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SOME EXTENSIONS OF RHOADES FIXED POINT THEOREMS

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ABSTRACT

Nadler found a fixed point for the mapping defined on product of metric spaces which are uniformly continuous and also contraction in the first variable. Tarafdar generalized the Banach contraction principle on a complete Hausdorff uniform spaces. In this paper we generalize the result of Nadler according to technique of Tarafdar in some contractive conditions of Rhoades. Here we discuss only those conditions which involve a single mapping.

Keywords: Metric Space, Uniform space, Locally Compact Space, Product Space, Uniformly Continuous Mapping, Contraction in the first variaSble.

I. INTRODUCTION

Definition: A topological space X is said to have fixed point property(f. p. p.) if every continuous function f: X$\rightarrow$ X has a fixed point.

The problem of whether the f. p. p. is or is not necessary invariant under cartesian products is an old one (see [2] and [3] for its history). The f. p. p. is preserved when the maps f: XxZ$\rightarrow$XxZ have special contraction properties. Nadler [5] main results are as follows:

1.1 Theorem

Let (X, d) be a metric space. Let $A_i : X \rightarrow X$ be a function with at least one fixed point a_i for each i =1,2,……., and let $A_0 : X \rightarrow X$ be a contraction mapping with fixed point a_0. If the sequence $\{A_i\}$ converges uniformly to A_0, then the sequence $\{a_i\}$ converges to a_0.

1.2 Theorem

Let (X, d) be a locally compact metric space. Let $A_i : X \rightarrow X$ be a contraction mapping with fixed point a_i for each i =1,2,…….., and let $A_0 : X \rightarrow X$ be a contraction mapping with fixed point a_0. If the sequence $\{A_i\}$ converges pointwise to A_0, then the sequence $\{a_i\}$ converges to a_0.

1.3 Theorem

Let (X, d) be a complete metric space, Z a metric space which has the f. p. p. and f:XxZ$\rightarrow$XxZ be a contraction in the first variable.

(a) If f is uniformly continuous, then f has a fixed point.

(b) If (X, d) is locally compact, f is continuous, then f has a fixed point.

We extend the class of complete metric spaces X to the class of complete Hausdorff uniform spaces and the class of metric spaces Z to the class of uniform spaces in which sequences are adequate.

II. SOME DEFINITIONS FROM RHOADES [6]

Let (X, d) be a complete metric space and f:X $\to$ X be a mapping. For x $\in$X, let 0(x)={x, f(x), f^2(x), ……} be the orbit of x under f. Consider the following conditions on f and (X, d):

(1)' (Dass and Gupta) – There exist numbers α, β>0, α+ β<1 and for each x, x_* $\in$ X, x_* $\in$0 (x) such that

$$d(f(x), f(x_*)) \le \alpha \frac{d(x_*, f(x_*))[1+d(x, f(x))]}{1+d(x,x_*)} + \beta d(x,x_*)$$

(2)' (Jaggi and Dass) – There exist numbers α, β≥0, α+β<1 and for each x, x_* $\in$ X, x $\neq x_*$, $x_*\in$0(x) such that

$$d(f(x), f(x_*)) \le \alpha \frac{d(x, f(x))d(x_*, f(x_*))}{d(x, f(x_*))+d(x_*, f(x))+d(x,x_*)} + \beta d(x,x_*)$$

(3)' (Gupta and Saxena)– There exist numbers a, b, c≥0, a+b+c<1 and for each x,$x_*\in$X,$x_*\in$0(x) such that

$$d(f(x), f(x_*)) \le \frac{a[1+d(x, f(x))]d(x_*, f(x_*))}{1+d(x,x_*)} + \frac{bd(x, f(x))d(x_*, f(x_*))}{d(x,x_*)} + cd(x,x_*)$$

(4)' (Jaggi) – There exist numbers α, β≥0, α+ β<1 and for each x, x_* $\in$ X, x $\neq x_*$, $x_*\in$0 (x) such that

$$d(f(x), f(x_*)) \le \alpha \frac{d(x, f(x))d(x_*, f(x_*))}{d(x,x_*)} + \beta d(x,x_*)$$

