intersection curve of four parametric hypersurfaces in $\mathbb{R}^5$.

## 2. Preliminaries

**Definition 2.1.** Let $\{e_1, e_2, e_3, e_4, e_5\}$ be the standard basis of five dimensional Euclidean space $E^5$. Then the vector product of the vectors $x = \sum_{i=1}^{5} x_i e_i$, $y = \sum_{i=1}^{5} y_i e_i$, $z = \sum_{i=1}^{5} z_i e_i$ and $w = \sum_{i=1}^{5} w_i e_i$ is defined by

$$x \otimes y \otimes z \otimes w = \begin{vmatrix} e_1 & e_2 & e_3 & e_4 & e_5 \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ y_1 & y_2 & y_3 & y_4 & y_5 \\ z_1 & z_2 & z_3 & z_4 & z_5 \\ w_1 & w_2 & w_3 & w_4 & w_5 \end{vmatrix}. \tag{1}$$

The vector product $x \otimes y \otimes z \otimes w$ yields a vector that is orthogonal to $x$, $y$, $z$, $w$.

let $R \subset E^5$ be a regular hypersurface given by $\Phi = \Phi(u_1, u_2, u_3, u_4)$ and $\gamma : I \subset \mathbb{R} \to \Phi$ be an arbitrary curve with arc length parametrisation. If $\{t, n, b_1, b_2, b_3\}$ is the Frenet Frame along $\gamma$, then we have

$$\begin{cases} t' = \kappa_1 n, \\ n' = -\kappa_1 t + \kappa_2 b_1, \\ b_1' = -\kappa_2 n + \kappa_3 b_2, \\ b_2' = -\kappa_3 b_1 + \kappa_4 b_3, \\ b_3' = -\kappa_4 b_2, \end{cases} \tag{2}$$

where $t$, $n$, $b_1$, $b_2$ and $b_3$ denote the tangent, the principal normal, the first binormal, the second binormal and third binormal vector fields. The normal vector $n$ is the normalised acceleration vector $\gamma''$. The unit vector $b_1$ is determined such that $n'$ can be decomposed into two components, a tangent one in the direction of $t$ and a normal one in the direction of $b_1$. The unit vector $b_2$ is determined such that $b_1'$ can be decomposed into two components - a normal and another in the direction of $b_2$. The unit vector $b_3$ is the unique unit vector field perpendicular to four dimensional subspace $\{t, n, b_1, b_2\}$. The functions $\kappa_1$, $\kappa_2$, $\kappa_3$ and $\kappa_4$ are the first, second, third and fourth curvatures of $\gamma(s)$. The first, second, third and fourth curvatures measure how rapidly the curve pulls away in a neighbourhood of $s$, from the tangent line, from planar curve, from three dimensional curve and from the four dimensional curve at $s$, respectively.
Now, using the Frenet Frame we have the derivatives of $\gamma$ as

$$\begin{aligned} \gamma' &= t, \qquad \gamma'' = t' = \kappa_1 n, \qquad \gamma''' = -\kappa_1^2 t + \kappa_1' n + \kappa_1 \kappa_2 b_1, && (3) \\ \gamma^{(4)} &= -3\kappa_1 \kappa_1' t + (-\kappa_1^3 + \kappa_1'' - \kappa_1 \kappa_2^2) n + (2\kappa_1' \kappa_2 + \kappa_1 \kappa_2') b_1 + \kappa_1 \kappa_2 \kappa_3 b_2, && (4) \\ \gamma^{(5)} &= (-3(\kappa_1')^2 - 4\kappa_1 k_1'' + \kappa_1^4 + \kappa_1^2 \kappa_2^2) t + (-6\kappa_1^2 \kappa_1' + \kappa_1''' - \kappa_1' \kappa_2^2 \\ &\quad -3\kappa_1 \kappa_2 \kappa_2' - 2\kappa_1' \kappa_{22}^2) n + (\kappa_1^3 \kappa_2 - \kappa_1 \kappa_2^3 + 3\kappa_1'' \kappa_2 + 3\kappa_1' \kappa_2' + \kappa_1 \kappa_2'' - \kappa_1 \kappa_2 \kappa_3^2) b_1 \\ &\quad (3\kappa_1' \kappa_2 \kappa_3 + 2\kappa_1 \kappa_2' \kappa_3 + 2\kappa_1 \kappa_2' \kappa_3 + \kappa_1 \kappa_2 \kappa_3') b_2 + \kappa_1 \kappa_2 \kappa_3 \kappa_4 b_3. && (5) \end{aligned}$$

Also since $\Phi$ is regular, the partial derivatives $\Phi_1$, $\Phi_2$, $\Phi_3$, $\Phi_4$, where $(\Phi_i = \frac{\partial \Phi}{\partial u_i})$ are linearly independent at every point of $\Phi$ , i.e., $\Phi_1 \otimes \Phi_2 \otimes \Phi_3 \otimes \Phi_4 \neq 0$. Thus, the unit normal vector of $\Phi$ is given by

$$N = \frac{\Phi_1 \otimes \Phi_2 \otimes \Phi_3 \otimes \Phi_4}{\|\Phi_1 \otimes \Phi_2 \otimes \Phi_3 \otimes \Phi_4\|}.$$

Furthermore, the first, second, and the third binormal vectors of the curve are given by

$$b_3 = \frac{\gamma' \otimes \gamma'' \otimes \gamma''' \otimes \gamma^{(4)}}{\|\gamma' \otimes \gamma'' \otimes \gamma''' \otimes \gamma^{(4)}\|}, \qquad b_2 = \frac{b_3 \otimes \gamma' \otimes \gamma'' \otimes \gamma'''}{\|b_3 \otimes \gamma' \otimes \gamma'' \otimes \gamma'''\|}, \qquad b_1 = \frac{b_2 \otimes b_3 \otimes \gamma' \otimes \gamma''}{\|b_2 \otimes b_3 \otimes \gamma' \otimes \gamma''\|} \tag{6}$$

and the curvatures are obtained by

$$\kappa_1 = \|\gamma''\|, \quad \kappa_2 = \frac{\langle \gamma''', b_1 \rangle}{\kappa_1}, \quad \kappa_3 = \frac{\langle \gamma^{(4)}, b_2 \rangle}{\kappa_1 \kappa_2}, \quad \kappa_4 = \frac{\langle \gamma^{(5)}, b_3 \rangle}{\kappa_1 \kappa_2 \kappa_3}. \tag{7}$$

On the other hand, since the curve $\gamma(s)$ lies on $\Phi$, we may write

$$\gamma(s) = \Phi(u_1(s), u_2(s), u_3(s), u_4(s)).$$

Then, we have

$$\gamma'(s)=\sum_{i=1}^{4}\Phi_i u_i', \tag{8}$$

$$\gamma''(s)=\sum_{i=1}^{4}\Phi_i u_i''+\sum_{i,j=1}^{4}\Phi_{ij}u_i'u_j', \tag{9}$$

$$\gamma'''(s)=\sum_{i=1}^{4}\Phi_i u_i'''+3\sum_{i,j=1}^{4}\Phi_{ij}u_i''u_j'+\sum_{i,j,k=1}^{4}\Phi_{ijk}u_i'u_j'u_k', \tag{10}$$

$$\gamma^{(4)}(s)=\sum_{i=1}^{4}\Phi_i u_i^{(4)}+4\sum_{i,j=1}^{4}\Phi_{ij}u_i'''u_j'+3\sum_{i,j=1}^{4}\Phi_{ij}u_i''u_j''+6\sum_{i,j,k=1}^{4}\Phi_{ijk}u_i''u_j'u_k'+\sum_{i,j,k,l=1}^{4}\Phi_{ijkl}u_i'u_j'u_k'u_l' \tag{11}$$

$$\begin{aligned}\gamma^{(5)} &= \sum_{i=1}^{4}\Phi_i u_i^{(5)}+5\sum_{i,j=1}^{4}\Phi_{ij}u_i^{(4)}u_j'+10\sum_{i,j=1}^{4}\Phi_{ij}u_i'''u_j''+10\sum_{i,j,k=1}^{4}\Phi_{ijk}u_i'''u_j'u_k'+15\sum_{i,j,k=1}^{4}\Phi_{ijk}u_i''u_j''u_k'\\&+10\sum_{i,j,k,l=1}^{4}\Phi_{ijkj}u_i''u_j'u_k'u_l'+\sum_{i,j,k,l,m=1}^{4}\Phi_{ijklm}u_i'u_j'u_k'u_l'u_m'.\end{aligned}$$

$$\begin{aligned}\gamma^{(6)} &= \sum_{i=1}^{4}\Phi_i u_i^{(6)}+6\sum_{i,j=1}^{4}\Phi_{ij}u_i^{(5)}u_j'+15\sum_{i,j=1}^{4}\Phi_{ij}u_i^{(4)}u_j''+15\sum_{i,j,k=1}^{4}\Phi_{ijk}u_i^{(4)}u_j'u_k'+10\sum_{i,j=1}^{4}\Phi_{ij}u_i'''u_j'''+60\sum_{i,j,k=1}^{4}\Phi_{ijk}u_i'''u_j''u_k'\\&+20\sum_{i,j,k,l=1}^{4}\Phi_{ijkl}u_i'''u_j'u_k'u_l'+15\sum_{i,j,k=1}^{4}\Phi_{ijk}u_i''u_j''u_k''+45\sum_{i,j,k,l=1}^{4}\Phi_{ijkl}u_i''u_j''u_k'u_l'+15\sum_{i,j,k,l,m=1}^{4}\Phi_{ijklm}u_i''u_j'u_k'u_l'u_m'\\&+\sum_{i,j,k,l,m,n=1}^{4}\Phi_{ijklmn}u_i'u_j'u_k'u_l'u_m'u_n'.\end{aligned} \tag{12}$$

