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Bounds for the Zeros of Complex Polynomials

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ABSTRACT

In this paper we find certain bounds for the zeros of a polynomial with complex coefficients. Our results give generalizations and refinements of many known results in the field.

Mathematics Subject Classification : 30 C 10, 30 C 15

Keywords: Bound, Polynomial, Zeros

I.INTRODUCTION

Locating a region containing all or some of the zeros of a polynomial plays an important role in various branches of mathematics such as communication theory, coding theory, control theory, cryptography, signal processing, graph theory, mathematical biology etc. The problem of locating regions containing the zeros of a polynomial has a long history dating back to Gauss[2,3] who gave a bound for all the zeros of a polynomial in terms of its coefficients. A fresh start was made by Cauchy [2,3] who proved the following result:

Theorem A. Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n. If

$$M=\max_{0\le j\le n-1}\left|\frac{a_j}{a_n}\right|,$$

Then all the zeros of P(z) lie in $|z|<1+M$.

Another optimal bound for all the zeros of a polynomial was given by Fujiwara [2,3] in the following theorem:

Theorem B. If $P(z)=\sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n, then all the zeros of P(z) lie in

$$|z|\le 2\max\left[\left|\frac{a_{n-1}}{a_n}\right|,\left|\frac{a_{n-2}}{a_n}\right|^{1/2},\ldots\ldots,\left|\frac{a_0}{a_n}\right|^{1/n}\right].$$

Several improvements and generalizations of the above results are available in the literature. Recently Gulzar et al. [1] obtained a refinement of the zero bounds given by Cauchy, Toya, Carmichael and Mason, Williams (see [2,p. 122-126]) by proving the following result:

Theorem C. All the zeros of the polynomial $P(z)=\sum_{j=0}^{n} a_j z^j$ of degree n lie in the disk $|z|\le (1+M_p{}^q)^{1/q}$,

where

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$$M_p = \inf_{\lambda \in C} \left\{ \sum_{j=0}^{n} \left| \frac{\lambda a_j - a_{j-1}}{a_n} \right|^p \right\}^{1/p}, a_{-1} = 0,$$

p>1,q>1 with $p^{-1} + q^{-1} = 1.$

II .MAIN RESULTS

In this paper we prove the following theorem.

Theorem 1. All the zeros of the polynomial $P(z) = \sum_{j=0}^{n} a_j z^j$ of degree n lie in the disk

$$|z| \le \left(\frac{1 + \sqrt{1 + 4A^q}}{2} \right)^{\frac{1}{q}},\text{where}$$

$$A = \left(\sum_{j=0}^{n} \left| \frac{a_j \lambda_1 \lambda_2 + a_{j-1}(\lambda_1 - \lambda_2) - a_{j-2}}{a_n} \right|^p \right)^{\frac{1}{p}}, a_{-2} = a_{-1} = 0,$$

p>1,q>1 with $p^{-1} + q^{-1} = 1$ and $\lambda_1, \lambda_2 \in C$ with $a_n(\lambda_1 - \lambda_2) - a_{n-1} = 0$.

Taking $\lambda_1 = \lambda, \lambda_2 = 0$ in Theorem 1, we get the following interesting refinement of Theorem C:

Corollary 1. All the zeros of the polynomial $P(z) = \sum_{j=0}^{n} a_j z^j$ of degree n lie in the disk

$$|z| \le \left(\frac{1 + \sqrt{1 + 4B^q}}{2} \right)^{\frac{1}{q}},\text{where}$$

$$B = \left(\sum_{j=0}^{n} \left| \frac{\lambda a_{j-1} - a_{j-2}}{a_n} \right|^p \right)^{\frac{1}{p}} \ge M_p,$$

where p>1,q>1 with $p^{-1} + q^{-1} = 1$, $\lambda \in C$ and M_p is defined as in Theorem C.

Remark 1. Note that

$$\left(\frac{1 + \sqrt{1 + 4B^q}}{2} \right)^{\frac{1}{q}} < \left(1 + M_p{}^q\right)^{1/q}$$

if

$$\left(\frac{1+\sqrt{1+4B^q}}{2}\right) < \left(1+M_p{}^q\right) < \left(1+B^q\right)$$

i.e. if

$$1+\sqrt{1+4B^q} < 2+2B^q$$

or

$$1+4B^q < 1+4B^{2q}+4B^q$$

which is true.