(5)' (Khan) – There exists a number k, 0 $\le$ k<1 and for each x, x_* $\in$ X, x_* $\in$0 (x) such that

$$d(f(x), f(x_*)) \le k \frac{d(x, f(x))d(x, f(x_*))+d(x_*, f(x_*)).d(x_*, f(x))}{d(x, f(x_*))+d(x_*, f(x))}$$

(6)' (Jain and Dixit) – There exist α_i , β_i $\ge 0, \alpha_1 + 2\alpha_3 + 2\alpha_4 + \beta_1 + \beta_2 + \beta_3 + 2\beta_5 < 1$, $\alpha_2 + \beta_1 + \beta_4 + \beta_5 <1$ and for each x, $x_*\in$ X, x $\neq x_*$, $x_*\in$0 (x) such that

$$d(f(x), f(x_*)) \le \alpha_1 \frac{d(x, f(x)).d(x_*, f(x_*))}{d(x,x_*)} + \alpha_2 \frac{d(x, f(x_*)).d(x_*, f(x))}{d(x,x_*)} + \alpha_3 \frac{d(x_*, f(x)).d(x_*, f(x_*))}{d(x,x_*)}$$
$$+\alpha_4 \frac{d(x, f(x)).d(x_*, f(x_*))}{d(x,x_*)} + \beta_1 d(x,x_*) + \beta_2 d(x, f(x)) + \beta_3 d(x_*, f(x_*)) + \beta_4 d(x, f(x_*))$$
$$+\beta_5 d(x_*, f(x))$$

(7)' (Sharma and Bajaj) – There exist a number β, 0 < β < ½ and for each x, $x_*\in$ X, x_* $\in$0 (x) such that

$$d(f(x), f(x_*)) \le \beta \frac{d(x, f(x)).d(x, f(x_*))}{d(x, f(x))+d(x, f(x_*))}$$

(8)' (Dass) – There exist numbers α_i , β_j>0 with $\alpha_1+\alpha_2+\alpha_3+\sum_{j=1}^{5}\beta_j<1$ for each positive integer m, and for each x, $x_* \in X$, $x \neq x_*$, $x_* \in 0$ (x) such that

$$d(f^m(x), f^m(x_*)) \le \alpha_1 \frac{d(x, f^m(x)).d(x_*, f^m(x_*))}{d(x,x_*)} + \alpha_2 \frac{d(x, f^m(x)).d(x_*, f^m(x))}{d(f^m(x), f^m(x_*))}$$
$$+\alpha_3 \frac{d(x, f^m(x_*)).d(x_*, f^m(x_*))}{d(f^m(x), f^m(x_*))} + \beta_1 d(x,x_*) + \beta_2 d(x, f^m(x))$$
$$+\beta_3 d(x_*, f^m(x_*)) + \beta_4 d(x, f^m(x_*)) + \beta_5 d(x_*, f^m(x))$$

(9)' (Pachpatte Thm.1) – There exists a number $q_1 \in (0,1)$, and for each x, $x_* \in X$, $x \neq x_*$, $x_* \in 0$ (x) such that

$$d(f(x), f(x_*)) \le q_1 \max\left\{d(x,x_*), \frac{d(x, f(x)).d(x_*, f(x_*))}{d(x,x_*)}, \frac{d(x, f(x_*)).d(x_*, f(x))}{d(x,x_*)}, \frac{d(x, f(x)).d(x, f(x_*))}{2d(x,x_*)}\right\}$$