**Definition 2.2.** Let $\Phi^1$, $\Phi^2$, $\Phi^3$ and $\Phi^4$ be the regular hypersurfaces, respectively. Then the unit normal of these hyprsurfaces is obtained by

$$N_i=\frac{\Phi_1^i\otimes\Phi_2^i\otimes\Phi_3^i\otimes\Phi_4^i}{\|\Phi_1^i\otimes\Phi_2^i\otimes\Phi_3^i\otimes\Phi_4^i\|},\qquad i=1,2,3,4.$$

Assuming that the intersection of these hypersurfaces is a smooth curve $\gamma(s)$ with arc length parametrisation $s$. Let $\gamma(s_0)=p$. Now, if $N_1,N_2,N_3,N_4$ are linearly dependent we non-transversal intersection at $p$ with the following subcases:

(1) Almost tangential intersection

$$N_4=aN_1+bN_2+cN_3,\ a,b,c\in\mathbb{R}. \tag{13}$$

(2) Tangential intersection

$$N_1=N_2=N_3=N_4. \tag{14}$$

## 3. Almost tangential intersection of four parametric hypersurfaces

Since the normal vectors are not linearly independent, the calculation of geometric properties are not straightforward. In the following theorem, we find the algorithm for tangential intersection of four parametric hypersurfaces.

### 3.1. *Tangential direction in case of almost tangential intersection.*

In the following theorem, we derive the algorithm for obtaining the tangential direction in case of of almost tangential intersection of four parametric hypersurfaces.

**Theorem 3.1.** *Let $\Phi^1,\Phi^2,\Phi^3$ and $\Phi^4$ be the four parametric hypersurfaces with almost tangential, then unit tangent is given by*

$$t=\frac{\mu(\Phi_1^1+\Phi_4^1\gamma_1+\xi_1\Phi_3^1+\xi_1\gamma_3\Phi_4^1)+(\Phi_{12}^1+\Phi_4^1\gamma_2+\xi_3\Phi_3^1+\xi_2\gamma_3\Phi_4^1)}{\|\mu(\Phi_1^1+\Phi_4^1\gamma_1+\xi_1\Phi_3^1+\xi_1\gamma_3\Phi_4^1)+(\Phi_{12}^1+\Phi_4^1\gamma_2+\xi_3\Phi_3^1+\xi_2\gamma_3\Phi_4^1)\|}.$$

*or,*

$$t=\frac{(\Phi_1^1+\Phi_4^1\gamma_1+\xi_1\Phi_3^1+\xi_1\gamma_3\Phi_4^1)+\upsilon(\Phi_{12}^1+\Phi_4^1\gamma_2+\xi_3\Phi_3^1+\xi_2\gamma_3\Phi_4^1)}{\|(\Phi_1^1+\Phi_4^1\gamma_1+\xi_1\Phi_3^1+\xi_1\gamma_3\Phi_4^1)+\upsilon(\Phi_{12}^1+\Phi_4^1\gamma_2+\xi_3\Phi_3^1+\xi_2\gamma_3\Phi_4^1)\|}.$$

*Proof.* For the unit tangent, we have

$$t=\gamma'(s)=\sum_{i=1}^{4}\Phi_i^1u_i'=\sum_{i=1}^{4}\Phi_i^2v_i'=\sum_{i=1}^{4}\Phi_i^3w_i'=\sum_{i=1}^{4}\Phi_i^4r_i'. \tag{15}$$

Using (15), we get

$$\langle\gamma'',N_4\rangle=\langle\gamma'',aN_1+bN_2+cN_3\rangle, i.e.,$$

$$\sum_{i=1}^{4}h_{ij}^4r_i'r_j'=a\sum_{i=1}^{4}h_{ij}^1u_i'u_j'+b\sum_{i=1}^{4}h_{ij}^2v_i'v_j'+c\sum_{i=1}^{4}h_{ij}^3w_i'w_j' \tag{16}$$

where $h_{ij}^k, k=1,2,3,4$ are the second fundamental form coefficients of the hypersurfaces $\Phi^k$.
Now, taking the dot product of $\sum_{i=1}^4\Phi_i^1u_i'=\sum_{i=1}^4\Phi_i^2v_i'$, $\sum_{i=1}^4\Phi_i^1u_i'=\sum_{i=1}^4\Phi_i^3w_i'$ and $\sum_{i=1}^4\Phi_i^1u_i'=\sum_{i=1}^4\Phi_i^4r_i'$ with $\Phi_j^2,\Phi_j^3,\Phi_j^4$, $j=1,2,3,4$ we obtain

$$\sum_{i=1}^{4}\langle\Phi_i^1,\Phi_j^2\rangle=\sum_{i=1}^{4}g_{ij}^2v_i',\quad \sum_{i=1}^{4}\langle\Phi_i^1,\Phi_j^3\rangle=\sum_{i=1}^{4}g_{ij}^3w_i' \quad \text{and} \quad \sum_{i=1}^{4}\langle\Phi_i^1,\Phi_j^4\rangle=\sum_{i=1}^{4}g_{ij}^4r_i',$$

where $g_{ij}^k$ are the first fundamental form coefficients of the hypersurface $\Phi^k$. Thus, denoting the unknowns $v_i'$, $w_i'$ and $r_i'$ in terms of linear combinations of $u_i'$ yields from (16)

$$\begin{aligned}&a_{11}(u_1')^2+a_{12}u_1'u_2'+a_{13}u_1'u_3'+a_{14}u_1'u_4'+a_{22}(u_2')^2+a_{23}u_2'u_3'+a_{24}u_2'u_4'\\&+a_{33}(u_3')^2+a_{34}u_3'u_4'+a_{44}(u_4')^2=0,\end{aligned} \tag{17}$$

where $a_ij$ are scalars.
Again, since $\langle t,N_2\rangle=\sum_{i=1}^4\langle\Phi_i^1,N_2\rangle u_i'\neq 0$, then at least one of the inner products is non-zero. Suppose $\langle\Phi_4^1,N_2\rangle u_i'\neq 0$, then, we obtain

$$u_4'=\beta_1u_1'+\beta_2u_2'+\beta_3u_3', \tag{18}$$

where $\beta_i=-\frac{\langle\Phi_i^1,N_2\rangle}{\langle\Phi_4^1,N_2\rangle}$., $i=1,2,3$
Using (18) in (17), we obtain

$$b_{11}(u_1')^2+b_{12}u_1'u_2'+b_{13}u_1'u_3'+b_{22}(u_2')^2+b_{23}u_2'u_3'+b_{33}(u_3')^2=0, \tag{19}$$

where $a_ij$ are scalars.
Taking cross product of $t=\Phi_1^1u_1'+\Phi_2^1u_2'+\Phi_3^1u_3'+\Phi_4^1u_4'$ with $(\Phi_4^1\times\Phi_1^2\times\Phi_2^2)$, and then projecting onto $N_1$, we obtain

$$\begin{aligned}&(\Phi_1^1\times\Phi_4^1\times\Phi_1^2\times\Phi_2^2)\cdot N_1u_1'+(\Phi_2^1\times\Phi_4^1\times\Phi_1^2\times\Phi_2^2)\cdot N_1u_2'+(\Phi_3^1\times\Phi_4^1\times\Phi_1^2\times\Phi_2^2)\cdot N_1u_3'\\&+(\Phi_4^1\times\Phi_4^1\times\Phi_1^2\times\Phi_2^2)\cdot N_1u_4'=0.\end{aligned}$$

or,

$$(\Phi_1^1\times\Phi_4^1\times\Phi_1^2\times\Phi_2^2)\cdot N_1u_1'+(\Phi_2^1\times\Phi_4^1\times\Phi_1^2\times\Phi_2^2)\cdot N_1u_2'+(\Phi_3^1\times\Phi_4^1\times\Phi_1^2\times\Phi_2^2)\cdot N_1u_3'=0. \tag{20}$$

