Characterization of Transmuted Size-biased Exponential distribution and its Applications

Kaisar Ahmad[1], S.P. Ahmad[2]

Department of Statistics, University of Kashmir, Srinagar(India)

ABSTRACT

In this paper, two parameter Transmuted Size-biased Exponential distribution (TSED) is introduced. The characterizing properties of Transmuted Size-biased Exponential distribution are derived. The performance of TSED is examined by using real data sets and the results are compared with their sub families.

Key words: Transmuted Size-biased Exponential distribution, characteristic function, Reliability measures of TSED, real life applications.

I.INTRODUCTION

The exponential distribution is the most widely used lifetime model in reliability theory, because of its simplicity and mathematical feasibility. The exponential distribution occurs when describing the lengths of the inter-arrival times in a homogeneous Poisson process. Exponential variables can also be used to model situations where certain events occur with a constant probability per unit length, such as the distance between mutations on a DNA strand, or between road kills on a given road. Work by Epstein and Sobel [1] gave numerous results and popularized the exponential distribution as a lifetime distribution, especially in the area of industrial life testing.

The most important properties of the exponential distribution is the memory less property i.e., probability of its surviving an additional h hours is exactly the same as the probability of surviving h hours of a new item. This property says that events happens during a time interval of length x is independent of how much time has already elapsed without the event happening. The exponential distribution, because of memory-less property, is used for life testing of the products that do not age with time. There are several electronic devices whose failure rate does not depend upon their age and, therefore, the exponential distribution is considered.

The weighted distributions arise when the observations generated from a stochastic process are not given equal chance of being recorded; instead they are recorded according to some weighted function. When the weight function depends on measure of the unit size, resulting distribution is called size-biased. Size biased distributions are a special case of the more general form known as weighted distributions. First introduced by Fisher [2] to model ascertainment bias, these were later formalized in a unifying theory by Rao [3]. Ghitany and Al-Mutairi [4] studied the Size-biased Poisson-Lindley distribution.

A random variable x is said to have a Size biased exponential distribution with parameter $\alpha > 0$, if its pdf is given by

$$g(y) = \alpha^2 y e^{-\alpha y} \qquad ; y > 0,\ \alpha > 0 \tag{1.1}$$

The cdf of Size biased exponential distribution is given by

$$G(y) = \int_0^y g(y)dy$$

$$G(y) = \int_0^y \alpha^2 y e^{-\alpha y} dy$$

solving the above expression, we get

$$G(y) = 1 - (1+\alpha y)e^{-\alpha y}. \tag{1.2}$$

II.TRANSMUTED SIZE-BIASED EXPONENTIAL DISTRIBUTION (TSED)

The transmutation approach for the Size-biased exponential distribution is being executed in the same manner as proposed by Aryal and Chris [5] for transmuted Weibull distribution. Aryal et al. [6] constructed the transmuted Gumbel distribution and it has been observed that transmuted Gumbel distribution can be used to model climate data. Ahmad et al. [7] gave some contribution on transmuted Weibull distribution. A new transmuted additive Weibull distribution was proposed by Mansour et. al [8].

A random variable x is said to have transmuted distribution if its cumulative distribution function (cdf), $Z(y)$, is given by

$$Z(y) = (1+\lambda)G(y) - \lambda G(y)^2 \tag{2.1}$$

where G(y) is the cdf of base distribution. Observe that at $\lambda = 0$ we have the distribution of the base random variable.

The cdf, $Z(y)$, of transmuted Size-biased exponential distribution is given by

$$Z(y) = (1+\lambda)G(y) - \lambda G(y)^2; \qquad |\lambda| \le 1.$$

Combining eq. (1.2) and eq. (2.1), gives the cdf of the transmuted Size-biased exponential distribution as

$$F(y) = \left(1 - e^{-\alpha y}(1+\alpha y)\right)\left(1 + \lambda(1+\alpha y)e^{-\alpha y}\right) \tag{2.2}$$

where $\alpha > 0$ and $|\lambda| \leq 1$ are the scale and transmuted parameters respectively.