(10)' (Pachpatte Thm.2) – There exists a number $q_2 \in (0,1)$, and for each x, $x_* \in X$, $x \neq x_*$, $x_* \in 0(x)$ such that

$$\min\left\{d(f(x), f(x_*)), d(x, f(x)), d(x_*, f(x_*)), \frac{d(x, f(x)).d(x_*.f(x_*))}{d(x,x_*)}\right\} -$$
$$\min\left\{\frac{d(x, f(x_*)).d(x_*, f(x))}{d(x,x_*)}, \frac{d(x, f(x)).d(x, f(x_*))}{d(x,x_*)}\right\} \le q_2 d(x,x_*)$$

In what follows, X will denote a complete Hausdorff uniform space, Z a uniform space in which sequences are adequate and f:XxZ→XxZ be a mapping. For a fixed $z \in Z$, f_z: X→X be a mapping which is defined as $f_z(x)=\pi_1 f(x,z)$ for all $x \in X$, where π_1 is the projection of XxZ on X along Z. (m)', $1 \le m \le 10$; will denote the condition (m)' in Rhoades [6] with the modification that constant or functions that appear in (m)' depend on z.

2.1 Theorem

Let (X, **u**) be a complete Hausdorff uniform space, Z a uniform space in which sequences are adequate which has the f. p. p. and let f: XxZ→XxZ be a mapping and **u**=$\{\rho_\alpha : \alpha \in I\}$.

(a) If f is uniformly continuous and $f_z \in (3)'$ for all $\alpha \in I$ and all $z \in Z$, then f has a fixed point.

(b) If X is locally compact, f is continuous and $f_z \in (3)'$ for all $\alpha \in I$ and all $z \in Z$, then f has a fixed point.

Proof: We prove (a) and (b) simultaneously:

Step I: $\{t_n\}$ is a ρ_α - Cauchy sequence for each $\alpha \in I$

We construct a sequence $t_n(z) = t_n$ in X as follows:

For a fixed x_0 in X and for any $z \in Z$, $t_0 = x_0$, $t_n = \pi_1 f(t_{n-1}, z) = f_z(t_{n-1}) = f_z^n(t_0)$; $n \ge 1$

Let A^*(**u**)=$\{\rho_\alpha : \alpha \in I\}$ be the augmented associated family of pseudometrics for **u** on X,

Since $f_z \in (3)'$. Let $\alpha \in I$ be arbitrary. Then for x, $x_* \in X$, $x_* \in 0(x)$, there exist a, b, c ≥ 0 with a + b+ c < 1, we have

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$$\rho_\alpha(f_z(x), f_z(x_*)) \le \frac{a[1+\rho_\alpha(x, f_z(x))].\rho_\alpha(x_*, f_z(x_*))}{1+\rho_\alpha(x, x_*)} + \frac{b.\rho_\alpha(x, f_z(x)).\rho_\alpha(x_*, f_z(x_*))}{\rho_\alpha(x, x_*)} + c.\rho_\alpha(x, x_*)$$

Set $x_* = f_z(x)$ in the above inequality to obtain

$$\rho_\alpha(f_z(x), f_z^2(x)) \le \left(\frac{c}{1-a-b}\right).\rho_\alpha(x, f_z(x))$$

Now, set $x = x_*$, then we have

$$\rho_\alpha(f_z(x_*), f_z^2(x_*)) \le \left(\frac{c}{1-a-b}\right).\rho_\alpha(x_*, f_z(x_*))$$

Repeating above substitute we obtain

$$\rho_\alpha(f_z^2(x), f_z^3(x)) \le \left(\frac{c}{1-a-b}\right)^2.\rho_\alpha(x, f_z(x))$$

Using induction, we get

$$\rho_\alpha(f_z^n(x), f_z^{n+1}(x)) \le \left(\frac{c}{1-a-b}\right)^n.\rho_\alpha(x, f_z(x))$$

Finally set $x = x_0$, we get

$$\rho_\alpha(t_n, t_{n+1}) \le h_\alpha^n.\rho_\alpha(t_0, t_1), \quad \text{where } h_\alpha = \left(\frac{c}{1-a-b}\right) < 1$$