Hence it follows that the bound given by Theorem 1 is sharper than the bound given by Theorem C.

III. PROOFS OF THEOREMS

Proof of Theorem 1: Consider the polynomial

$$F(z)=(\lambda_1 - z)(\lambda_2 + z)P(z)$$
$$=(\lambda_1 - z)(\lambda_2 + z)(a_n z^n + a_{n-1}z^{n-1} + ...+ a_1 z + a_0)$$
$$=\{\lambda_1\lambda_2 + (\lambda_1 - \lambda_2)z - z^2\}(a_n z^n + a_{n-1}z^{n-1} + ...+ a_1 z + a_0)$$
$$=-a_n z^{n+2} + \{a_n(\lambda_1-\lambda_2) - a_{n-1}\}z^{n+1} + \{a_n\lambda_1\lambda_2 + a_{n-1}(\lambda_1-\lambda_2) - a_{n-2}\} + ...$$
$$+\{a_2\lambda_1\lambda_2 + a_1(\lambda_1-\lambda_2) - a_0\}z^2 + \{a_1\lambda_1\lambda_2 + a_0\}z + a_0\lambda_1\lambda_2$$
$$=-a_n z^{n+2} + \{a_n\lambda_1\lambda_2 + a_{n-1}(\lambda_1-\lambda_2) - a_{n-2}\}z^n + ...+ \{a_2\lambda_1\lambda_2 + a_1(\lambda_1-\lambda_2) - a_0\}z^2$$
$$+\{a_1\lambda_1\lambda_2 + a_0\}z + a_0\lambda_1\lambda_2 .$$

Then , for $|z|>1$, we have, by using the hypothesis and the Holder's inequality,

$$|F(z)| \geq |a_n||z|^{n+2} - [\ |a_n\lambda_1\lambda_2 + a_{n-1}(\lambda_1-\lambda_2) - a_{n-2}||z|^n + ...+ |a_2\lambda_1\lambda_2 + a_1(\lambda_1-\lambda_2) - a_0||z|^2$$

$$+|a_2\lambda_1\lambda_2 + a_1(\lambda_1-\lambda_2) - a_0||z|^2 + |a_1\lambda_1\lambda_2 + a_0|z + |a_0|\lambda_1\lambda_2\ ]$$

$$=|a_n||z|^{n+2}\left[1-\sum_{j=0}^{n}|a_j\lambda_1\lambda_2 + a_{j-1}(\lambda_1-\lambda_2) - a_{j-2}|.\frac{1}{|z|^{n-j+2}}\right]$$

$$\geq |a_n||z|^{n+2}\left[1-\{\sum_{j=0}^{n}|a_j\lambda_1\lambda_2+a_{j-1}(\lambda_1-\lambda_2)-a_{j-2}|^p\}^{1/p}\{\sum_{j=0}^{n}\frac{1}{|z|^{(n-j+2)q}}\}^{1/q}\right]$$

$$=|a_n||z|^{n+2}\left[1-A\{\sum_{j=0}^{n}\frac{1}{|z|^{(n-j+2)q}}\}^{1/q}\right]$$

$$=|a_n||z|^{n+2}\left[1-A\left\{\frac{1}{|z|^{2q}}+\frac{1}{|z|^{3q}}+......+\frac{1}{|z|^{(n+2)q}}\right\}^{1/q}\right]$$

$$>|a_n||z|^{n+2}\left[1-A\left\{\frac{1}{|z|^{2q}}+\frac{1}{|z|^{3q}}+......+\frac{1}{|z|^{(n+2)q}}+.......\right\}^{1/q}\right]$$

$$=|a_n||z|^{n+2}\left[1-A.\frac{1}{|z|(|z|^q-1)^{1/q}}\right]$$

$$>0$$

if

$$|z|^{2q}-|z|^q-A^q>0.$$

Since the positive root of the real equation $x^2-x-A^q=0$ is $x=\frac{1+\sqrt{1+4A^q}}{2}$, it follows that

$|F(z)|>0$ if $|z|>\frac{1+\sqrt{1+4A^q}}{2}$. In other words, all the zeros of F(z) and hence P(z)lie in

$|z|\leq\frac{1+\sqrt{1+4A^q}}{2}$ and the result follows.

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