Graphical representation for Cdfs of TSED:

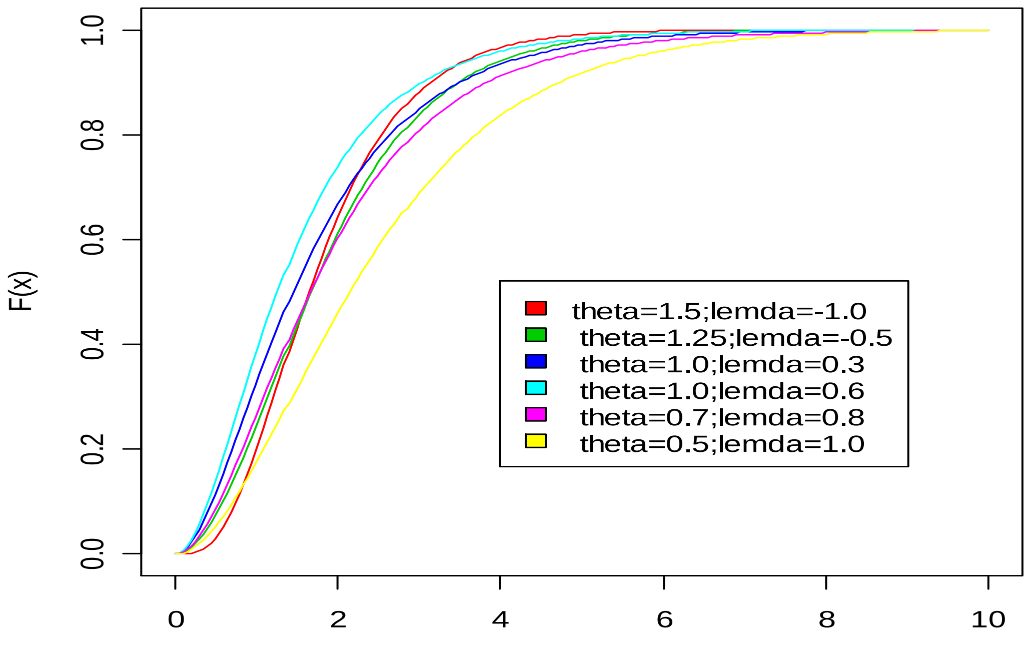

Figure 5.2.2: The cdfs of TSED for different values of theta and lemda

Differentiating eq. (5.2.2) with respect to y, we have

$$\frac{\partial}{\partial y}(F(y)) = \frac{\partial}{\partial y}\left[\left(1-e^{-\alpha y}(1+\alpha y)\right)\left(1+\lambda(1+\alpha y)e^{-\alpha y}\right)\right]$$

Thus the probability density function of transmuted Size-biased exponential distribution which is given by

$$f(y;\alpha,\lambda) = \alpha^2 y e^{-\alpha y}\left(1-\lambda+2\lambda(1+\alpha y)e^{-\alpha y}\right) \quad y \geq 0,\ \alpha > 0 \ ;|\lambda| \leq 1 \qquad (2.3)$$

Graphical representation for the pdfs of TSED:

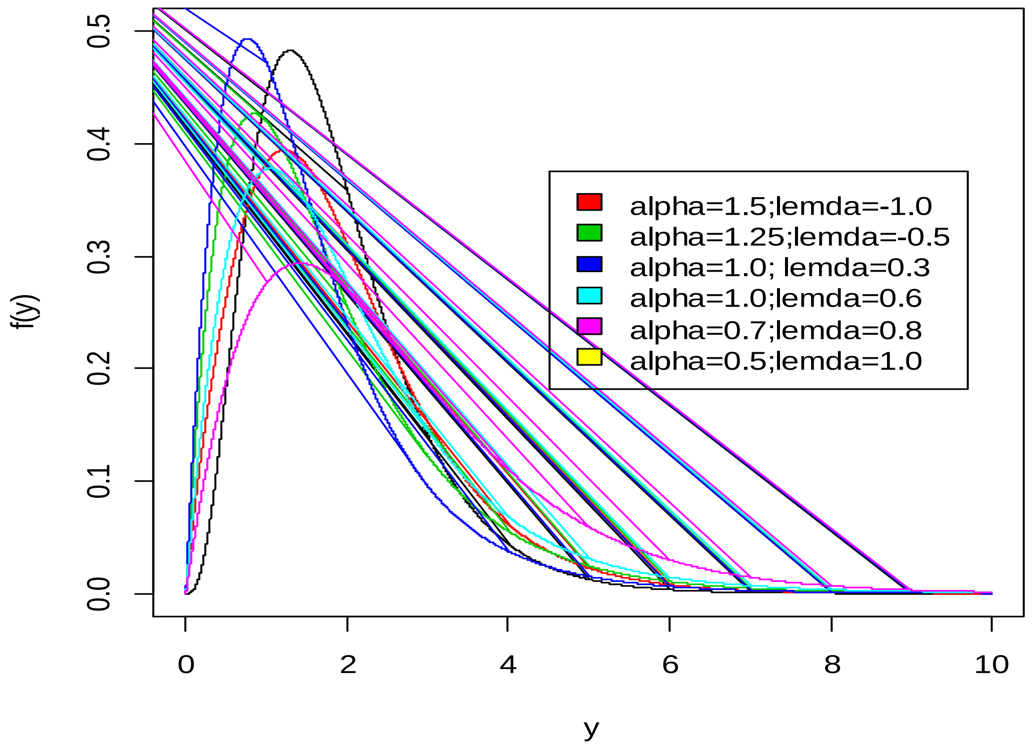

Figure 2:The pdf's of TSED under various values of alpha and lemda

III.STRUCTURAL PROPERTIES OF TSED

In this section, some structural properties of Transmuted Size-biased Exponential are discussed. We have used the similar approach as used by Ahmad et. al [9] for deriving the characterizing properties of Weibull distribution.

Theorem 3.1: Let $(y_1, y_2, \ldots, y_n)$ be a random sample of size n from the TSED with probability density function (2.3), then

$$E(y)^r = \frac{(1-\lambda)\Gamma(r+2)}{\alpha^r} + \frac{\lambda\Gamma(r+2)}{2(2\alpha)^r} + \frac{\lambda\Gamma(r+3)}{4(2\alpha)^r}.$$

Proof: Since we know that

$$E(y)^r = \int_0^\infty y^r f(y;\alpha,\lambda)dy \qquad (3.1.1)$$

using eq. (5.2.3) in eq. (5.3.1.1), we have

$$E(y)^r = \alpha^2 \int_0^\infty y^{r+1} e^{-\alpha y} \{1-\lambda+2\lambda(1+\alpha y)e^{-\alpha y}\} dy$$

$$E(y)^r = \alpha^2 \left[(1-\lambda)\int_0^\infty y^{r+1} e^{-\alpha y} dy + 2\lambda \left(\int_0^\infty y^{r+1} e^{-2\alpha y} dy + \alpha \int_0^\infty y^{r+2} e^{-2\alpha y} dy \right) \right]$$

on solving the above equation, we get

$$E(y)^r = \frac{(1-\lambda)\Gamma(r+2)}{\alpha^r} + \frac{\lambda\Gamma(r+2)}{2(2\alpha)^r} + \frac{\lambda\Gamma(r+3)}{4(2\alpha)^r}. \quad (3.1.2)$$

Remarks:

If r=1 in eq. (3.1.2), we get the mean of Transmuted Size-biased Exponential distribution which is given by

$$\mu_1' = \frac{(8-3\lambda)}{4\alpha}. \quad (3.1.3)$$

If r=2 in eq. (3.1.2), we have

$$\mu_2' = \frac{(24-15\lambda)}{4\alpha^2}. \quad (3.1.4)$$

Thus the variance of Transmuted Size-biased Exponential distribution is given by

$$\mu_2 = \frac{(8+3\lambda)(4-3\lambda)}{16\alpha^2}. \quad (3.1.5)$$

If r=3 in eq. (3.1.2), we have

$$\mu_3' = \frac{96-87\lambda}{4\alpha^3} \quad (3.1.6)$$

Thus $$\mu_3 = \frac{4(32-30\lambda)-9\lambda^2(6+3\lambda)}{32\alpha^3}. \quad (3.1.7)$$

If r=4 in eq. (3.1.2), we have

$$\mu_4' = \frac{960-915\lambda}{8\alpha^4} \quad (3.1.8)$$

Thus $$\mu_4 = \frac{9\lambda(112-27\lambda^3-216\lambda^2-192)+3415}{256\alpha^4}. \quad (3.1.9)$$