For if $(\Phi_3^1\times\Phi_4^1\times\Phi_1^2\times\Phi_2^2)\cdot N_1=det(\Phi_3^1,\Phi_4^1,\Phi_1^2,\Phi_2^2,N_1)\neq 0$, denote

$$\xi_i=-\frac{(\Phi_i^1\times\Phi_4^1\times\Phi_1^2\times\Phi_2^2)\cdot N_1}{(\Phi_3^1\times\Phi_4^1\times\Phi_1^2\times\Phi_2^2)\cdot N_1}=-\frac{det(\Phi_i^1,\Phi_4^1,\Phi_1^2,\Phi_2^2,N_1)}{det(\Phi_3^1,\Phi_4^1,\Phi_1^2,\Phi_2^2,N_1)}. \tag{21}$$

Then we have

$$u_3'=\xi_1u_1'+\xi_2u_2'. \tag{22}$$

Using (22) in (19), we get

$$c_{11}(u_1')^2+c_{12}u_1'u_2'+c_{22}(u_2')^2=0 \tag{23}$$

Denoting $\mu=\frac{c_{11}}{c_{22}}$ when $c_{22}\neq 0$ or $\upsilon=\frac{c_{22}}{c_{11}}$ when $a_{11}=0$ and $c_{22}\neq 0$ yields

$$c_{11}\mu^2+c_{12}\mu+c_{22}=0, \tag{24}$$

$$c_{22}\upsilon^2+c_{12}\upsilon=0. \tag{25}$$

Now if $c_{11}\neq 0$, then on solving (24) for $\mu$, the unit tangent vector from (15) is obtained as

$$t=\frac{\mu(\Phi_1^1+\beta_1\Phi_4^1+\xi_1\Phi_3^1+\xi_1\beta_3\Phi_4^1)+(\Phi_{12}^1+\beta_2\Phi_4^1+\xi_3\Phi_3^1+\xi_2\beta_3\Phi_4^1)}{\|\mu(\Phi_1^1+\beta_1\Phi_4^1+\xi_1\Phi_3^1+\xi_1\beta_3\Phi_4^1)+(\Phi_{12}^1+\beta_2\Phi_4^1+\xi_3\Phi_3^1+\xi_2\beta_3\Phi_4^1)\|}. \tag{26}$$

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Similarly if $c_{22} \neq 0$ and $c_{11} = 0$, then on solving (25) for $\upsilon$, the unit tangent vector from (15) is obtained as

$$t = \frac{(\Phi_1^1 + \beta_1\Phi_4^1 + \xi_1\Phi_3^1 + \xi_1\beta_3\Phi_4^1) + \upsilon(\Phi_{12}^1 + \beta_2\Phi_4^1 + \xi_3\Phi_3^1 + \xi_2\beta_3\Phi_4^1)}{\|(\Phi_1^1 + \beta_1\Phi_4^1 + \xi_1\Phi_3^1 + \xi_1\beta_3\Phi_4^1) + \upsilon(\Phi_{12}^1 + \beta_2\Phi_4^1 + \xi_3\Phi_3^1 + \xi_2\beta_3\Phi_4^1)\|}. \tag{27}$$

□

There are four distinct cases for the solution of (23) depending upon the discriminant $D = C_{12}^2 - 4c_{11}c_{22}$;

1. If $D < 0$, then (23) does not have anyn solution. In this case $p$ is and isolated contact point of the hyprsurfaces.
2. If $D > 0$ then (23) has distinct roots, consequently $p$ is a branch point of the intersection curve.
3. If $D = 0$ and $c_{11}^2 + c_{12}^2 + c_{22}^2 \neq 0$, then (23) has a double root. Thus the hypersurfaces intersect at $p$ and at its neighbourhood.
4. If $c_{11} = c_{12} = c_{22} = 0$, then (23) vanishes for any value of $u_1'$ and $u_2\prime$. Thus the hypersurfaces have contact of at least second order at $p$.

*Remark* 3.1. For $\langle\Phi_4^1, N_2\rangle \neq 0$ in (18), following two cases arise

1. If we write $u_1' = \eta_1 u_1' + \eta_3 u_3'$ insted of (23), the corresponding equation for (23) is $c_{11}(u_1')^2 + c_{12}u_1'u_3' + c_{22}(u_3')^2 = 0$. In this case the unit tangent vector is given by
$$t = \frac{\mu(\Phi_1^1 + \beta_1\Phi_4^1 + \eta_1\Phi_3^1 + \eta_1\beta_3\Phi_4^1) + (\Phi_{12}^1 + \beta_2\Phi_4^1 + \eta_3\Phi_3^1 + \eta_2\beta_3\Phi_4^1)}{\|\mu(\Phi_1^1 + \beta_1\Phi_4^1 + \eta_1\Phi_3^1 + \eta_1\beta_3\Phi_4^1) + (\Phi_{12}^1 + \beta_2\Phi_4^1 + \eta_3\Phi_3^1 + \eta_2\beta_3\Phi_4^1)\|}$$
or,
$$t = \frac{(\Phi_1^1 + \beta_1\Phi_4^1 + \eta_1\Phi_3^1 + \eta_1\beta_3\Phi_4^1) + \upsilon(\Phi_{12}^1 + \beta_2\Phi_4^1 + \eta_3\Phi_3^1 + \eta_2\beta_3\Phi_4^1)}{\|(\Phi_1^1 + \beta_1\Phi_4^1 + \eta_1\Phi_3^1 + \eta_1\beta_3\Phi_4^1) + \upsilon(\Phi_{12}^1 + \beta_2\Phi_4^1 + \eta_3\Phi_3^1 + \eta_2\beta_3\Phi_4^1)\|},$$
where $\mu = \frac{u_1'}{u_3'}$, $\upsilon = \frac{u_3'}{u_1'}$ and $\eta_i = -\frac{det(\Phi_i^1, \Phi_4^1, \Phi_1^2, \Phi_2^2, N_1)}{det(\Phi_2^1, \Phi_4^1, \Phi_1^2, \Phi_2^2, N_1)}$, $i = 1, 3$.
2. If we write $u_1' = \eta_2 u_1' + \eta_3 u_3'$ insted of (23), the corresponding equation for (23) is $c_{11}(u_1')^2 + c_{12}u_1'u_3' + c_{22}(u_3')^2 = 0$. In this case the unit tangent vector is given by
$$t = \frac{\mu(\Phi_1^1 + \beta_1\Phi_4^1 + \eta_1\Phi_3^1 + \eta_1\beta_3\Phi_4^1) + (\Phi_{12}^1 + \beta_2\Phi_4^1 + \eta_3\Phi_3^1 + \eta_2\beta_3\Phi_4^1)}{\|\mu(\Phi_1^1 + \beta_1\Phi_4^1 + \eta_1\Phi_3^1 + \eta_1\beta_3\Phi_4^1) + (\Phi_{12}^1 + \beta_2\Phi_4^1 + \eta_3\Phi_3^1 + \eta_2\beta_3\Phi_4^1)\|}$$
or,
$$t = \frac{(\Phi_1^1 + \beta_1\Phi_4^1 + \eta_1\Phi_3^1 + \eta_1\beta_3\Phi_4^1) + \upsilon(\Phi_{12}^1 + \beta_2\Phi_4^1 + \eta_3\Phi_3^1 + \eta_2\beta_3\Phi_4^1)}{\|(\Phi_1^1 + \beta_1\Phi_4^1 + \eta_1\Phi_3^1 + \eta_1\beta_3\Phi_4^1) + \upsilon(\Phi_{12}^1 + \beta_2\Phi_4^1 + \eta_3\Phi_3^1 + \eta_2\beta_3\Phi_4^1)\|},$$
where $\mu = \frac{u_1'}{u_3'}$, $\upsilon = \frac{u_3'}{u_1'}$ and $\eta_i = -\frac{det(\Phi_i^1, \Phi_4^1, \Phi_1^2, \Phi_2^2, N_1)}{det(\Phi_1^1, \Phi_4^1, \Phi_1^2, \Phi_2^2, N_1)}$, $i = 2, 3$.

As above, for each $\langle\Phi_i^1, N_2\rangle \neq 0$, $(i = 1, 2, 3)$, there arises two possibilities for each $i$.