Using triangle inequality we find, for $m > n$

$$\rho_\alpha(t_n, t_m) \le \rho_\alpha(t_n, t_{n+1}) + \rho_\alpha(t_{n+1}, t_{n+2}) + \ldots. + \rho_\alpha(t_{m-1}, t_m)$$

$$\le (h_\alpha^n + h_\alpha^{n+1} + \ldots. + h_\alpha^{m-1}).\rho_\alpha(t_0, t_1)$$

$$= \frac{h_\alpha^n(1-h_\alpha^{m-n}).\rho_\alpha(t_0, t_1)}{1-h_\alpha}$$

$$< \frac{h_\alpha^n.\rho_\alpha(t_0, t_1)}{1-h_\alpha}$$

Since $h_\alpha^n \to 0$ as $n \to \infty$, this inequality shows that $\{t_n\}$ is a Cauchy sequence (i.e. a Cauchy sequence in ρ_α-topology). Since $\alpha \in I$ is arbitrary, $\{t_n\}$ is a ρ_α- Cauchy sequence for each $\alpha \in I$.

Step II: Fixed point of f_z in X [7]

Let $S_p = \{t_n : n \ge p\}$ for all positive integer p and let $B = \{S_p : p = 1, 2\ldots.\}$be the filter basis. It is easy to see the filter basis B is a Cauchy in the uniform space (X, **u**). To see this we first note that the family $\{H(\alpha, \in) : \alpha \in I, \in > 0\}$ is a base for **u**. Now let $H \in$ **u** be an arbitrary entourage. Then there exists a $v \in I$ and $\in > 0$ such that $H(v, \in) \subset H$. Since $\{t_n\}$ is a ρ_v-Cauchy sequence in X, there exists a positive integer p such that $\rho_v(t_n, t_m) < \in$ for

$m \geq p$, $n \geq p$ this implied that $S_p \times S_p \subset H(v, \in)$.Thus given any $H \in \mathbf{u}$ we can find a $S_p \in B$ such that $S_p \times S_p \subset H$. Hence B is a Cauchy filter in (X, **u**). Since (X, **u**) is complete and Hausdorff, the Cauchy filter B $=\{S_p\}$converges to a unique point $p_1 \in X$ in the τ_u topology (uniform topology induced by uniformity **u**). Thus $\tau_u \lim S_p = p_1$. Now since f_z is ρ_α - continuous for each $\alpha \in I$, it follows that f_z is τ_u continuous. Hence $f_z(p_1) = f_z(\tau_u \lim S_p) = \tau_u \lim f_z(S_p) = \tau_u \lim S_{p+1} = p_1$.Thus p_1 is a fixed point of f_z. Here p_1 is unique fixed point of f_z as if we assume p_2 is another fixed point of f_z such that $p_1 \neq p_2$. Since (X, **u**) is a Hausdorff space and $p_1 \neq p_2$ there is an index $\beta \in I$ such that $\rho_\beta(p_1, p_2) \neq 0$. Since f_z is a contraction on X, we have

$\rho_\beta(p_1, p_2) = \rho_\beta(f_z(p_1), f_z(p_2)) \leq h_{\beta.}\, \rho_\beta(p_1, p_2)$

Which is absurd as $0 < h_\beta < 1$ and $\rho_\beta(p_1, p_2) \neq 0$. Hence p_1 is unique fixed point of f_z.

Step III: Fixed point of f in XxZ

Let $F : Z \to X$ be given by $F(z) = p_1$ the unique fixed point of f_z. Now let $z_0 \in Z$ and let $\{z_i\}$ be a sequence of points of Z which converges to z_0. By the assumption of (a) for this theorem, the sequence $\{f_{zi}\}$ converges uniformly to f_{z0} and hence, by Theorem 1.1, the sequence $\{F(z_i)\}$ converges to $F(z_0)$. Under the assumption of (b) we may apply Theorem 1.2, to conclude that the sequence $\{F(z_i)\}$ converges to $F(z_0)$. Hence in either case, this proves that F is continuous on Z. Next let $G: Z \to Z$ be the continuous mapping defined by $G(z) = \pi_2 f(F(z),z)$ for each $z \in Z$, where π_2 is the projection of X x Z on Z along X. Since Z has the f.p.p. there is a point $p \in Z$ Such that $G(p) = p$. Therefore $p = G(p) = \pi_2 f(F(p),p)$. It follows that (F(p), p) is a fixed point of f. This completes the proof of the theorem.