International Journal of Advance Research in Science and Engineering
Volume No.06, Special Issue No.(03), December 2017
www.ijarse.com
IJARSE
ISSN: 2319-8354

Theorem 3.2: Let $(y_1, y_2, \ldots, y_n)$ be n positive independent and identically distributed random samples drawn from TSED having probability density function (2.3), then

$$M_y(t) = \sum_{r=0}^{\infty} \frac{(t)^r}{r!} \left\{ \frac{(1-\lambda)\Gamma(r+2)}{\alpha^r} + \frac{\lambda\Gamma(r+2)}{2(2\alpha)^r} + \frac{\lambda\Gamma(r+3)}{4(2\alpha)^r} \right\}.$$

Proof: By the definition of moment generating function, we have

$$M_Y(t) = \int_0^{\infty} e^{ty} f(y)dy \qquad (3.2.1)$$

$$M_Y(t) = \int_0^{\infty} \left\{ 1 + (ty) + \frac{(ty)^2}{2!} + \frac{(ty)^3}{3!} + \ldots + \frac{(ty)^n}{n!} + \ldots \right\} f(y)dy$$

$$M_Y(t) = \int_0^{\infty} \sum_{r=0}^{\infty} \frac{(ty)^r}{r!} f(y)dy$$

$$M_Y(t) = \sum_{r=0}^{\infty} \frac{(t)^r}{r!} E(y)^r \qquad (3.2.2)$$

using eq. (3.1.2) in eq. (3.2.2), we have

$$M_Y(t) = \sum_{r=0}^{\infty} \frac{(t)^r}{r!} \left\{ \frac{(1-\lambda)\Gamma(r+2)}{\alpha^r} + \frac{\lambda\Gamma(r+2)}{2(2\alpha)^r} + \frac{\lambda\Gamma(r+3)}{4(2\alpha)^r} \right\}.$$

Theorem 3.3: Let $(y_1, y_2, \ldots, y_n)$ be n positive independent and identically distributed random samples drawn from TSED having probability density function (2.3), then

$$\phi_Y(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \left\{ \frac{(1-\lambda)\Gamma(r+2)}{\alpha^r} + \frac{\lambda\Gamma(r+2)}{2(2\alpha)^r} + \frac{\lambda\Gamma(r+3)}{4(2\alpha)^r} \right\}.$$

Proof: By the definition of characteristic function, we have

$$\phi_Y(t)=\int_0^\infty e^{ity} f(y)dy \qquad (3.3.1)$$

$$\phi_Y(t)=\int_0^\infty \left\{1+(ity)+\frac{(ity)^2}{2!}+\frac{(ity)^3}{3!}+\ldots+\frac{(ity)^n}{n!}+\ldots\right\} f(y)dy$$

$$\phi_Y(t)=\int_0^\infty \sum_{r=0}^\infty \frac{(ity)^r}{r!} f(y)dy$$

$$\phi_Y(t)=\sum_{r=0}^\infty \frac{(it)^r}{r!} E(y)^r \qquad (3.3.2)$$

using eq. (3.1.2) in eq. (3.3.2), we have

$$\phi_Y(t)=\sum_{r=0}^\infty \frac{(it)^r}{r!}\left\{\frac{(1-\lambda)\Gamma(r+2)}{\alpha^r}+\frac{\lambda\Gamma(r+2)}{2(2\alpha)^r}+\frac{\lambda\Gamma(r+3)}{4(2\alpha)^r}\right\}.$$

Theorem 3.4: Let $(y_1, y_2, \ldots, y_n)$ be a random sample of size n from the TSED with probability density function (2.3), then

$$H.M.=\left(\frac{1}{\alpha(2+\lambda)/2}\right)$$

Proof: By the definition of harmonic mean, we have

$$\frac{1}{H}=\int_0^\infty \frac{1}{y} f(y;\alpha,\lambda)dy. \qquad (3.4.1)$$

Using eq. (2.3) in eq. (3.4.1), we have

$$\frac{1}{H}=\int_0^\infty \frac{1}{y}\alpha^2 ye^{-\alpha y}\left(1-\lambda+2\lambda(1+\alpha y)e^{-\alpha y}\right)dy$$

$$\frac{1}{H}=\alpha^2(1-\lambda)\int_0^\infty e^{-\alpha y}dy+2\lambda\alpha^2\int_0^\infty e^{-2\alpha y}(1+\alpha y)dy \qquad (3.4.2)$$