### 3.2. *First curvature of almost tangential intersection curve*

In case of almost tangential intersection, to find the first curvature at the intersection point $p$, we have to find the second order derivative of $\gamma(s)$. For that, we develop the following algorithm: The curvature vector of the intersection curve can be written as

$$\begin{aligned} t' = \gamma'' &= \sum_{i=1}^{4}\Phi_i^1 u_i'' + \sum_{i,j=1}^{4}\Phi_{ij}^1 u_i'u_j' = \sum_{i=1}^{4}\Phi_i^2 v_2'' + \sum_{i,j=1}^{4}\Phi_{ij}^2 v_i'v_j' \\ &= \sum_{i=1}^{4}\Phi_i^3 w_2'' + \sum_{i,j=1}^{4}\Phi_{ij}^3 w_i'w_j' = \sum_{i=1}^{4}\Phi_i^4 r_2'' + \sum_{i,j=1}^{4}\Phi_{ij}^4 r_i'r_j' \end{aligned} \tag{28}$$

To find the curvature vector, since $u_i'$ are known from theorem (3.1), , we need to find only $u_i''$. Considering $t' = \gamma'' = \sum_{i=1}^{4}\Phi_i^1 u_i'' + \sum_{i,j=1}^{4}\Phi_{ij}^1 u_i'u_j'$, we see that (28) is an equation in four unknowns $u_i'', i = 1, 2, 3, 4$. For that the first equation is given by taking the dot product of

$$\sum_{i=1}^{4}\Phi_i^1 u_i'' + \sum_{i,j=1}^{4}\Phi_{ij}^1 u_i'u_j' = \sum_{i=1}^{4}\Phi_i^2 v_2'' + \sum_{i,j=1}^{4}\Phi_{ij}^2 v_i'v_j'$$

with $N_2$, then we have

$$\sum_{i=1}^{4}\langle\Phi_i^1,N_2\rangle u_i''+\sum_{i,j=1}^{4}\langle\Phi_{ij}^1,N_2\rangle u_i'u_j'=\sum_{i,j=1}^{4}\langle\Phi_{ij}^2,N_2\rangle v_i'v_j'. \tag{29}$$

Second equation can be found in the same way by taking the dot product of

$$\sum_{i=1}^{4}\Phi_i^1 u_i''+\sum_{i,j=1}^{4}\Phi_{ij}^1 u_i'u_j'=\sum_{i=1}^{4}\Phi_i^3 w_2''+\sum_{i,j=1}^{4}\Phi_{ij}^3 w_i'w_j'$$

with $N_3$, i.e.,

$$\sum_{i=1}^{4}\langle\Phi_i^1,N_3\rangle u_i''+\sum_{i,j=1}^{4}\langle\Phi_{ij}^1,N_3\rangle u_i'u_j'=\sum_{i,j=1}^{4}\langle\Phi_{ij}^3,N_3\rangle w_i'w_j'. \tag{30}$$

On taking the dot product of 28 with $t$, we obtain the third required equation as

$$\langle t',t\rangle=\sum_{i=1}^{4}\langle\Phi_{i=1}^4,t\rangle u_i''+\sum_{i,j=1}^{4}\langle\Phi_{ij=1}^4,t\rangle u_i'u_j' \tag{31}$$

Since $\langle\gamma''',N_4\rangle=\langle\gamma''',aN_1+bN_2+cN_3\rangle$, we have the additional equation from

$$\begin{aligned}\sum_{i,j=1}^{4}\langle\Phi_i^4,N_4\rangle r_i''r_j'+\sum_{i,j,k=1}^{4}\langle\Phi_{ijk}^2,N_4\rangle r_i'r_j'r_k'&=a\left(\sum_{i,j=1}^{4}\langle\Phi_i^1,N_1\rangle u_i''u_j'+\sum_{i,j,k=1}^{4}\langle\Phi_{ijk}^1,N_1\rangle u_i'u_j'u_k'\right)\\+b\left(\sum_{i,j=1}^{4}\langle\Phi_i^2,N_2\rangle v_i''v_j'+\sum_{i,j,k=1}^{4}\langle\Phi_{ijk}^2,N_2\rangle v_i'v_j'v_k'\right)&+c\left(\sum_{i,j=1}^{4}\langle\Phi_i^3,N_3\rangle w_i''w_j'+\sum_{i,j,k=1}^{4}\langle\Phi_{ijk}^3,N_3\rangle w_i'w_j'w_k'\right).\end{aligned} \tag{32}$$

Now as in theorem (3.1), representing $v_i''$, $w_i''$ and $r_i''$, $1=1,2,3,4$ in term of linear combination of $u_i''$, we obtain the fourth equation from (32). Thus, if the obtained linear system of equations from (29) – (32) has a non-zero coefficient determinant, then after solving this system of equations, we can find the curvature vector and consequently the principal normal vector by $n=\frac{\gamma''}{\kappa_1}$. Depending upon the nature of $\langle\gamma''',N_4\rangle=\langle\gamma''',aN_1+bN_2+cN_3\rangle$, if $\langle\gamma''',N_4\rangle=\langle\gamma''',aN_1+bN_2+cN_3\rangle$ vanishes, then (29), (30) and (31) constitutes a system whose coefficient matrix has rank 3. In this case, we solve the three unknowns by means of fourth one. The fourth unknown can be obtained by differentiating $\langle\gamma'',N_4\rangle=\langle\gamma'',aN_1+bN_2+cN_3\rangle$, i.e., from

$$\langle\gamma'',N_4'\rangle=\langle\gamma'',aN_1'+bN_2'+cN_3'\rangle,\quad\text{or,}\quad\langle\gamma'',N_4''\rangle=\langle\gamma''aN_1''+bN_2''+cN_3''\rangle \tag{33}$$

### *3.3. Third order derivative and the second curvature of almost tangential intersection*

Using (10), we have

$$\gamma'''=\sum_{i=1}^{4}\Phi_i^1u_i'''+3\sum_{i,j=1}^{4}\Phi_{ij}^1u_i''u_j'+\sum_{i,j,k=1}^{4}\Phi_{ijk}^1u_i'u_j'u_k'=\sum_{i=1}^{4}\Phi_i^2v_i'''+3\sum_{i,j=2}^{4}\Phi_{ij}^2v_i''v_j'+\sum_{i,j,k=2}^{4}\Phi_{ijk}^2v_i'v_j'v_k' \tag{34}$$

$$=\sum_{i=1}^{4}\Phi_i^3w_i'''+3\sum_{i,j=3}^{4}\Phi_{ij}^3w_i''w_j'+\sum_{i,j,k=3}^{4}\Phi_{ijk}^3w_i'w_j'w_k'=\sum_{i=1}^{4}\Phi_i^4r_i'''+3\sum_{i,j=3}^{4}\Phi_{ij}^4r_i''r_j'+\sum_{i,j,k=3}^{4}\Phi_{ijk}^4r_i'r_j'r_k' \tag{35}$$

Since $u_i'$ and $u_i''$ are already known. To find the third derivative, it remains to find our $u_i'''$. Thus considering $\gamma'''=\sum_{i=1}^4\Phi_i^1u_i'''+3\sum_{i,j=1}^4\Phi_{ij}^1u_i''u_j'+\sum_{i,j,k=1}^4\Phi_{ijk}^1u_i'u_j'u_k'$, we see that $\gamma'''$ is an equation in four unknowns $u_1'''$, $u_2'''$, $u_3'''$, $u_4'''$. For that, we constitute a system of four equations.

The first equation is given by taking the dot product of $\sum_{i=1}^4\Phi_i^1u_i'''+3\sum_{i,j=1}^4\Phi_{ij}^1u_i''u_j'+\sum_{i,j,k=1}^4\Phi_{ijk}^1u_i'u_j'u_k'=\sum_{i=1}^4\Phi_i^2v_i'''+3\sum_{i,j=2}^4\Phi_{ij}^2v_i''v_j'+\sum_{i,j,k=2}^4\Phi_{ijk}^2v_i'v_j'v_k'$ with $N_2$, i.e.,

$$\sum_{i=1}^{4}\langle\Phi_i^1,N_2\rangle u_i'''+3\sum_{i,j=1}^{4}\langle\Phi_{ij}^1,N_2\rangle u_i''u_j'+\sum_{i,j,k=1}^{4}\langle\Phi_{ijk}^1,N_2\rangle u_i'u_j'u_k'=3\sum_{i,j=2}^{4}\langle\Phi_{ij}^2,N_2\rangle v_i''v_j'+\sum_{i,j,k=2}^{4}\langle\Phi_{ijk}^2,N_2\rangle v_i'v_j'v_k'. \tag{36}$$