We observe that condition $(1)'$, $(4)'$ are stronger than $(3)'$, therefore the above theorem 2.1 has two corollaries corresponding to each of these two conditions.

2.2 Corollary

(a) If f is uniformly continuous on XxZ and if for each $z \in Z$, there exist numbers $\alpha, \beta > 0$, $\alpha + \beta < 1$ and for each $x, x_* \in X$, $x_* \in 0(x)$ such that

$$\rho_\alpha(f_z(x), f_z(x_*)) \leq \alpha \frac{\rho_\alpha(x_*, f_z(x_*))[1+\rho_\alpha(x, f_z(x))]}{1+\rho_\alpha(x, x_*)} + \beta.\rho_\alpha(x, x_*)$$

Then f has a fixed point.

(b) If X is locally compact, f is continuous and for each $z \in Z$, there exist numbers $\alpha, \beta > 0$, $\alpha + \beta < 1$ such that the inequality in (a) is satisfied, then f has a fixed point.

2.3 Corollary

(a) If f is uniformly continuous on XxZ and if for each $z \in Z$, there exist numbers $\alpha, \beta \geq 0$, $\alpha + \beta < 1$ and for each $x, x_* \in X$, $x_* \in 0(x)$ such that

$$\rho_\alpha(f_z(x), f_z(x_*)) \leq \alpha \frac{\rho_\alpha(x, f_z(x)).\rho_\alpha(x_*, f_z(x_*))]}{\rho_\alpha(x, x_*)} + \beta.\rho_\alpha(x, x_*)$$

Then f has a fixed point.

(b) If X is locally compact, f is continuous and for each $z \in Z$, there exist numbers $\alpha, \beta \geq 0$, $\alpha + \beta < 1$ such that the inequality in (a) is satisfied, then f has a fixed point.

2.4 Theorem: Let (X, **u**) be a complete Hausdorff uniform space, Z a uniform space in which sequences are adequate which has the f. p. p. and let $f:XxZ \to XxZ$ be a mapping and $\mathbf{u}=\{\rho_\alpha : \alpha \in I\}$

(a) If f is uniformly continuous such that for each $z \in Z$, f_z satisfies any one of the conditions $(2)'$, $(5)'$, $(6)'$, $(7)'$, $(8)'$, $(9)'$ and $(10)'$, then f has a fixed point.

(b) If X is locally compact, f is continuous such that for each $z \in Z$, f_z satisfies any one of the conditions $(2)'$, $(5)'$, $(6)'$, $(7)'$, $(8)'$, $(9)'$ and $(10)'$, then f has a fixed point.

Proof: We prove (a) and (b) simultaneously:

Step I: $\{t_n\}$ is a ρ_α - Cauchy sequence for each $\alpha \in I$

We define a sequence $t_n(z)= t_n$ in X as follows:

For a fixed x_0 in X and any $z \in Z$,

$$f_z^0(x_0) = t_0,\ t_n = f_z^n(x_0) = \pi_1 f\,(f_z^{n-1}(x_0), z); n \geq 1$$

Let $A^*(\mathbf{u})=\{\rho_\alpha : \alpha \in I\}$ be the augmented associated family of pseudo-metrics for **u** on X,