Solving the eq. (3.4.2), we have

$$\frac{1}{H}=\frac{\alpha(2+\lambda)}{2}$$

Therefore $H.M.=\left(\frac{1}{\alpha(2+\lambda)/2}\right)$

IV.RELIABILITY MEASURES OF TSED

In this section, the reliability function and the hazard function for the proposed transmuted Size-biased Exponential distribution are discussed. The reliability function is also known as the survival or survivor function. It is the probability that a system will survive beyond a specified time and it is obtained mathematically as the complement of the cumulative density function (cdf).

The survival function is given by

$$S(y)=1-F(y) \tag{4.1}$$

Using eq. (2.2) in eq. (4.1), we have

$$s(y,\alpha,\lambda)=1-\left(1-e^{-\alpha y}(1+\alpha y)\right)\left(1+\lambda(1+\alpha y)e^{-\alpha y}\right) \quad y>0,\alpha>0,-1\le\lambda\le 1 \tag{4.2}$$

The hazard function also known as the hazard rate or failure rate. This is an important quantity characterizing life phenomenon. It can be interpreted as the conditional probability of failure, given it has survived to time y.

The hazard rate function of Transmuted Size-biased Exponential distribution is given by

$$h(y)=\frac{f(y)}{s(y)} \tag{4.3}$$

Using eq. (2.3) and eq. (4.2) in eq. (4.3), we have

$$h(y,\alpha,\lambda)=\frac{\alpha^2 ye^{-\alpha y}\left(1-\lambda+2\lambda(1+\alpha\, y)e^{-\alpha y}\right)}{1-\left(1-e^{-\alpha y}(1+\alpha y)\right)\left(1+\lambda(1+\alpha y)e^{-\alpha y}\right)} \quad y>0,\alpha>0,-1\le\lambda\le 1.$$

V.APPLICATIONS OF TRANSMUTED SIZE-BIASED EXPONENTIAL DISTRIBUTION (TSED)

The flexibility and potentiality of the transmuted Size-biased exponential distribution in fitting various data sets is examined with the help of following illustrations.

Illustration I: The data set given by Lawless [10] which represents the repair times (in hours) for 46 failures of an airborne communications receiver. The data are as follows

International Journal of Advance Research in Science and Engineering
Volume No.06, Special Issue No.(03), December 2017
www.ijarse.com
IJARSE
ISSN: 2319-8354

(0.2, 0.3, 0.5, 1.5, 0.5, 0.5, 0.6, 0.6, 0.7, 0.7, 0.7, 0.11, 0.8 ,1.0, 1.0, 1.0, 1.0, 1.1, 1.3, 1.5 ,1.5, 1.5, 1.5,2.0,2.0, 2.2, 2.5, 2.7 ,3.0 ,3.0, 3.3, 3.3 ,4.0,4.0,4.5, 4.7, 5.0 ,5.4 ,5.4 ,7.0 ,7.5 ,8.8 ,9.0 ,10.3, 22.0 and 24.5).

Treating the time as exact continuous observations fit a distribution to the data. Check whether the two largest repair times seem consistent with the Transmuted Size-Biased Exponential model.

Table 5.1: AIC, BIC and AICC criterion for SED and TSED:

Model	Parameter Estimate	Standard Error	Measures			
			$-2\log l$	AIC	BIC	AICC
Size-Biased Exponential	$\hat{\alpha}$ = 0.55351	0.05770	234.0282	236.0282	237.8569	236.1191
Transmuted Size-Biased Exponential	$\hat{\alpha}$ =0.46210	0.06093	227.5016	231.5017	235.1590	231.7808
	$\hat{\lambda}$ =0.60515	0.20660				

Illustration II: The data set is taken from Xu et al. [11] and it represents the time to failure ($10^3 h$) of turbocharger of one type of engine. The data are as follows

(1.6, 3.5, 4.8, 5.4, 6.0, 6.5 ,7.0 ,7.3 ,7.7 ,8.0 ,8.4 ,2.0 ,3.9 ,5.0 ,5.6 ,6.1 ,6.5 ,7.1, 7.3, 7.8, 8.1, 8.4, 2.6, 4.5, 5.1, 5.8, 6.3, 6.7, 7.3, 7.7, 7.9, 8.3, 8.5, 3.0 ,4.6 ,5.3 ,6.0 ,8.7 ,8.8 and 9.0).