The second equation can be obtained by taking the dot product of $\sum_{i=1}^4\Phi_i^1u_i'''+3\sum_{i,j=1}^4\Phi_{ij}^1u_i''u_j'+\sum_{i,j,k=1}^4\Phi_{ijk}^1u_i'u_j'u_k'=\sum_{i=1}^4\Phi_i^3w_i'''+3\sum_{i,j=1}^4\Phi_{ij}^3w_i''w_j'+\sum_{i,j,k=1}^4\Phi_{ijk}^3w_i'w_j'w_k'$ with $N_3$, i.e.,

$$\sum_{i=1}^{4}\langle\Phi_i^1,N_3\rangle u_i'''+3\sum_{i,j=1}^{4}\langle\Phi_{ij}^1,N_3\rangle u_i''u_j'+\sum_{i,j,k=1}^{4}\langle\Phi_{ijk}^1,N_3\rangle u_i'u_j'u_k'=3\sum_{i,j=1}^{4}\langle\Phi_{ij}^3,N_3\rangle w_i''w_j'+\sum_{i,j,k=1}^{4}\langle\Phi_{ijk}^3,N_3\rangle w_i'w_j'w_k'. \tag{37}$$

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On differentiating $\langle \gamma', \gamma'' \rangle = 0$, we obtain $\langle \gamma''', t \rangle = -\kappa_1^2$. This gives us the third required equation as

$$\sum_{i=1}^{4} \langle \Phi_i^1, t \rangle u_i''' + 3 \sum_{i,j=1}^{4} \langle \Phi_{ij}^1, t \rangle u_i'' u_j' + \sum_{i,j,k=1}^{4} \langle \Phi_{ijk}^1, t \rangle u_i' u_j' u_k' = -\kappa_1^2 \tag{38}$$

Now, using $\langle \gamma^{(4)}, N_4 \rangle = \langle \gamma^{(4)}, aN_1 + bN_2 + cN_3 \rangle$, we get

$$\begin{aligned}
&4\sum_{i=1}^{4} \langle \Phi_{ij}^4, N_4 \rangle r_i''' r_j' + 3 \sum_{i,j=1}^{4} \langle \Phi_{ij}^4, N_4 \rangle r_i'' r_j'' + 6 \sum_{i,j,k=1}^{4} \langle \Phi_{ijk}^4, N_4 \rangle r_i' r_j' r_k'' + \sum_{i,j,k,l=1}^{4} \langle \Phi_{ijkl}^4, N_4 \rangle r_i' r_j' r_k' r_l' \\
&= a \left( 4\sum_{i=1}^{4} \langle \Phi_{ij}^1, N_1 \rangle u_i''' u_j' + 3 \sum_{i=1}^{4} \langle \Phi_{ij}^1, N_1 \rangle u_i'' u_j'' + 6 \sum_{i,j,k=1}^{4} \langle \Phi_{ijk}^1, N_1 \rangle u_i' u_j' u_k'' + \sum_{i,j,k,l=1}^{4} \langle \Phi_{jkl}^1, N_1 \rangle u_i' u_j' u_k' u_l' \right) \\
&+ b \left( 4\sum_{i=1}^{4} \langle \Phi_{ij}^2, N_2 \rangle v_i''' v_j' + 3 \sum_{i,j=1}^{4} \langle \Phi_{ij}^2, N_2 \rangle v_i'' v_j'' + 6 \sum_{i,j,k=1}^{4} \langle \Phi_{ijk}^2, N_2 \rangle v_i' v_j' v_k'' + \sum_{i,j,k,l=1}^{4} \langle \Phi_{ijkl}^2, N_2 \rangle v_i' v_j' v_k' v_l' \right) \\
&+ c \left( 4\sum_{i=1}^{4} \langle \Phi_{ij}^3, N_3 \rangle w_i''' w_j' + 3 \sum_{i,j=1}^{4} \langle \Phi_{ij}^3, N_3 \rangle w_i'' w_j'' + 6 \sum_{i,j,k=1}^{4} \langle \Phi_{ijk}^3, N_3 \rangle w_i' w_j' w_k'' + \sum_{i,j,k,l=1}^{4} \langle \Phi_{ijkl}^3, N_3 \rangle w_i' w_j' w_k' w_l' \right).
\end{aligned} \tag{39}$$

Now, as in the above subsection, representing $v_i''$, $w_i''$, $r_i''$, $i = 1,2,3,4$ in terms of $u_i''$, the equations (36–39) constitutes a linear system of equations in four unknowns $u_i''$, $i = 1,2,3,4$. If this system has a nonzero coefficient determinant, we can find the required unknowns, which enables us to compute third order derivative of the intersection curve. Noting that, if the equation $\langle \gamma^{(4)}, N_4 \rangle = \langle \gamma^{(4)}, aN_1 + bN_2 + cN_3 \rangle$ vanishes, we can replace the equation with $\langle \gamma''', N_4^{(i)} \rangle = \langle \gamma''', aN_1^{(i)} + bN_2^{(i)} + cN_3^{(i)} \rangle$, $i = 1,2,3,4$.

### 3.4. *The fourth order derivative and the third curvature*

On computing the fourth order derivative of the intersection curve, we are able to find the third, second and first binormal vectors from (6), consequently the second and third curvature follows from (7).

Using (11) and taking the dot product of both sides of

$$\begin{aligned}
&\sum_{i=1}^{4} \Phi_i^1 u_i^{(4)} + 4\sum_{i=1}^{4} \Phi_{ij}^1 u_i''' u_j' + 3\sum_{i=1}^{4} \Phi_{ij}^1 u_i'' u_j'' + 6 \sum_{i,j,k=1}^{4} \Phi_{ijk}^1 u_i' u_j' u_k'' + \sum_{i,j,k,l=1}^{4} \Phi_{ijkl}^1 u_i' u_j' u_k' u_l' \\
&= \sum_{i=1}^{4} \Phi_i^2 v_i^{(4)} + 4\sum_{i=1}^{4} \Phi_{ij}^2 v_i''' v_j' + 3 \sum_{i,j=1}^{4} \Phi_{ij}^2 v_i'' v_j'' + 6 \sum_{i,j,k=1}^{4} \Phi_{ijk}^2 v_i' v_j' v_k'' + \sum_{i,j,k,l=1}^{4} \Phi_{ijkl}^2, N_2 v_i' v_j' v_k' v_l'
\end{aligned}$$

with $N_2$, we get the first required linear equation

$$\begin{aligned}
&4\sum_{i=1}^{4} \langle \Phi_{ij}^1, N_2 \rangle u_i''' u_j' + 3\sum_{i=1}^{4} \langle \Phi_{ij}^1, N_2 \rangle u_i'' u_j'' + 6 \sum_{i,j,k=1}^{4} \langle \Phi_{ijk}^1, N_2 \rangle u_i' u_j' u_k'' + \sum_{i,j,k,l=1}^{4} \langle \Phi_{ijkl}^1, N_2 \rangle u_i' u_j' u_k' u_l' \\
&= 4\sum_{i=1}^{4} \langle \Phi_{ij}^2, N_2 \rangle v_i''' v_j' + 3 \sum_{i,j=1}^{4} \langle \Phi_{ij}^2, N_2 \rangle v_i'' v_j'' + 6 \sum_{i,j,k=1}^{4} \langle \Phi_{ijk}^2, N_2 \rangle v_i' v_j' v_k'' + \sum_{i,j,k,l=1}^{4} \langle \Phi_{ijkl}^2, N_2 \rangle v_i' v_j' v_k' v_l'.
\end{aligned}$$

The second linear equation is given in the same fashion by taking the dot product of

$$\begin{aligned}
&\sum_{i=1}^{4} \Phi_i^1 u_i^{(4)} + 4\sum_{i=1}^{4} \Phi_{ij}^1 u_i''' u_j' + 3\sum_{i=1}^{4} \Phi_{ij}^1 u_i'' u_j'' + 6 \sum_{i,j,k=1}^{4} \Phi_{jk}^1 u_i' u_j' u_k'' + \sum_{i,j,k,l=1}^{4} \Phi_{jkl}^1 u_i' u_j' u_k' u_l' \\
&= \sum_{i=1}^{4} \Phi_i^3 w_i^{(4)} + 4\sum_{i=1}^{4} \Phi_{ij}^3 w_i''' w_j' + 3 \sum_{i,j=1}^{4} \Phi_{ij}^3 w_i'' w_j'' + 6 \sum_{i,j,k=1}^{4} \Phi_{ijk}^3 w_i' w_j' w_k'' + \sum_{i,j,k,l=1}^{4} \Phi_{ijkl}^3, N_3 w_i' w_j' w_k' w_l'
\end{aligned}$$

with $N_3$, i.e.,

$$\begin{aligned}
&4\sum_{i=1}^{4} \langle \Phi_{ij}^1, N_3 \rangle u_i''' u_j' + 3\sum_{i=1}^{4} \langle \Phi_{ij}^1, N_3 \rangle u_i'' u_j'' + 6 \sum_{i,j,k=1}^{4} \langle \Phi_{ijk}^1, N_3 \rangle u_i' u_j' u_k'' + \sum_{i,j,k,l=1}^{4} \langle \Phi_{ijkl}^1, N_3 \rangle u_i' u_j' u_k' u_l' \\
&= 4\sum_{i=1}^{4} \langle \Phi_{ij}^3, N_3 \rangle w_i''' w_j' + 3 \sum_{i,j=1}^{4} \langle \Phi_{ij}^3, N_3 \rangle w_i'' w_j'' + 6 \sum_{i,j,k=1}^{4} \langle \Phi_{ijk}^3, N_3 \rangle w_i' w_j' w_k'' + \sum_{i,j,k,l=1}^{4} \langle \Phi_{ijkl}^3, N_3 \rangle w_i' w_j' w_k' w_l'.
\end{aligned}$$