If f is such that $f_z \in (2)'$ and apply $x_* = f_z(x)$ then we have

$$\rho_\alpha(f_z(x), f_z^2(x)) \leq \alpha.\frac{\rho_\alpha(x, f_z(x)).\rho_\alpha(f_z(x), f_z^2(x))}{\rho_\alpha(x, f_z^2(x)) + \rho_\alpha(f_z(x), f_z^2(x)) + \rho_\alpha(x, f_z(x))} + \beta.\rho_\alpha(x, f_z(x))$$

$$\leq \alpha.\rho_\alpha(f_z(x), f_z^2(x)) + \beta.\rho_\alpha(x, f_z(x))$$

or $\rho_\alpha(f_z(x), f_z^2(x)) \leq \left(\dfrac{\beta}{1-\alpha}\right).\rho_\alpha(x, f_z(x))$

Let $x = x_*$ in above inequality we have

$$\rho_\alpha(f_z(x_*), f_z^2(x_*)) \leq \left(\frac{\beta}{1-\alpha}\right).\rho_\alpha(x_*, f_z(x_*))$$

Again set $x_* = f_z(x)$, then we can obtain

$$\rho_\alpha(f_z^2(x), f_z^3(x)) \leq \left(\frac{\beta}{1-\alpha}\right)^2.\rho_\alpha(x, f_z(x))$$

By the induction we can write above relation as

$$\rho_\alpha(f_z^n(x), f_z^{n+1}(x)) \leq \left(\frac{\beta}{1-\alpha}\right)^n.\rho_\alpha(x, f_z(x))$$

Finally set $x = x_0$, then we obtain

$$\rho_\alpha(t_n, t_{n+1}) \leq \left(\frac{\beta}{1-\alpha}\right)^n.\rho_\alpha(t_0, t_1) \tag{1}$$

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Here we note that if the function f: XxZ→XxZ is such that $f_z \in (5)'$ then by using similar arguments, we get

$$\rho_\alpha(t_n, t_{n+1}) \le k^n . \rho_\alpha(t_0, t_1) \qquad (2)$$

Similarly if f is such that $f_z \in (6)'$ then we can obtain, the condition

$$\rho_\alpha(t_n, t_{n+1}) \le \left(\frac{\beta_1 + \beta_2 + \beta_4}{1 - \alpha_1 - \beta_3 - \beta_4}\right)^n . \rho_\alpha(t_0, t_1) \qquad (3)$$

Likewise if f is such that $f_z \in (7)'$ then we obtain

$$\rho_\alpha(t_n, t_{n+1}) \le \beta^n . \rho_\alpha(t_0, t_1) \qquad (4)$$

Now, if f is such that $f_z \in (9)'$ then we can obtain

$$\rho_\alpha(t_n, t_{n+1}) \le q_1^n . \rho_\alpha(t_0, t_1) \qquad (5)$$

If f is such that $f_z \in (10)'$ then we can obtain

$$\rho_\alpha(t_n, t_{n+1}) \le q_2^n . \rho_\alpha(t_0, t_1) \qquad (6)$$

Finally if the function f is such that $f_z \in (8)'$, then to obtain a condition of the type above, we proceed as follows:

Define $g_1 = f_z^m$, then we have

$$\rho_\alpha(g_1(x), g_1(x_*)) \le \alpha_1 \frac{\rho_\alpha(x, g_1(x)).d(x_*, g_1(x_*))}{\rho_\alpha(x, x_*)} + \alpha_2 \frac{\rho_\alpha(x, g_1(x)).d(x_*, g_1(x))}{\rho_\alpha(g_1(x), g_1(x_*))}$$

$$+ \alpha_3 \frac{\rho_\alpha(x, g_1(x_*)).\rho_\alpha(x_*, g_1(x_*))}{\rho_\alpha(g_1(x), g_1(x_*))} + \beta_1\rho_\alpha(x, x_*) + \beta_2\rho_\alpha(x, g_1(x)) \qquad (7)$$