Treating the time as exact continuous observations fit a suitable distribution to the data.

Table 5.2: AIC, BIC and AICC criterion for SED and TSED:

Model	Parameter Estimate	Standard Error	Measures			
			$-2\log l$	AIC	BIC	AICC
Size-Biased Exponential	$\hat{\alpha}$ = 0.31987	0.03576	201.0261	203.0261	204.7150	203.1261
Transmuted Size-Biased Exponential	$\hat{\alpha}$ = 0.43118	0.05055	185.2386	189.2386	192.6163	189.5386
	$\hat{\lambda}$ = -1.0000	0.40289				

5.3 Results and Discussion for TSED:

On examining the summary statistics of the data sets which are fitted to the transmuted Size-biased exponential distribution and Size-biased exponential distribution. We estimate the unknown parameters of each distribution. Also Akaike information criterion (AIC), Bayesian information criterion (BIC), and the corrected Akaike information criterion (AICC) are used to compare the distributions. The best distribution corresponds to lower -2logL, AIC, BIC, AICC value. The results presented in the above tables shows that transmuted Size-biased exponential distribution provides adequate fit to the data among the models considered.

VI.CONCLUSION

In this study we have obtained some structural properties of TSED and results are presented in the above tables. From the above results, we may conclude that transmuted Size-biased exponential distribution provides good fit to the data and is better as compare to its sub model.

VII.ACKNOWLEDGEMENT

Authors wish to thank the referees for their valuable comments which led to the improvement of the paper and paper is accepted for publication.

REFERENCES

[1] Epstein, B. And Sobel, M. (1955). Sequential Life Testing In Exponential Case. *Annals of the Mathematical Statistics*, 26, 82-93.

[2] Fisher, R.A (1934).The effects of methods of ascertainment upon the estimation of frequencies. *Ann. Eugenics*, 6, 13-25.

[3] Rao, C.R (1965): On discrete distributions arising out of method of ascertainment, in classical and Contagious Discrete, *G.P. Patil .ed ;Pergamum Press and Statistical publishing Society, Calcutta,* pp-320-332.

[4] Ghitany, M. E., and Al-Mutairi, D. K. (2008): Size-biased Poisson-Lindley distribution and its application, *METRON-International Journal of Statistics*. Vol. LXVI, n. 3, pp. 299-311.

[5] Gokarna R. Aryal and Chris P. Tsokos (2011): Transmuted Weibull Distribution: A Generalization of the Weibull Probability Distribution; *European Journal of Pure and Applied Mathematics* Vol. 4, No. 2, 89-102.

[6] Gokarna R. Aryal and Chris P. Tsokos (2009). On the transmuted extreme value distribution with application. Nonlinear Analysis: *Theory, Methods and Applications,* 71:1401–1407.

[7] Ahmad, K., Ahmad, S. P., and Ahmed, A. (2015). "Structural Properties of Transmuted Weibull Distribution". *Journal of Modern Applied Statistical Methods:* Vol. 14(2), 141-158.

[8] Mansour, M.M., Enayat, M. A., Mohamed S.H. and Salah, M. M. (2015): A New Transmuted Additive Weibull Distribution Based on a New Method for Adding a Parameter to a Family of Distributions. *International Journal of Applied Mathematical Sciences.*

[9] Kaisar Ahmad, S.P. Ahmad and A. Ahmed (2014): Some Important Characterizing Properties, Information Measures and Estimations of Weibull Distribution. *International Journal of Modern Mathematical Sciences, 12(2): 88-97.*

[10] Lawless, J. F., (2003): *Statistical models and methods for life time data*. *Second edition, Wiley, New York.*

[11] Xu, K., Xie, M., Tang, L.C. and Ho, S.L (2003): Application of neural networks in forecasting engine systems reliability, *Applied Soft Computing*, 2 (4), 255-268.