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On the other hand, differentiating $\langle \gamma''', t\rangle = -\kappa_1^2$ gives the third equation as $\langle \gamma^{(4)}, t\rangle = -3k_1k_1'$, i.e.,

$$\sum_{i=1}^{4}\langle \Phi_i^1, t\rangle u_i^{(4)} + 4\sum_{i,j=1}^{4}\langle \Phi_{ij}^1, t\rangle u_i''' u_j' + 3\sum_{i,j=1}^{4}\langle \Phi_{ij}^1, t\rangle u_i'' u_j'' + 6\sum_{i,j,k=1}^{4}\langle \Phi_{ijk}^1, t\rangle u_i' u_j' u_k'' \\ + \sum_{i,j,k,l=1}^{4}\langle \Phi_{ijkl}^1, t\rangle u_i' u_j' u_k' u_l' = -3k_1k_1'. \tag{40}$$

Finally, if we use $\langle \gamma^{(5)}, N_4\rangle = \langle \gamma^{(5)}, aN_1 + bN_2 + cN_3\rangle$, we obtain

$$5\sum_{i,j=1}^{4}\langle \Phi_{ij}^4 r_i^{(4)} r_j', N_4\rangle + 10\sum_{i,j=1}^{4}\langle \Phi_{ij}^4 r_i''' r_j'', N_4\rangle + 10\sum_{i,j,k=1}^{4}\langle \Phi_{ijk}^4 r_i''' r_j' r_k', N_4\rangle + 15\sum_{i,j,k=1}^{4}\langle \Phi_{ijk}^4 r_i'' r_j'' r_k', N_4\rangle \\ +10\sum_{i,j,k,l=1}^{4}\langle \Phi_{ijkl}^4 r_i'' r_j' r_k' r_l', N_4\rangle + \sum_{i,j,k,l,r=1}^{4}\langle \Phi_{ijklr}^4 r_i' r_j' r_k' r_l' r_r', N_4\rangle \\ = a\Bigg(5\sum_{i,j=1}^{4}\langle \Phi_{ij}^1 u_i^{(4)} u_j', N_4\rangle + 10\sum_{i,j=1}^{4}\langle \Phi_{ij}^1 u_i''' u_j'', N_4\rangle + 10\sum_{i,j,k=1}^{4}\langle \Phi_{ijk}^1 u_i''' u_j' u_k', N_4\rangle + 15\sum_{i,j,k=1}^{4}\langle \Phi_{ijk}^1 u_i'' u_j'' u_k', N_4\rangle \\ +10\sum_{i,j,k,l=1}^{4}\langle \Phi_{ijkl}^1 u_i'' u_j' u_k' u_l', N_4\rangle + \sum_{i,j,k,l,r=1}^{4}\langle \Phi_{ijklr}^1 u_i' u_j' u_k' u_l' u_r', N_4\rangle\Bigg) + b\Bigg(5\sum_{i,j=1}^{4}\langle \Phi_{ij}^1 u_i^{(4)} u_j', N_4\rangle \\ +10\sum_{i,j=1}^{4}\langle \Phi_{ij}^1 u_i''' u_j'', N_4\rangle + 10\sum_{i,j,k=1}^{4}\langle \Phi_{ijk}^1 u_i''' u_j' u_k', N_4\rangle + 15\sum_{i,j,k=1}^{4}\langle \Phi_{ijk}^1 u_i'' u_j'' u_k', N_4\rangle \\ +10\sum_{i,j,k,l=1}^{4}\langle \Phi_{ijkl}^1 u_i'' u_j' u_k' u_l', N_4\rangle + \sum_{i,j,k,l,r=1}^{4}\langle \Phi_{ijklr}^1 u_i' u_j' u_k' u_l' u_r', N_4\rangle\Bigg) + c\Bigg(5\sum_{i,j=1}^{4}\langle \Phi_{ij}^1 u_i^{(4)} u_j', N_4\rangle \\ +10\sum_{i,j=1}^{4}\langle \Phi_{ij}^1 u_i''' u_j'', N_4\rangle + 10\sum_{i,j,k=1}^{4}\langle \Phi_{ijk}^1 u_i''' u_j' u_k', N_4\rangle + 15\sum_{i,j,k=1}^{4}\langle \Phi_{ijk}^1 u_i'' u_j'' u_k', N_4\rangle + 10\sum_{i,j,k,l=1}^{4}\langle \Phi_{ijkl}^1 u_i'' u_j' u_k' u_l', N_4\rangle \\ + \sum_{i,j,k,l,r=1}^{4}\langle \Phi_{ijklr}^1 u_i' u_j' u_k' u_l' u_r', N_4\rangle\Bigg). \tag{41}$$

Now, as in the above subsections, writing $v_i^{(4)}$, $w_i^{(4)}$ and $r_i^{(4)}$, $i = 1,2,3,4$ in terms of linear combinations of $u_i^{(4)}$. Also, since $u_i'$, $u_i''$, $u_i'''$ are already known, we obtain a system of four linear equations in $u_i^{(4)}$, $i = 1,2,34$. On solving this system of linear equations we obtain the desired quantities. Again noting that if $\langle \gamma^{(5)}, N_4\rangle = \langle aN_1 + bN_2 + cN_3\rangle$ vanishes, then this equation can be replaced by $\langle \gamma^{(4)}, N_4\rangle = \langle \gamma^{(5)}, aN_1^{(i)} + bN_2^{(i)} + cN_3^{(i)}\rangle$, $i = 1,2,3,4$.

### 3.5. *Fifth order derivative and fourth curvature of the almost tangential intersection*

To find the fifth derivative of the intersection curve we need to find $u_i^{(5)}$, $i = 1,2,3,4$. For the required linear system of equations, the first two equations can be found in the same way as in above subsections by taking the dot product of

$$\sum_{i=1}^{4}\Phi_i^1 u_i^{(5)} + 5\sum_{i,j=1}^{4}\Phi_{ij}^1 u_i^{(4)} u_j' + 10\sum_{i,j=1}^{4}\Phi_{ij}^1 u_i''' u_j'' + 10\sum_{i,j,k=1}^{4}\Phi_{ijk}^1 u_i''' u_j' u_k' + 15\sum_{i,j,k=1}^{4}\Phi_{ijk}^1 u_i'' u_j'' u_k' + 10\sum_{i,j,k,l=1}^{4}\Phi_{ijkl}^1 u_i'' u_j' u_k' u_l' \\ + \sum_{i,j,k,l,r=1}^{4}\Phi_{ijklr}^1 u_i' u_j' u_k' u_l' u_r' = \sum_{i=1}^{4}\Phi_i^2 v_i^{(5)} + 5\sum_{i,j=1}^{4}\Phi_{ij}^2 v_i^{(4)} v_j' + 10\sum_{i,j=1}^{4}\Phi_{ij}^2 v_i''' v_j'' + 10\sum_{i,j,k=1}^{4}\Phi_{ijk}^2 v_i''' v_j' v_k' + 15\sum_{i,j,k=1}^{4}\Phi_{ijk}^2 v_i'' v_j'' v_k' \\ +10\sum_{i,j,k,l=1}^{4}\Phi_{ijkj}^2 v_i'' v_j' v_k' v_l' + \sum_{i,j,k,l,r=1}^{4}\Phi_{ijklr}^2 v_i' v_j' v_k' v_l' v_r'$$

with $N_2$ and

$$\sum_{i=1}^{4}\Phi_i^1 u_i^{(5)} + 5\sum_{i,j=1}^{4}\Phi_{ij}^1 u_i^{(4)} u_j' + 10\sum_{i,j=1}^{4}\Phi_{ij}^1 u_i''' u_j'' + 10\sum_{i,j,k=1}^{4}\Phi_{ijk}^1 u_i''' u_j' u_k' + 15\sum_{i,j,k=1}^{4}\Phi_{ijk}^1 u_i'' u_j'' u_k' + 10\sum_{i,j,k,l=1}^{4}\Phi_{ijkj}^1 u_i'' u_j' u_k' u_l' \\ + \sum_{i,j,k,l,r=1}^{4}\Phi_{ijklr}^1 u_i' u_j' u_k' u_l' u_r' = \sum_{i=1}^{4}\Phi_i^3 w_i^{(5)} + 5\sum_{i,j=1}^{4}\Phi_{ij}^3 w_i^{(4)} w_j' + 10\sum_{i,j=1}^{4}\Phi_{ij}^3 w_i''' w_j'' + 10\sum_{i,j,k=1}^{4}\Phi_{ijk}^3 w_i''' w_j' w_k' \\ +15\sum_{i,j,k=1}^{4}\Phi_{ijk}^3 w_i'' w_j'' w_k' + 10\sum_{i,j,k,l=1}^{4}\Phi_{ijkj}^3 w_i'' w_j' w_k' w_l' + \sum_{i,j,k,l,r=1}^{4}\Phi_{ijklr}^3 w_i' w_j' w_k' w_l' w_r'$$

with $N_3$.