Using symmetry in above equation (7), we have

$$+ \beta_3\rho_\alpha(x, g_1(x)) + \beta_4\rho_\alpha(x, g_1(x_*)) + \beta_5\rho_\alpha(x_*, g_1(x))$$

$$\rho_\alpha(g_1(x_*), g_1(x)) \le \alpha_1 \frac{\rho_\alpha(x_*, g_1(x_*)).\rho_\alpha(x, g_1(x))}{\rho_\alpha(x_*, x)} + \alpha_2 \frac{\rho_\alpha(x_*, g_1(x_*)).\rho_\alpha(x, g_1(x_*))}{\rho_\alpha(g_1(x_*), g_1(x))}$$

$$+ \alpha_3 \frac{\rho_\alpha(x_*, g_1(x)).\rho_\alpha(x, g_1(x))}{\rho_\alpha(g_1(x_*), g_1(x))} + \beta_1\rho_\alpha(x_*, x) + \beta_2\rho_\alpha(x_*, g_1(x_*)) \qquad (8)$$

Adding equation $+ \beta_3\rho_\alpha(x, g_1(x)) + \beta_4\rho_\alpha(x_*, g_1(x)) + \beta_5\rho_\alpha(x, g_1(x_*))$ (7) and (8) above, we get

$$\rho_\alpha(g_1(x), g_1(x_*)) \le \gamma_1 \frac{\rho_\alpha(x, g_1(x)).\rho_\alpha(x_*, g_1(x_*))}{\rho_\alpha(x_*, x)}$$

$$+ \gamma_2 \frac{[\rho_\alpha(x, g_1(x)).\rho_\alpha(x_*, g_1(x)) + \rho_\alpha(x, g_1(x_*))\rho_\alpha(x_*, g_1(x_*))]}{\rho_\alpha(g_1(x), g_1(x_*))} + \gamma_3\rho_\alpha(x, x_*)$$

where,

$$+ \gamma_4[\rho_\alpha(x, g_1(x)) + \rho_\alpha(x_*, g_1(x_*))] + \gamma_5[\rho_\alpha(x, g_1(x_*)) + \rho_\alpha(x_*, g_1(x))]$$

$$\gamma_1 = \alpha_1, \quad \gamma_2 = \frac{\alpha_2 + \alpha_3}{2}, \quad \gamma_3 = \beta_1, \quad \gamma_4 = \frac{\beta_2 + \beta_3}{2}, \quad \text{and} \quad \gamma_5 = \frac{\beta_4 + \beta_5}{2}$$

with $\gamma_1 + 2\gamma_2 + \gamma_3 + 2\gamma_4 + 2\gamma_5 = \alpha_1 + \alpha_2 + \alpha_3 + \sum_{i=1}^{5} \beta_i < 1$

(9)

In equation (9), we apply similar procedure described above for equation (1) with g_1 referred as f_z then we have

$$\rho_\alpha(t_n, t_{n+1}) \le \left(\frac{\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5}{1 - \gamma_1 - \gamma_2 - \gamma_4 - \gamma_5} \right)^n . \rho_\alpha(t_0, t_1) \qquad (10)$$

According to conditions $(2)'$, $(5)'$, $(6)'$, $(7)'$, $(9)'$, $(10)'$and $(8)'$ we obtain equations (1), (2), (3), (4), (5), (6) and (10) respectively. In each of these cases if the concern constant replaced by h_α then by step-I of theorem 2.1 we see that $\{t_n\}$ is a ρ_α-Cauchy sequence in X. However by the completeness of X, there is a point p_1 in X such that $t_n \to p_1$. We can easily see that p_1 is a unique fixed point of f_z. By the help of steps-II, III of the above theorem 2.1, we can conclude the theorem 2.4.

III. CONCLUSION

We observe that condition $(1)'$, $(4)'$ are stronger than $(3)'$ and condition $(4)'$ is stronger than conditions $(6)'$ and $(8)'$ therefore the Theorem 2.1 has two corollaries corresponding to each of these two conditions $(1)'$, $(4)'$ and Theorem 2.4 has one corollary corresponding to $(4)'$ which is already mentioned as a corollary to the Theorem 2.1. This paper is extension of Nadler [5] as well as Gupta [8] according to some contractive conditions of Rhoades [6] which involve a single mapping.

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