The third equation is obtained by projecting $\gamma^{(5)}$ onto $t$ and the last equation is obtained by using $\langle\gamma^{(6)}, N_4\rangle = \langle\gamma^{(6)}, aN_1 + bN_2 + cN_3\rangle$. As in the above cases if $\langle\gamma^{(6)}, N_4\rangle = \langle\gamma^{(6)} aN_1 + bN_2 + cN_3\rangle$ vanishes, then this equation can be replaced by $\langle\gamma^{(5)}, N_4\rangle = \langle\gamma^{(5)}, aN_1^{(i)} + bN_2^{(i)} + cN_3^{(i)}\rangle$, $i = 1,2,3,4,5$.

## 4. Tangential intersection of four parametric hypersurfaces

Here we assume that $N_1 = N_2 = N_3 = N_4 = N$. For the unit tangent vector, the algorithm is provided by the following theorem:

**Theorem 4.1.** *Let $\Phi_1, \Phi_2, \Phi_3, \Phi_4$ be the four parametric hypersurfaces in $\mathbb{R}^5$, then we have*

$$t = \frac{(\Phi_1^1 + \lambda_1\Phi_4^1) + (\Phi_2^1 + \lambda_2\Phi_4^1)\varepsilon + (\Phi_3^1 + \lambda_3\Phi_4^1)\varpi}{\|(\Phi_1^1 + \lambda_1\Phi_4^1) + (\Phi_2^1 + \lambda_2\Phi_4^1)\varepsilon + (\Phi_3^1 + \lambda_3\Phi_4^1)\varpi\|},$$

*or*

$$t = \frac{(\Phi_1^1 + \lambda_1\Phi_4^1)\varepsilon + (\Phi_2^1 + \lambda_2\Phi_4^1) + (\Phi_3^1 + \lambda_3\Phi_4^1)\varpi}{\|(\Phi_1^1 + \lambda_1\Phi_4^1)\varepsilon + (\Phi_2^1 + \lambda_2\Phi_4^1) + (\Phi_3^1 + \lambda_3\Phi_4^1)\varpi\|},$$

*or*

$$t = \frac{(\Phi_1^1 + \lambda_1\Phi_4^1)\varepsilon + (\Phi_2^1 + \lambda_2\Phi_4^1)\varpi + (\Phi_3^1 + \lambda_3\Phi_4^1)}{\|(\Phi_1^1 + \lambda_1\Phi_4^1)\varepsilon + (\Phi_2^1 + \lambda_2\Phi_4^1)\varpi + (\Phi_3^1 + \lambda_3\Phi_4^1)\|}.$$

*Proof.* Projecting $\gamma''$ onto the common normal vector, we obtain

$$\left\{\begin{array}{l} \langle\gamma'', N_1\rangle = \langle\gamma'', N_2\rangle \Rightarrow \sum_{i=1}^4 h_{ij}^1 u_i' u_j' = \sum_{i=1}^4 h_{ij}^2 v_i' v_j', \\ \langle\gamma'', N_1\rangle = \langle\gamma'', N_3\rangle \Rightarrow \sum_{i=1}^4 h_{ij}^1 u_i' u_j' = \sum_{i=1}^4 h_{ij}^3 w_i' w_j', \\ \langle\gamma'', N_1\rangle = \langle\gamma'', N_4\rangle \Rightarrow \sum_{i=1}^4 h_{ij}^1 u_i' u_j' = \sum_{i=1}^4 h_{ij}^4 r_i' r_j'. \end{array}\right. \tag{42}$$

As in the above section replacing $v_i', w_i', r_i'$ by $u_i'$, we denote the first two equations of (42) as system

$$e_{11}(u_1')^2 + e_{12}u_1'u_2' + e_{13}u_1'u_3' + e_{14}u_1'u_4' + e_{22}(u_2')^2 + e_{23}u_2'u_3' + e_{24}u_2'u_4' + e_{33}(u_3')^2 + e_{34}u_3'u_4' + e_{44}(u_4')^2 = 0 \tag{43}$$

$$f_{11}(u_1')^2 + f_{12}u_1'u_2' + f_{13}u_1'u_3' + f_{14}u_1'u_4' + f_{22}(u_2')^2 + f_{23}u_2'u_3' + f_{24}u_2'u_4' + f_{33}(u_3')^2 + f_{34}u_3'u_4' + f_{44}(u_4')^2 = 0 \tag{44}$$

respectively, where $e_{ij}, f_{ij}, i,j = 1,2,3,4$ are scalars.

Since, $\langle t, N\rangle = \sum_{i=1}^4 \langle\Phi_i^1, N\rangle u_i' = 0$, this means that at least one of the products is non-vanishing. Suppose $\langle\Phi_4^1, N\rangle \neq 0$, this implies that $u_4' = \lambda_1 u_1' + \lambda_1 u_2' + \lambda_1 u_3'$, where $\lambda = \frac{\langle\Phi_i^1, N\rangle}{\langle\Phi_4^1, N\rangle}$, $i = 1,2,3$. Thus (43) and (44) reduces to

$$m_{11}(u_1')^2 + m_{12}u_1'u_2' + m_{13}u_1'u_3' + m_{22}(u_2')^2 + m_{23}u_2'u_3' + m_{33}(u_3')^2 = 0, \tag{45}$$

$$n_{11}(u_1')^2 + n_{12}u_1'u_2' + n_{13}u_1'u_3' + n_{22}(u_2')^2 + n_{23}u_2'u_3' + n_{33}(u_3')^2 = 0 \tag{46}$$

where $m_{ij}, n_{ij}, i,j = 1,2,3$ are scalars. Now, if we denote

$$\varepsilon = \frac{u_2'}{u_1'}, \quad \varpi = \frac{u_3'}{u_1'} \quad \text{when} \quad \chi_1 = \begin{vmatrix} m_{22} & m_{33} \\ n_{22} & n_{33} \end{vmatrix} \neq 0,$$

or

$$\varepsilon = \frac{u_1'}{u_2'}, \quad \varpi = \frac{u_3'}{u_2'} \quad \text{when} \quad \chi_2 = \begin{vmatrix} m_{11} & m_{33} \\ n_{11} & n_{33} \end{vmatrix} \neq 0,$$

or

$$\varepsilon = \frac{u_1'}{u_3'}, \quad \varpi = \frac{u_2'}{u_3'} \quad \text{when} \quad \chi_3 = \begin{vmatrix} m_{11} & m_{22} \\ n_{11} & n_{22} \end{vmatrix} \neq 0$$

we obtain

$$\left.\begin{array}{l} m_{22}\varepsilon^2 + m_{33}\varpi^2 + m_{12}\varepsilon + m_{13}\varpi + m_{23}\varepsilon\varpi + m_{11} = 0, \\ n_{22}\varepsilon^2 + n_{33}\varpi^2 + n_{12}\varepsilon + n_{13}\varpi + n_{23}\varepsilon\varpi + n_{11} = 0, \end{array}\right\} \tag{47}$$

or

$$\left.\begin{array}{l} m_{11}\varepsilon^2 + m_{33}\varpi^2 + m_{12}\varepsilon + m_{23}\varpi + m_{13}\varepsilon\varpi + m_{22} = 0, \\ n_{11}\varepsilon^2 + m_{33}\varpi^2 + n_{12}\varepsilon + n_{23}\varpi + n_{13}\varepsilon\varpi + n_{22} = 0, \end{array}\right\} \tag{48}$$

or

$$\left.\begin{array}{l} m_{11}\varepsilon^2 + m_{22}\varpi^2 + m_{13}\varepsilon + m_{23}\varpi + m_{12}\varepsilon\varpi + m_{33} = 0, \\ n_{11}\varepsilon^2 + n_{22}\varpi^2 + n_{13}\varepsilon + n_{23}\varpi + n_{12}\varepsilon\varpi + n_{33} = 0 \end{array}\right\} \tag{49}$$

respectively. We, see that (47),(48) and (49) are pairs of conics with respect to $\varepsilon$ and $\varpi$. The intersection point $(\varepsilon, \varpi)$ can be found by any known methods of conic solutions, thus the unit tangent vector is obtained by

$$t = \frac{(\Phi_1^1 + \lambda_1\Phi_4^1) + (\Phi_2^1 + \lambda_2\Phi_4^1)\varepsilon + (\Phi_3^1 + \lambda_3\Phi_4^1)\varpi}{\|(\Phi_1^1 + \lambda_1\Phi_4^1) + (\Phi_2^1 + \lambda_2\Phi_4^1)\varepsilon + (\Phi_3^1 + \lambda_3\Phi_4^1)\varpi\|}, \tag{50}$$

or

$$t = \frac{(\Phi_1^1 + \lambda_1\Phi_4^1)\varepsilon + (\Phi_2^1 + \lambda_2\Phi_4^1) + (\Phi_3^1 + \lambda_3\Phi_4^1)\varpi}{\|(\Phi_1^1 + \lambda_1\Phi_4^1)\varepsilon + (\Phi_2^1 + \lambda_2\Phi_4^1) + (\Phi_3^1 + \lambda_3\Phi_4^1)\varpi\|}, \tag{51}$$

or

$$t = \frac{(\Phi_1^1 + \lambda_1\Phi_4^1)\varepsilon + (\Phi_2^1 + \lambda_2\Phi_4^1)\varpi + (\Phi_3^1 + \lambda_3\Phi_4^1)}{\|(\Phi_1^1 + \lambda_1\Phi_4^1)\varepsilon + (\Phi_2^1 + \lambda_2\Phi_4^1)\varpi + (\Phi_3^1 + \lambda_3\Phi_4^1)\|}, \tag{52}$$

respectively. □

Note that, we can use this method if $\chi_i \neq 0$, $i = 1,2,3$, or if $\sum_{i,j=1}^{3}(m_{ij}^2 + n_{ij}^2) \neq 0$ and $\chi_1 = \chi_2 = \chi_3 = 0$, then $t$ may not exist.

*Remark* 4.1. Depending upon the real intersection points of the conic pairs, we have following

1. When the conics do not have any point in common $p$ is an isolated contact point.
2. When the conics have one point in common, $t$ is unique.
3. When the conics have two or more points in common, then $p$ is a branch point.
4. When $\sum_{i,j=1}^{3}(m_{ij}^2 + n_{ij}^2) = 0$, then conic pair vanishes for any values of $u_i'$, i.e., $p$ is a higher order contact point. If all the second fundamental coefficients of the hypersurfaces vanishes at $p$, then $p$ is a flat point of the hypersurfaces.

### *4.1. Curvature vector of tangential intersection*

To find the curvature vector, we need to find $u_i''$, which needs a system of four linear equations in $u_1'', u_2'', u_3'', u_4''$ . As in the above section representing $v_i''$, $w_i''$, $r_i''$ in terms of linear combination of $u_i''$. Also denoting $\gamma^i$ be the curve associated with $\Phi^i, i = 1,2,3,4$ hypersurface. Thus the first three equations of the required system of equations is given by the following projections

$$\left\{\begin{array}{l} \langle(\gamma^1)''', N_1\rangle = \langle(\gamma^2)''', N_2\rangle, \\ \langle(\gamma^1)''', N_1\rangle = \langle(\gamma^3)''', N_3\rangle, \\ \langle(\gamma^1)''', N_1\rangle = \langle(\gamma^4)''', N_4\rangle, \end{array}\right. \tag{53}$$

and the last equation is given by $\langle(\gamma^1)', (\gamma^1)''\rangle = 0$. If the coefficient determinant of the system (53) is non-zero, then we can find $u_i'', i = 1,2,3,4$. substituting the results into (9) gives the curvature vector. Consequently, the first curvature can be found from (7) and $n, \kappa_1'$ can be found from (3). Otherwise, if the coefficient determinant of (53) is zero, then among the first three equations of (53) one or more equations vanishes, or one equation is proportional to some other. In this case that equation(vanishing) is replaced by $\langle(\gamma^1)'', N_1'\rangle = \langle(\gamma^i)'', N_i'\rangle$, or $\langle(\gamma^1)'', N_1''\rangle = \langle(\gamma^i)'', N_i''\rangle, i = 2,3,4$.

### *4.2. Second curvature of the tangential intersection*

For the the second curvature, we need to find $\gamma'''$. To find this, we need to evaluate $u_i''', i = 1,2,3,4$. Hence a system of four linear equations in $u_i''', i = 1,2,3,4$ is needed. On representing $v_i''', w_i''', r_i'''$ in terms of linear combination of $u_i'''$, the first three equations are given by the following system

$$\left\{\begin{array}{l} \langle(\gamma^1)^{(4)}, N_1\rangle = \langle(\gamma^2)^{(4)}, N_2\rangle, \\ \langle(\gamma^1)^{(4)}, N_1\rangle = \langle(\gamma^3)^{(4)}, N_3\rangle, \\ \langle(\gamma^1)^{(4)}, N_1\rangle = \langle(\gamma^4)^{(4)}, N_4\rangle, \end{array}\right. \tag{54}$$

and the last equation is obtained as $\langle(\gamma^1)', (\gamma^1)'''\rangle = -\kappa_1^2$. If the coefficient matrix is non-singular, then $u_i'''$ are easily found, while as if any of the equations in (54) vanishes, then that equation is replaced by $\langle(\gamma^1)''', N_1'\rangle = \langle(\gamma^i)''', N_i'\rangle$, or $\langle(\gamma^1)''', N_1''\rangle = \langle(\gamma^i)''', N_i''\rangle$, or $\langle(\gamma^1)''', N_1'''\rangle = \langle(\gamma^i)''', N_i'''\rangle i = 2,3,4$.

### 4.3. Third curvature of the tangential intersection

To find the fourth curvature, we calculate $u_i^{(4)}, i = 1,2,3,4$. Representing $v_i^{(4)}$, $w_i^{(4)}$, $r_i^{(4)}$ in terms of linear combination of $u_i^{(4)}$, then the first three equations are

$$\begin{cases} \langle(\gamma^1)^{(5)}, N_1\rangle = \langle(\gamma^2)^{(5)}, N_2\rangle, \\ \langle(\gamma^1)^{(5)}, N_1\rangle = \langle(\gamma^3)^{(5)}, N_3\rangle, \\ \langle(\gamma^1)^{(5)}, N_1\rangle = \langle(\gamma^5)^{(5)}, N_5\rangle. \end{cases} \tag{55}$$

Also the last equation depending on $u_i^{(4)}$ is $\langle(\gamma^1)', (\gamma^1)^{(4)}\rangle = -3\kappa_1\kappa_1'$. If the determinant of coefficient matrix is non-vanishing, then $u_i^{(4)}$ are easily found, while as if any of the equations in (55) vanishes, then that equation is replaced by $\langle(\gamma^1)^{(4)}, N_1^{(j)}\rangle = \langle(\gamma^i)^{(4)}, N_i^{(j)}\rangle$, $i = 2,3,4$. and $j = 1\cdots 4$.

### 4.4. Fourth curvature of the tangential intersection

Finally to find the fourth curvature, we find $u_i^{(5)}, i = 1,2,3,4$. Writting $v_i^{(5)}$, $w_i^{(5)}$, $r_i^{(5)}$ in terms of linear combination of $u_i^{(5)}$, we have

$$\begin{cases} \langle(\gamma^1)^{(6)}, N_1\rangle = \langle(\gamma^2)^{(6)}, N_2\rangle, \\ \langle(\gamma^1)^{(6)}, N_1\rangle = \langle(\gamma^3)^{(6)}, N_3\rangle, \\ \langle(\gamma^1)^{(6)}, N_1\rangle = \langle(\gamma^6)^{(6)}, N_6\rangle. \end{cases} \tag{56}$$

The fourth equation depending on $u_i^{(5)}$ is $\langle(\gamma^1)', (\gamma^1)^{(5)}\rangle = -3(\kappa_1)^2 - 4\kappa_1'\kappa_1'' + k_1^{(4)} + k_1^2 k_2^2$. If the determinant of coefficient matrix is non-zero, then $u_i^{(5)}$ are easily found, while as if any of the equations in (56) vanishes, then that equation is replaced by $\langle(\gamma^1)^{(5)}, N_1^{(j)}\rangle = \langle(\gamma^i)^{(5)}, N_i^{(j)}\rangle$, $i = 2,3,4$. and $j = 1\cdots 5$.

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